SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

A FURTHER EXTENSION OF THE LEIBNIZ RULE TO FRACTIONAL DERIVATIVES AND ITS RELATION TO PARSEVAL'S FORMULA*

THOMAS J. OSLER†

Abstract. The familiar Leibniz rule for the Nth derivative of the product of two functions is $D^N uv = \sum \binom{N}{n} D^{N-n}uD^n v$. A generalization of this formula for fractional derivatives is given as $D^\alpha uv = \sum a \binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma}uD^{an+\gamma}v$, where α need not be a natural number and $0 < a \leqq 1$. (The special case, $a = 1$, appeared previously.) Further generalizations of the Leibniz rule are also given and are derived from a generalization of Taylor's series given previously by the author. It is shown that these new series are generalizations of Parseval's formula from the study of Fourier series. Finally, new series expansions relating the special functions of mathematical physics are derived as special cases of the generalizations of the Leibniz rule. These series include a generalized Dougall's formula, several series of the Cardinal type, and a series related to a problem of Ramanujan.

1. Introduction. The fractional derivative of order α of $f(z)$ with respect to $g(z)$ is written $D^\alpha_{g(z)}f(z)$ and is an extension of the familiar derivative $d^\alpha f(z)/dg(z)^\alpha$ to nonintegral values of α. Fractional derivatives have been employed successfully in finding solutions to ordinary [9], partial [6], [18], and integral [5] equations. In these applications, the fractional derivative is advantageous because certain critical operations which are not obvious in a classical formulation are suggested by the notation itself. Consider, for example, the result

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x (x-t)^{\alpha-1}\int_0^t f(u)(t-u)^{\beta-1}\,du\,dt = \frac{1}{\Gamma(\alpha+\beta)}\int_0^x f(t)(x-t)^{\alpha+\beta-1}\,dt,$$

$\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$. In the notation of fractional derivatives, this last result reads

$$D_x^{-\alpha}D_x^{-\beta}f(x) = D_x^{-\alpha-\beta}f(x),$$

a result which students of the calculus would guess.

Fractional derivatives are also of value in exploring the properties of the higher transcendental functions. Consider the known, but not commonly seen, formula for the Bessel function of order v:

$$J_v(z) = (2z)^{-v}\pi^{-1/2}D_{z^2}^{-v-1/2}\frac{\cos z}{z}.$$

When v is $-1/2, -3/2, -5/2, \cdots$, this formula shows that $J_v(z)$ is an elementary function. Since $J_v(z)$, and many of the important special functions, can be represented as fractional derivatives of elementary functions, it seems reasonable that important properties of the higher transcendental functions could be derived from a knowledge of rules for manipulating fractional derivatives. This observation has appeared previously [7], [9], [11], [12], [13], [14]. The author's papers [11], [12], [13], [14], [15] have been concerned with extending familiar rules for derivatives from the elementary calculus (chain rule, Leibniz rule, Taylor's series) to the higher calculus of fractional derivatives.

---

* Received by the editors October 14, 1970, and in revised form February 12, 1971.

† Department of Mathematics, Rensselaer Polytechnic Institute, Troy, New York 12181.

As early as 1859, George Boole [2, Preface] wrote: "This question of the true value and proper place of symbolical methods is undoubtedly of great importance. Their convenient simplicity—their condensed power—must ever constitute their first claim upon attention." It is in this spirit that the Leibniz rule from the elementary calculus is extended in this paper and used in conjunction with fractional derivative representations of the special functions. In this way the "simplicity and condensed power" of the fractional derivative notation is exploited.

We list below successively more complex extensions of the familiar Leibniz rule,

$$D^N uv = \sum_{n=0}^{N} \binom{N}{n} D^{N-n} u D^n v.$$

Extension 1. If α is not a natural number, the Leibniz rule admits the simple generalization

$$D_z^\alpha u(z)v(z) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_z^{\alpha-n} u(z) D_z^n v(z) \tag{1.1}$$

which was known to Grunwald [8] as early as 1867. Other authors have also considered this formula [1], [11], [12], [15], [16], [19]. A simple derivation employing complex variable techniques and Taylor's series is given in the author's expository paper [15].

Extension 2. Equation (1.1) has a disturbing feature. If we interchange u and v, the formula remains unchanged on the left side, while on the right side this is not obvious since u is differentiated fractionally and v is differentiated in the usual elementary sense. A generalization of (1.1) in which the interchanging of u and v appears permissible on both sides is

$$D_z^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{n+\gamma} D_z^{\alpha-n-\gamma} u(z) D_z^{n+\gamma} v(z), \tag{1.2}$$

where

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)\Gamma(\beta+1)},$$

and γ is an arbitrary real or complex number. This series was first published by Watanabe [19] in 1931. The region of convergence of the series in the z-plane was first determined by the author [11], [12].

Extension 3. Our next extension shows that we can also differentiate fractionally with respect to an arbitrary function $g(z)$, and even more, the sum need not be over the integers n, but can be over a times n, where $0 < a \leqq 1$:

$$D_{g(z)}^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an+\gamma} D_{g(z)}^{\alpha-an-\gamma} u(z) D_{g(z)}^{an+\gamma} v(z). \tag{1.3}$$

This result is new.

Extension 4. By introducing the function $\theta(\zeta; z) = (g(\zeta) - g(z))q(\zeta)$, we can generalize the previous result to

$$D^{\alpha}_{g(z)}u(z)v(z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma}_{g(z)}[u(z)q(z)^{an+\gamma}] \cdot D^{an+\gamma}_{g(\zeta)}[v(\zeta)\theta_{g(\zeta)}(\zeta; z)q(\zeta)^{-an-\gamma-1}]\Big|_{\zeta=z}, \tag{1.4}$$

where again $0 < a \leqq 1$ and γ is arbitrary. This new result can be simplified provided $v(g^{-1}(0)) = 0$ as

$$D^{\alpha}_{g(z)}u(z)v(z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma}_{g(z)}[u(z)q(z)^{an+\gamma}] \cdot D^{an+\gamma-1}_{g(z)}\left[\frac{dv(z)}{dg(z)}q(z)^{-an-\gamma}\right]. \tag{1.4a}$$

Extension 5. The product uv can be replaced by a general function of two variables. This leads to the generalization

$$D^{\alpha}_{g(z)}f(z, z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma,an+\gamma}_{g(\xi),g(\zeta)} \cdot [f(\xi, \zeta)\theta_{g(\zeta)}(\zeta; z)q(\xi)^{an+\gamma}q(\zeta)^{-an-\gamma-1}]\Big|_{\substack{\xi=z\\ \zeta=z}}, \tag{1.5}$$

where again $0 < a \leqq 1$, and $\theta(\zeta : z) = (g(\zeta) - g(z))q(\zeta)$. Here the $D^{\alpha,\beta}_{g(\xi),h(\zeta)}f(\xi, \zeta)$ means operate on $f(\xi, \zeta)$ with $D^{\beta}_{h(\zeta)}$ holding ξ fixed followed by $D^{\alpha}_{g(\xi)}$ holding ζ fixed. If we set $f(\xi, \zeta) = u(\xi)v(\zeta)$ in (1.5), we obtain (1.4). If, in addition, $f(\xi, g^{-1}(0)) = 0$, (1.5) simplifies to a form corresponding to (1.4a):

$$D^{\alpha}_{g(z)}f(z, z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D^{\alpha-an-\gamma,an+\gamma-1}_{g(\xi),g(\zeta)} \cdot [f_{g(\zeta)}(\xi, \zeta)[q(\xi)/q(\zeta)]^{an+\gamma}]\Big|_{\substack{\xi=z\\ \zeta=z}}. \tag{1.5a}$$

The special case of (1.5) in which $q(\zeta) = 1$ and $a = 1$ (and thus $\theta_{g(\zeta)}(\zeta; z) = 1$) was given by the author in [11], [12]. All the preceding generalizations of the Leibniz rule are special cases of (1.5).

While derivations of special cases of our generalized Leibniz rule (1.5) have been given previously, the derivation presented in this paper is not an extension of previous methods. An entirely new technique is employed based on the author's previous generalization of Taylor's series to fractional derivatives [14].

The relationship between the generalized Leibniz rule and the familiar Parseval's formula [21, p. 37] from Fourier series is examined. We discover the interesting fact that Parseval's formula is a special case of the Leibniz rule in much the same way that a Fourier series is a special case of a Laurent series.

The paper concludes with an examination of several infinite series expansions derived from (1.5) by introducing specific functions for f, g, q, θ, and specific parameters for α, γ, and a. These series which relate the higher transcendental functions show one way in which fractional derivatives can be exploited in the study of the special functions.

In summary, then, this paper contributes the following items in mathematical analysis:

(i) The generalization of Leibniz rule (1.5) as well as its special cases (1.3), (1.4), (1.4a), and (1.5a) are new.

(ii) The derivation of the generalized Leibniz rule (1.5), based on a generalized Taylor's series, is new. (See § 4.)

(iii) The observation of the relation between the Leibniz rule and Parseval's formula is new. (See § 3.)

(iv) Several of the series expansions relating the special functions (see Table 5.2) appear to be new.

2. Fractional derivatives and special functions. In this section we review the definition of fractional differentiation and give examples of common special functions of mathematical physics represented by fractional derivatives of elementary functions.

The most common definition for the fractional derivative of $f(z)$ of order α found in the literature is the "Riemann–Liouville integral" [4], [5], [6], [7], [8], [9], [18]

$$D_z^\alpha f(z) = \Gamma(-\alpha)^{-1} \int_0^z f(t)(z-t)^{-\alpha-1}\, dt,$$

where $\operatorname{Re}(\alpha) < 0$. The concept of a fractional derivative with respect to an arbitrary function $g(z)$, $D_{g(z)}^\alpha f(z)$, was apparently introduced for the first time in the author's papers [11], [12], while the idea appeared earlier for certain specific functions $g(z)$ in [6]. The most convenient form of the definition for our purposes is given through a generalization of Cauchy's integral formula. A thorough motivation for the following precise definition is found in [11], [12].

DEFINITION 2.1. Let $f(z)$ be analytic in the simply connected region R. Let $g(z)$ be regular and univalent on R, and let $g^{-1}(0)$ be an interior or boundary point of R. Assume also that $\int_C f(z)g'(z)\, dz = 0$ for any simple closed contour C in $R \cup \{g^{-1}(0)\}$ through $g^{-1}(0)$. Then if α is not a negative integer, and z is in R, we define the *fractional derivative of order α of $f(z)$ with respect to $g(z)$* to be

$$D_{g(z)}^\alpha f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \int_{g^{-1}(0)}^{(z^+)} \frac{f(\zeta)g'(\zeta)\, d\zeta}{(g(\zeta)-g(z))^{\alpha+1}}. \tag{2.1}$$

For nonintegral α, the integrand has a branch line which begins at $\zeta = z$ and passes through $\zeta = g^{-1}(0)$. The limits of integration imply that the contour of integration starts at $g^{-1}(0)$, encloses z once in the positive sense, and returns to $g^{-1}(0)$ without cutting the branch line or leaving $R \cup \{g^{-1}(0)\}$. (See Fig. 2.1).

If α is a negative integer $-N$, $\Gamma(\alpha+1) = \infty$ while the integral in (2.1) vanishes. If we interpret (2.1) as the limit as α approaches $-N$, it then defines the derivative of order $-N$, or perhaps we should say the "Nth iterated integral of $f(z)$ with respect to $g(z)$."

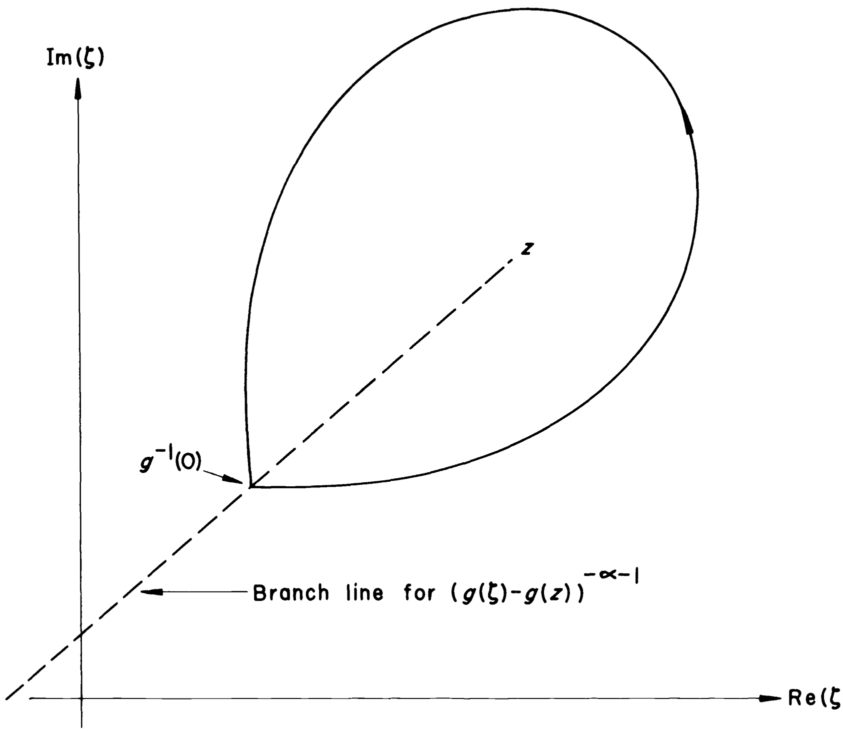

FIG. 2.1. *Branch line and contour of integration for Definition* 2.1 *of fractional differentiation*

It is important to notice that with the restrictions on $g(z)$ as given in Definition 2.1, the substitution $w = g(z)$ maintains the equality $D^{\alpha}_{w} f(w) = D^{\alpha}_{g(z)} f(g(z))$.

It is particularly interesting to set $g(z) = z - a$, for we find that

$$D^{\alpha}_{z-a} f(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{a}^{(z+)} f(\zeta)(\zeta - z)^{-\alpha - 1} \, d\zeta. \tag{2.2}$$

While ordinary derivatives with respect to z and $z - a$ are equal, (2.2) shows that this is not the case for fractional derivatives, since the value of the contour integral depends on the point $\zeta = a$ at which the contour crosses the branch line.

We also require fractional partial derivatives.

DEFINITION 2.2. Let $f(z, w)$ be an analytic function of two variables for z and w in the simply connected region R. Let $g(z)$ be regular and univalent on R, and let $g^{-1}(0)$ be an interior or boundary point of R. Assume also that $\int_C f(z, w) g'(w) \, dw = 0$ and $\int_C D^{\beta}_{g(w)} f(z, w) g'(z) \, dz = 0$ for any simple closed contour C in $R \cup \{g^{-1}(0)\}$ through $g^{-1}(0)$. Then if α and β are not negative integers, and z and w are in R we write

$$\begin{aligned} D^{\alpha,\beta}_{g(z),g(w)} f(z, w) &= D^{\alpha}_{g(z)}[D^{\beta}_{g(w)} f(z, w)] \\ &= \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{-4\pi^2} \int_{g^{-1}(0)}^{(z+)} \frac{g'(\zeta)}{(g(\zeta) - g(z))^{\alpha + 1}} \\ &\quad \cdot \int_{g^{-1}(0)}^{(w+)} \frac{f(\zeta, \xi) g'(\xi) \, d\xi \, d\zeta}{(g(\xi) - g(w))^{\beta + 1}}. \end{aligned} \tag{2.3}$$

Contour integrals of the type (2.1) occur often in the representations of special functions. These are particularly convenient for use with the generalized Leibniz rule (1.5). Fractional derivative representations of special functions are also found in [11], [12] and can be easily constructed from the tables in [4]. A few examples follow:

$$F(a,b;c;z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(b)} D_z^{b-c} z^{b-1}(1-z)^{-a},$$

$$_1F_1(a;c;z) = \frac{\Gamma(c)z^{1-c}}{\Gamma(a)} D_z^{a-c} e^z z^{a-1},$$

$$J_v(z) = \frac{z^{-v}}{2^v\sqrt{\pi}} D_{z^2}^{-v-1/2} \frac{\cos z}{z},$$

$$P_v^u(z) = \frac{(1-z^2)^{u/2}}{\Gamma(v+1)2^v} D_{1-z}^{v+u}(1-z^2)^v.$$

Having reviewed briefly the definition of fractional differentiation and its relation to the special functions, we proceed to show a formal correspondence between our generalized Leibniz rule and a familiar formula from the elementary study of Fourier series.

3. The Leibniz rule and Parseval's formula. In this section we formally examine the special case of the generalized Leibniz rule (1.4). By holding z fixed, and making a suitable change of variables, we shall see that Parseval's formula [21, p. 37], familiar from the study of Fourier series, emerges.

Let us begin by assuming that (1.4) is true. With $g(z) = z$ we have

$$\frac{1}{\Gamma(\alpha+1)} D_z^\alpha u(z)v(z) = \sum_{n=-\infty}^{\infty} \frac{a}{\Gamma(\alpha - an - \gamma + 1)} D_z^{\alpha-an-\gamma}[u(z)q(z)^{an+\gamma}] \cdot \frac{1}{\Gamma(an+\gamma+1)} D_\zeta^{an+\gamma}[v(\zeta)q(\zeta)^{-an-\gamma-1}\theta_\zeta(\zeta;z)]\Big|_{\zeta=z},$$

where we recall that $\theta(\zeta;z) = (\zeta - z)q(\zeta)$ and $0 < a \leqq 1$. Making use of the contour integral representation for fractional derivatives (2.2), we get

$$(3.1)\quad \frac{1}{2\pi i}\int_0^{(z^+)} \frac{u(t)v(t)\,dt}{(t-z)^{\alpha+1}} = \sum_{n=-\infty}^{\infty} \frac{-a}{4\pi^2}\int_0^{(z^+)} \frac{u(t)q(t)^{an+\gamma}\,dt}{(t-z)^{\alpha-an-\gamma+1}} \cdot \int_0^{(z^+)} \frac{v(t)q(t)^{-an-\gamma-1}\theta_t(t;z)\,dt}{(t-z)^{an+\gamma+1}}.$$

We now fix z and select the contours of integration appearing above to coincide with the curve defined by $|\theta(t;z)| = |\theta(0;z)|$; that is, the contour which passes through the origin (in the t-plane) on which $\theta(t;z)$ has constant modulus. This contour we assume is a closed curve which can be parametrized by the variable ϕ such that

$$(3.2)\quad \theta(t;z) = |\theta(0;z)|e^{i\phi},$$

with $\phi_0 < \phi < \phi_0 + 2\pi$. Using (3.2) to change the variable of integration from

t to ϕ in (3.1), and writing $u(t) = u[\phi], \cdots$, we get

$$(3.3)\qquad \begin{aligned} &\frac{a}{2\pi}\int_{\phi_0}^{\phi_0+2\pi/a} f(\phi)h(\phi)\,d\phi \\ &= \sum_{n=-\infty}^{\infty} \frac{a}{2\pi}\int_{\phi_0}^{\phi_0+2\pi/a} f(\phi)e^{ian\phi}\,d\phi \cdot \frac{a}{2\pi}\int_{\phi_0}^{\phi_0+2\pi/a} h(\phi)e^{-ian\phi}\,d\phi, \end{aligned}$$

where we have set

$$f(\phi) = \begin{cases} \dfrac{u[\phi]q[\phi]^{\alpha+1}e^{i(\gamma-\alpha)\phi}}{\theta_t[\phi\,;z]} & \text{for } \phi_0 < \phi < \phi_0 + 2\pi, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h(\phi) = \begin{cases} v[\phi]e^{-i\gamma\phi} & \text{for } \phi_0 < \phi < \phi_0 + 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

We recognize (3.3) as Parseval's formula [21, p. 37].

Now consider an analytic function $w(t)$ expanded in a Laurent series $w(t) = \sum a_n(t - z)^n$. If we restrict t to the circle $z + |z|e^{i\phi}$, the Laurent series becomes the Fourier series $w(z + |z|e^{i\phi}) = \sum a_n|z|^n e^{in\phi}$. Note the similarity between this and the previous calculation, in which we held z fixed and examined u and v on a particular closed curve in the t-plane. In fact, if $\theta(t\,;z) = t - z$, the contour of integration defined by (3.2) is identical to the circle on which the Laurent series was just examined. We conclude that by holding z fixed, the generalized Leibniz rule reduces to Parseval's formula in the same way that Laurent's series reduces to a Fourier series. Thus our extended Leibniz rule is a generalization of Parseval's formula.

4. The extended Leibniz rule. In formally examining the special case of the extended Leibniz rule (1.4) in the previous section, we have seen that it is related to the Parseval's formula familiar from the study of Fourier series. We now proceed to derive the extended Leibniz rule rigorously. We shall see that the derivation of the Leibniz rule follows from the generalized Taylor's series in much the same way that the Parseval's relation follows from the Fourier series.

We begin by stating and proving the special case of the extended Leibniz rule in which $g(z) = z$.

THEOREM 4.1. (i) *Let R be a simply connected region in the complex plane having the origin as an interior or boundary point.*

(ii) *Let $f(\xi, \zeta)$ satisfy the conditions of Definition 2.2 for the existence of $D^{\alpha,b}_{\xi,\zeta} f(\xi, \zeta)$ and $D^{\alpha}_{z} f(z, z)$ for ξ, ζ, and z in R.*

(iii) *Let $\theta(\zeta\,; z) = (\zeta - z)q(\zeta)$ be a given function such that $q(\zeta)$ is analytic for $\zeta \in R$, and $q(\zeta)$ is never zero on R.*

(iv) *Assume that the curves $C(z) = \{\zeta \mid |\theta(\zeta\,; z)| = |\theta(0\,; z)|\}$ are simple and closed for each z such that $C(z) \subset R \cup \{0\}$. Assume also that each curve defined by $\{\zeta \mid |\theta(\zeta\,; z)| = \text{const.}\}$ interior to $C(z)$ is simple and closed.*

(v) *Call $S = \{z \mid C(z) \subset R \cup \{0\}\}$.*

Then for $z \in S$, $0 < a \leqq 1$, *and all* α *and* γ *such that* $\binom{\alpha}{an+\gamma}$ *is defined,*

$$D_z^\alpha f(z,z) = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an+\gamma} D_{\xi,\zeta}^{\alpha-an-\gamma,an+\gamma} \cdot [f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\xi)^{an+\gamma}q(\zeta)^{-an-\gamma-1}]|_{\xi=\zeta=z}, \tag{4.1}$$

where $\theta_\zeta(\zeta;z) = d\theta(\zeta;z)/d\zeta$.

Proof. The $C(z)$ are the curves in the complex ζ-plane which pass through the origin, over which the amplitude of $\theta(\zeta;z)$ is constant. For example, if $\theta(\zeta;z) = \zeta - z$, then $C(z)$ is the circle centered at $\zeta = z$ passing through the origin. By restricting z to S (described in (v)), we insure that the curves $C(z)$ are contained in the region $R \cup \{0\}$ on which $f(\xi,\zeta)$ is sufficiently regular for manipulations which follow. In particular, $f(\xi,\zeta)$ can be expanded in a generalized Taylor's series for $\zeta \in C(z)$ in powers of $\theta(\zeta;z)$ since (ii), (iii) and (iv) are all that is required for its validity [14]. We obtain

$$f(\xi,\zeta) = \sum_{n=-\infty}^{\infty} \frac{aD_\zeta^{an+\gamma}[f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\zeta)^{-an-\gamma-1}]|_{\zeta=z}(q(\zeta)(\zeta-z))^{an+\gamma}}{\Gamma(an+\gamma+1)}.$$

Multiply both sides of this last expression by $\Gamma(\alpha+1)(\xi-z)^{-\alpha-1}/(2\pi i)$ and set $\zeta = \xi$:

$$\frac{\Gamma(\alpha+1)}{2\pi i}\frac{f(\xi,\xi)}{(\xi-z)^{\alpha+1}} = \sum_{n=-\infty}^{\infty} \frac{\alpha\Gamma(\alpha+1)q(\xi)^{an+\gamma}}{\Gamma(an+\gamma+1)2\pi i(\xi-z)^{\alpha-an-\gamma+1}} \cdot D_\zeta^{an+\gamma}[f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\zeta)^{-an-\gamma-1}]|_{\zeta=z}. \tag{4.2}$$

Since (4.2) converges for ξ on the curve $C(z)$ in the complex ξ-plane, we can integrate both sides along the contour $C(z)$ with respect to ξ starting and ending at $\xi = 0$. It is clear that we can integrate term by term along the contour $C(z)$, since (4.2) is really a Fourier series in the variable ϕ when we replace $\theta(\xi;z)$ by $|\theta(0;z)|e^{i\phi}$:

$$\frac{\Gamma(\alpha+1)}{2\pi i}\int_{C(z)} \frac{f(\xi,\xi)\,d\xi}{(\xi-z)^{\alpha+1}} = \sum_{n=-\infty}^{\infty} \frac{a\Gamma(\alpha+1)}{\Gamma(an+\gamma+1)2\pi i} \cdot \int_{C(z)} \frac{q(\xi)^{an+\gamma}D_\zeta^{an+\gamma}[f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\zeta)^{-an-\gamma-1}]|_{\zeta=z}\,d\xi}{(\xi-z)^{\alpha-an-\gamma+1}}.$$

Comparing the integrals above with the definitions of fractional differentiation (2.2) and (2.3) we see at once that the generalized Leibniz rule (4.1) is obtained.

Equation (4.1) can be simplified to

$$D_z^\alpha f(z,z) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D_{\xi,\zeta}^{\alpha-an-\gamma,an+\gamma-1}[f_\zeta(\xi,\zeta)[q(\xi)/q(\zeta)]^{an+\gamma}]|_{\xi=\zeta=z} \tag{4.3}$$

if we add the restriction that $f(\xi,0) = 0$.

COROLLARY 4.1. *With the hypothesis of Theorem* 4.1 *and* $f(\xi,0) = 0$, *the relation* (4.3) *is valid.*

Proof. Comparing (4.1) and (4.3) it is clear we must show that

$$E = D_{\xi,\zeta}^{\alpha-an-\gamma,an+\gamma}[f(\xi,\zeta)\theta_\zeta(\zeta;z)q(\xi)^{an+\gamma}q(\zeta)^{-an-\gamma-1}]|_{\zeta=\xi=z}$$
$$= D_{\xi,\zeta}^{\alpha-an-\gamma,an+\gamma-1}[f_\zeta(\xi,\zeta)[q(\xi)/q(\zeta)]^{an+\gamma}]|_{\xi=\zeta=z}.$$

The left-hand side of this last relation can be written as

$$E = D_\xi^{\alpha-an-\gamma}\left[q(\xi)^{an+\gamma}\frac{\Gamma(an+\gamma+1)}{2\pi i}\int_0^{(z+)}\frac{f(\xi,\zeta)\theta_\zeta(\zeta;z)\,d\zeta}{\theta(\zeta;z)^{an+\gamma+1}}\right]\Bigg|_{\xi=z}$$

using (2.2) and (2.3). Integrating by parts we get

$$E = D_\xi^{\alpha-an-\gamma}\left[q(\xi)^{an+\gamma}\frac{\Gamma(an+\gamma)}{2\pi i}\int_0^{(z+)} f_\zeta(\xi,\zeta)\theta(\zeta;z)^{-an-\gamma}\,d\zeta\right]\Bigg|_{\xi=z},$$

where the jump term vanished because $f(\xi,0) = 0$. Rewriting this last integral using the definitions of fractional differentiation (2.2) and (2.3), and $\theta(\zeta;z) = (\zeta - z)q(\zeta)$, we see at once that the corollary is proved.

We complete our derivation by extending the Leibniz rule to the case in which we differentiate with respect to an arbitary function $g(z)$.

COROLLARY 4.2. *Assume the hypothesis of Theorem* 4.1 *and the additional conditions*

(i) $g(w)$ *is regular and univalent for* $w \in g^{-1}(R)$,
(ii) $F(s,t) = f(g(s),g(t))$,
(iii) $\Xi(s;w) = \theta(g(s);g(w)) = (g(s) - g(w))Q(s)$,
(iv) $q(g(s)) = Q(s)$.

Then

$$D_{g(w)}^\alpha F(w,w) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D_{g(s),g(t)}^{\alpha-an-\gamma,an+\gamma} \cdot\left[F(s,t)\frac{d\Xi(t;w)}{dg(t)}Q(s)^{an+\gamma}Q(t)^{-an-\gamma-1}\right]\Bigg|_{s=t=w} \tag{4.4}$$

for $w \in g^{-1}(S)$, $0 < a \leqq 1$, *and all* α *and* γ *for which* $\binom{\alpha}{an+\gamma}$ *is defined.*

If in addition we have

(v) $F(s, g^{-1}(0)) = 0$,

then (4.4) *can be simplified to*

$$D_{g(w)}^\alpha F(w,w) = \sum_{n=-\infty}^{\infty} a\binom{\alpha}{an+\gamma} D_{g(s),g(t)}^{\alpha-an-\gamma,an+\gamma-1} \cdot\left[\frac{\partial F(s,t)}{\partial g(t)}[Q(s)/Q(t)]^{an+\gamma}\right]\Bigg|_{s=t=w}. \tag{4.5}$$

Proof. The proof of this corollary follows at once upon replacing z by $g(w)$ in Theorem 4.1 and Corollary 4.1, since $D_z^\alpha f(z) \equiv D_{g(w)}^\alpha f(g(w))$.

Remark. In Theorem 4.1 and the above corollaries, the Leibniz rule is valid for all α and γ for which

$$(4.6) \qquad \binom{\alpha}{an+\gamma} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-an-\gamma+1)\Gamma(an+\gamma+1)}$$

is defined. Since $\Gamma(z)$ is analytic except for poles at $z = 0, -1, -2, \cdots$; and since $1/\Gamma(z)$ is entire, the only values of α for which (4.6) is suspect are $\alpha = -1, -2, -3, \cdots$. It is well known that when α is a negative integer, $a = 1$, and $\gamma = 0$, the binomial coefficient (4.6) is defined; however, if α is a negative integer and $\gamma \neq 0$, then (4.6) is not defined for every integer n. A glance at (4.2) shows that the troublesome $\Gamma(\alpha + 1)$ appears in the numerator on both sides of the equation. If we divide both sides of our Leibniz rule by $\Gamma(\alpha + 1)$, this difficulty disappears. Thus we conclude that the restrictions on α and γ for the validity of the generalized Leibniz rule are needed only because the notation $\binom{\alpha}{an+\gamma}$ is convenient. When we use the Leibniz rule in the next section, to derive series expansions relating the special functions, we will divide by $\Gamma(\alpha + 1)$ and conclude that restrictions on α and γ are unnecessary.

We have completed our rigorous examination of the generalized Leibniz rule, and now turn to examples of infinite series relating special functions which are derived from it.

5. Examples. Before ending our discussion, it seems appropriate to examine direct consequences of our new formulas. We select specific functions for $f(\xi, \zeta)$, $g(z), q(\zeta)$, and specific parameters for α, γ and a in our generalized Leibniz rule (1.5). A list of the selections is given in Table 5.1. The fractional derivatives encountered can be computed with the help of the extensive table in [4, vol. 2, pp. 181–200] and also with the short table of fractional derivative representations for special functions in [12, p. 668]. The results of this simple procedure appear in Table 5.2. A similar table, restricted to the special case of (1.5) in which $q(\zeta) = 1$ and $a = 1$, appeared in [11], [12]. The notation for the special functions used is that of Erdélyi et al. [3], [4].

We call particular attention to the following series from Table 5.2.

Extension of Dougall's formula. Series 9 is a generalization of "Dougall's formula" [3, vol. 1, p. 7]. Dougall's formula is the special case of series 9 in which $a = 1$.

Series of the Cardinal type. Series 2 through 8 are of the Cardinal type [20, pp. 62–71]. A Cardinal series gives the values of a function $f(\alpha)$ when the values of $f(\alpha)$ are known only at $\alpha = an + \gamma$, where $0 < a \leqq 1$ and γ are fixed and $n = 0, \pm 1, \pm 2, \cdots$. If we set $u(z) = 1$ in (1.3), we obtain

$$(5.1) \qquad \frac{g(z)^{\alpha} D^{\alpha}_{g(z)} v(z)}{\Gamma(\alpha+1)} = \sum_{n=-\infty}^{\infty} \frac{a \sin \pi(\alpha - an - \gamma)}{\pi(\alpha - an - \gamma)} \frac{g(z)^{an+\gamma} D^{an+\gamma}_{g(z)} v(z)}{\Gamma(an+\gamma+1)}.$$

Thus if $f(\alpha)$ is of the form

$$f(\alpha) = \frac{g(z)^{\alpha} D^{\alpha}_{g(z)} v(z)}{\Gamma(\alpha+1)},$$

TABLE 5.1
Choices for functions and parameters in the generalized Leibniz rule (1.5) *from which the series in Table* 5.2 *are derived*

Series No.	$f(\xi, \zeta)$	$q(\zeta)$	$g(z)$	α	γ
1	$e^{A\xi}\zeta$	$e^{B\zeta}$	z	$N = 1, 2, 3, \cdots$	0
2	$\zeta^{B-1}(1-\zeta)^{-A}$	1	z	$B - C$	γ
3	$e^{\zeta}\zeta^{A-1}$	1	z	$A - B$	$A - C$
4	$(\cos \zeta)/\zeta$	1	z^2	$-\nu - 1/2$	$-1/2 - B$
5	$(\cosh \zeta)/\zeta$	1	z^2	$-\nu - 1/2$	$-1/2 - B$
6	$(\sin \zeta)/\zeta$	1	z^2	$-\nu - 1/2$	$-1/2 - B$
7	$(\sinh \zeta)/\zeta$	1	z^2	$-\nu - 1/2$	$-1/2 - B$
8	$(1-\zeta^2)^{\nu}$	1	$1 - z$	$\nu + \mu$	γ
9	$\xi^{B+C-2}\zeta^{A+D-2}$	1	z	$A + C - 2$	$A - 1$
10	$\xi^{b-1}(1-\xi)^{-e}\zeta^{B-1}(1-\zeta)^{-E}$	1	z	$b + B - d - D$	$B - D$
11	$\xi^{D-A-1}\zeta^{C-B-1}(\xi\zeta+1)^{-E}$	1	z	$C - A - 1$	$C - 1$
12	$e^{A\xi - B\zeta}\xi^{P-1}$	$e^{C\zeta}$	z	α	0
13	ξ^{B-1}	$\zeta + A$	z	α	0
14	$\xi^{A-1}\zeta^{B}$	$\zeta^k + P^k$	z	α	γ
15	$\xi^{A-1}\zeta^{B}$	$\exp(\zeta^k)$	z	α	γ
16	$\xi^{A}\zeta^{B-1}{}_rF_s(a_1, \cdots, a_r; b_1, \cdots, b_s; \zeta)$	1	z	α	γ

then (5.1) yields the Cardinal series

$$f(\alpha) = \sum_{n=-\infty}^{\infty} \frac{a \sin \pi(\alpha - an - \gamma)}{\pi(\alpha - an - \gamma)} f(an + \gamma).$$

A problem of Ramanujan. The series 12 is a generalization of the series

$$\varphi(z) = e^{-z} \sum_{n=0}^{\infty} \frac{(n+1)^n (ze^{-z})^n}{n!}$$

considered by Ramanujan [17, p. 332, Question 738]. Ramanujan set as a portion of a problem the demonstration that $\varphi(z) \equiv 1$ for $0 \leqq z \leqq 1$. This problem can be

TABLE 5.2

Series expansions derived from the generalized Leibniz rule
Note: Unless otherwise stated, $0 < a \leqq 1$ *in all series*

Series No.	Series Expansion
1	$NA^{N-1} = \sum_{n=1}^{N} \binom{N}{n} (A + Bn)^{N-n}(-Bn)^{n-1}, N = 1, 2, 3 \cdots,$ $a = 1$
2	$\frac{\pi {}_2F_1(A, B; C; z)}{a\Gamma(C)\Gamma(B - C + 1)} = \sum_{n=-\infty}^{\infty} \frac{\sin((an + \gamma + C - B)\pi) {}_2F_1(A, B; B - \gamma - an; z)}{(an + \gamma + C - B)\Gamma(an + \gamma + 1)\Gamma(B - \gamma - an)},$ $\operatorname{Re}(z) < 1/2, \quad 0 < \operatorname{Re}(B)$
3	$\frac{\pi {}_1F_1(A; B; z)}{a\Gamma(B)\Gamma(A - B + 1)} = \sum_{n=-\infty}^{\infty} \frac{\sin((an + B - C)\pi) {}_1F_1(A; C - an; z)}{(an + B - C)\Gamma(an + A - C + 1)\Gamma(C - an)},$ $\operatorname{Re}(A) > 0$
4 through 7	$\mathscr{F}_\nu(z) = \frac{a\Gamma(1/2 - \nu)(z/2)^{\nu - B}}{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin((an + \nu - B)\pi)\mathscr{F}_{B-an}(z)}{(an + \nu - B)\Gamma(an - B + 1/2)} \left(\frac{z}{2}\right)^{an},$ where $\mathscr{F}_\nu = J_\nu, I_\nu, \mathbf{H}_\nu$ and $\mathbf{L}_\nu$, respectively, for series 4, 5, 6, and 7
8	$\frac{\pi P_\nu^\mu(z)}{a\Gamma(\nu + \mu + 1)} = \sum_{n=-\infty}^{\infty} \frac{\sin((an + \gamma - \nu - \mu)\pi)}{(an + \gamma - \nu - \mu)} \cdot \frac{P_\nu^{an+\gamma-\nu}(z)}{\Gamma(an + \gamma + 1)} \left(\frac{1 - z}{1 + z}\right)^{(an+\gamma-\nu-\mu)/2},$ $-1 < \operatorname{Re}(\nu), \quad 0 < \operatorname{Re}(z)$
9	$\frac{\Gamma(A + B + C + D - 3)}{a\Gamma(A + C - 1)\Gamma(A + D - 1)\Gamma(B + C - 1)\Gamma(B + D - 1)} = \sum_{n=-\infty}^{\infty} \frac{1}{\Gamma(an + A)\Gamma(an + B)\Gamma(C - an)\Gamma(D - an)},$ $1 < \operatorname{Re}(B + C), \quad 1 < \operatorname{Re}(A + D), \quad 3 < \operatorname{Re}(A + B + C + D).$
10	$\frac{\Gamma(b + B - 1) {}_2F_1(e + E, b + B - 1; d + D - 1; z)}{a\Gamma(d + D - 1)\Gamma(b + B - d - D + 1)\Gamma(b)\Gamma(B)} = \sum_{n=-\infty}^{\infty} \frac{{}_2F_1(e, b; d + an; z) {}_2F_1(E, B; D - an; z)}{\Gamma(an + B - D + 1)\Gamma(an + d)\Gamma(b - d - an + 1)\Gamma(D - an)},$ $0 < \operatorname{Re}(b), \quad 0 < \operatorname{Re}(B), \quad 1/2 > \operatorname{Re}(z), \quad 1 < \operatorname{Re}(b + B)$
11	$\frac{\Gamma(C + D - A - B - 1) {}_3F_2\left[\begin{matrix} E, (C + D - A - B - 1)/2, (C + D - A - B)/2; -z^2 \\ (D - B)/2, (D - B + 1)/2 \end{matrix}\right]}{a\Gamma(C - A)\Gamma(C - B)\Gamma(D - A)\Gamma(D - B)} = \sum_{n=-\infty}^{\infty} \frac{{}_3F_2\left[\begin{matrix} E, C - B, D - A; -z^2 \\ D + an, 1 - B - an \end{matrix}\right]}{\Gamma(an + C)\Gamma(an + D)\Gamma(1 - A - an)\Gamma(1 - B - an)},$ $0 < \operatorname{Re}(D - A), \quad \operatorname{Re}(C - B), \quad 1 < \operatorname{Re}(C + D - A - B)$
12	${}_1F_1(P; P - \alpha; (A - B)z) = e^{-(C+B)z} \sum_{n=0}^{\infty} \frac{(-\alpha)_n {}_1F_1(P; P - \alpha + n; (A + Cn)z)}{(P - \alpha)_n n!(Cn + B + C)^{-n}(ze^{-Cz})^{-n}},$ $a = 1$ in (1.5), $\operatorname{Re}(P) > 0$

TABLE 5.2 (Cont.)

Series No.	Series Expansion
13	$\frac{1}{2} = \sum_{n=0}^{\infty} \frac{(2n)!(-\alpha)_n(-Az)^n(z+A)^{-2n}{}_2F_1(-n, B; B-\alpha+n; -z/A)}{(B-\alpha)_n(n!)^2},$ $a = 1$ in (1.5)
14	$\frac{\Gamma(A+B)}{a\Gamma(A+B-\alpha)\Gamma(\alpha+1)\Gamma(A)\Gamma(B+1)} = \sum_{n=-\infty}^{\infty} \frac{{}_{k+1}F_k\left[\begin{matrix}-an-\gamma, A/k, (A+1)/k, \cdots, (A+k-1)/k; -z^k/P^k \\ (an+A+\gamma-\alpha)/k, (an+A+\gamma-\alpha+1)/k, \cdots, (an+A+\gamma-\alpha+k-1)/k\end{matrix}\right]}{\Gamma(an+\gamma+1)\Gamma(an+A+\gamma-\alpha)\Gamma(\alpha-\gamma-an+1)\Gamma(B-\gamma-an+1)} \cdot {}_{k+1}F_k\left[\begin{matrix}an+\gamma, B/k, (B+1)/k, \cdots, (B+k-1)/k; -z^k/P^k \\ (B-an-\gamma+1)/k, \cdots, (B-an-\gamma+k)/k\end{matrix}\right],$ $-1 < \mathrm{Re}(A), \quad 0 < \mathrm{Re}(B), \quad k = 1, 2, 3, \cdots$
15	$\frac{\Gamma(A+B)}{a\Gamma(A+B-\alpha)\Gamma(\alpha+1)\Gamma(A)\Gamma(B+1)} = \sum_{n=-\infty}^{\infty} \frac{{}_kF_k\left[\begin{matrix}B/k, (B+1)/k, \cdots, (B+k-1)/k; -(an+\gamma)z^k \\ (B-an-\gamma+1)/k, (B-an-\gamma+2)/k, \cdots, (B-an-\gamma+k)/k\end{matrix}\right]}{\Gamma(an+A+\gamma-\alpha)\Gamma(an+\gamma+1)\Gamma(\alpha-\gamma-an+1)\Gamma(B-\gamma-an+1)} \cdot {}_kF_k\left[\begin{matrix}A/k, (A+1)/k, \cdots, (A+k-1)/k; (an+\gamma)z^k \\ (an+A+\gamma-\alpha)/k, (an+A+\gamma-\alpha+1)/k, \cdots, (an+A+\gamma-\alpha+k-1)/k\end{matrix}\right],$ $-1 < \mathrm{Re}(A), \quad 0 < \mathrm{Re}(B), \quad k = 1, 2, 3, \cdots$
16	$\frac{\Gamma(A+B)_{r+1}F_{s+1}\left[\begin{matrix}A+B, a_1, \cdots, a_r; z \\ A+B-\alpha, b_1, \cdots, b_s\end{matrix}\right]}{a\Gamma(A+B-\alpha)\Gamma(\alpha+1)\Gamma(A+1)\Gamma(B)} = \sum_{n=-\infty}^{\infty} \frac{{}_{r+1}F_{s+1}\left[\begin{matrix}B, a_1, \cdots, a_r; z \\ B-\gamma-an, b_1, \cdots, b_s\end{matrix}\right]}{\Gamma(an+\gamma+1)\Gamma(an+A-\alpha+\gamma+1)\Gamma(\alpha-\gamma-an+1)\Gamma(B-\gamma-an)},$ $-1 < \mathrm{Re}(A), \quad 0 < \mathrm{Re}(B)$

solved easily from series 12 if we set $\alpha = -1$, $B = 0$, $C = 1$, and $A = P + 1$ and obtain

(5.2)
$$
\begin{aligned}
&{}_1F_1(P; P+1; (P+1)z) \\
&= e^{-z} \sum_{n=0}^{\infty} \frac{(n+1)^n(ze^{-z})^n}{n!} \frac{n!}{(P+1)_n} {}_1F_1(P; P+n+1; (P+n+1)z)
\end{aligned}
$$

where $\mathrm{Re}(P) > 0$. If we could set $P = 0$ on both sides of (5.2), we would answer Ramanujan's question at once since ${}_1F_1(0; c; x) = 1$. However, the restriction $\mathrm{Re}(P) > 0$ does not permit us to set $P = 0$. Instead, we show that for fixed z,

$0 \leqq z < 1$, the series (5.2) converges uniformly in P, for $0 \leqq P \leqq 1$. This uniform convergence permits us to let P approach zero term by term in (5.2) and thereby solve Ramanujan's problem. As n approaches infinity, for fixed z, $0 \leqq z \leqq 1$, and all P such that $0 \leqq P \leqq 1$,

$$
\begin{aligned}
&{}_1F_1(P; P+n+1; (P+n+1)z) \\
&\quad = (1-z)^{-P}\left[1 - \frac{P(P+1)}{2(P+n+1)}\left(\frac{z}{1-z}\right)^2 + O(|P+n+1|^{-2})\right],
\end{aligned}
$$

[3, vol. 1, p. 280]. Thus

$$
\left| \frac{n!}{(P+1)_n} {}_1F_1(P; P+n+1; (P+n+1)z) \right|
$$

is bounded and the series (5.2) converges uniformly in P, for $0 \leqq P \leqq 1$, by the familiar test of Weierstrass. Thus we have shown that $\varphi(z) \equiv 1$ for $0 \leqq z < 1$ and have answered Ramanujan's question.

Note on restrictions in Table 5.2. The restrictions obtained from the hypothesis of Theorem 4.1 for the validity of the series in Table 5.2 are sometimes too strong. Consider, for example, series 9. It is known that only the restriction $\operatorname{Re}(A+B+C+D) > 3$ is necessary. The restrictions $\operatorname{Re}(B+C) > 1$ and $\operatorname{Re}(A+D) > 1$ are not needed and emerge from item (ii) of the hypothesis of Theorem 4.1 in which we require that $D^{a,b}_{\xi,\zeta} f(\xi, \zeta)$ be defined. Since Table 5.2 is provided to illustrate our general expansions, all restrictions emerging from the theorems of this paper are listed.

6.1. Concluding thoughts. In 1695 Leibniz [10], in a letter to J. Bernoulli, expressed his interest in the fact that the binomial series

$$
(A+B)^N = \sum_{n=0}^{N} \binom{N}{n} A^{N-n} B^n
$$

and the rule for the derivative of a product

$$
D^N uv = \sum_{n=0}^{N} \binom{N}{n} D^{N-n} u D^n v
$$

look so similar:

"There are yet many things latent in these progressions of summation and differentiation, which will gradually appear. There is thus notable agreement between the numerical powers of binomial and differential expansions; and I believe that I do not know what is hidden there."

Bernoulli answered: "Nothing is more elegant than the agreement which you have observed between the numerical power of the binomial and differential expansions; there is no doubt that something is hidden there."

Indeed the giants of analysis were correct. At the time of these letters, Newton had invented an extension of the binomial theorem to fractional powers, but the extension of the product rule had to await the invention of the fractional calculus.

Furthermore, the binomial series admits the generalization [14]

$$(6.1)\qquad (A + B)^\alpha = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an + \gamma} A^{\alpha - an - \gamma} B^{an + \gamma},$$

where $0 < a \leqq 1$ and $|A/B| = 1$. Equation (6.1) resembles our generalized Leibniz rule (6.2):

$$(6.2)\qquad D^\alpha uv = \sum_{n=-\infty}^{\infty} a \binom{\alpha}{an + \gamma} D^{\alpha - an - \gamma} u D^{an + \gamma} v.$$

Moreover, (6.1) is a special case of the generalized Taylor series from which (6.2) is derived in this paper. Thus a reason for the similarity in the two series is made evident.

It is already clear, however, that even further results "lay hidden." The Leibniz rule for functions of the operator D more general than D^α was given as early as 1930 by Emil Post [16]. Undoubtedly Post's form of the product rule can be generalized to reveal further connections prophesied by Leibniz.

Acknowledgments. The author wishes to thank the referees for several remarks which have improved the paper and for the reference to Ramanujan's problem. Thanks are due to Professor Joaquin B. Diaz of Rensselaer Polytechnic Institute for informative discussions concerning the Cardinal series.

REFERENCES

[1] M. A. BASSAM, *Some properties of the Holmgren-Riesz transform*, Ann. Scuola Norm. Sup. Pisa, 15 (1961), no. 3, pp. 1–24.
[2] G. BOOLE, *Treatise on Differential Equations*, Cambridge Univ. Press, Cambridge, 1859.
[3] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, 3 vols., McGraw-Hill, New York, 1953, 1955.
[4] ———, *Tables of Integral Transforms*, 2 vols., McGraw-Hill, New York, 1954.
[5] A. ERDÉLYI, *An integral equation involving Legendre functions*, J. Soc. Indust. Appl. Math., 12 (1964), pp. 15–30.
[6] ———, *Axially symmetric potentials and fractional integration*, Ibid., 13 (1965), pp. 216–228.
[7] I. M. GEL'FAND AND G. E. SHILOV, *Generalized Functions*, vol. 1, Academic Press, New York and London, 1964.
[8] A. K. GRUNWALD, *Über Begrenzte Derivation–und deren Anwendung*, Zeitschrift für Mathematik und Physik, 12 (1867), pp. 441–480.
[9] T. P. HIGGINS, *The use of fractional integral operators for solving nonhomogeneous differential equations*, Document D1-82-0677, Boeing Scientific Research Laboratories, Seattle, Wash., 1967.
[10] G. LEIBNIZ, *Mathematics Schriften*, vol. 1, Part 2, C. I. Gerhardt, ed., Halle, 1858, pp. 377–382.
[11] T. L. OSLER, *Leibniz rule, the chain rule and Taylor's theorem for fractional derivatives*, Doctoral thesis, New York Univ., 1970.
[12] ———, *Leibniz rule for fractional derivatives generalized and an application to infinite series*, SIAM J. Appl. Math., 18 (1970), pp. 658–674.
[13] ———, *The fractional derivative of a composite function*, this Journal, 1 (1970), pp. 288–293.
[14] ———, *Taylor's series generalized for fractional derivatives and applications*, this Journal, 2 (1971), pp. 37–48.
[15] ———, *Fractional derivatives and Leibniz rule*, Amer. Math. Monthly, 78 (1971), pp. 645–649.
[16] E. L. POST, *Generalized differentiation*, Trans. Amer. Math. Soc., 32 (1930), pp. 723–781.

[17] S. RAMANUJAN, *Collected Papers of Srinivasa Ramanujan*, Chelsea, New York, 1962.
[18] M. RIESZ, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta. Math., 81 (1949), pp. 1–233.
[19] Y. Watanabe, *Notes on the generalized derivative of Riemann-Liouville and its application to Leibnitz's formula, I and II*, Tohoku Math. J., 24 (1931), pp. 8–41.
[20] J. M. Whittaker, *Interpolatory Function Theory*, Stechert–Hafner Service Agency, New York and London, 1964.
[21] A. ZYGMUND, *Trigonometric Series*, vol. I, Cambridge University Press, New York, 1959.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

SCHUR'S THEOREM FOR HURWITZ POLYNOMIALS*

NORMAN LEVINSON† AND RAY REDHEFFER‡

Abstract. This paper contains a short proof of a theorem of Schur which may be used to decide recursively whether or not a given polynomial is a Hurwitz polynomial. The analysis is based on a new equivalence theorem for Hurwitz polynomials which has independent interest.

A polynomial with complex coefficients

$$f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n, \qquad a_0 \neq 0,$$

such that all its zeros lie in the left half-plane, is said to be a Hurwitz polynomial. Let

$$f^*(z) = (-1)^n \overline{f(-\bar{z})}, \tag{1}$$

so that

$$f^*(z) = \bar{a}_0 z^n - \bar{a}_1 z^{n-1} + \bar{a}_2 z^{n-2} - \cdots + (-1)^n \bar{a}_n. \tag{2}$$

The following theorem is a reformulation of a theorem of I. Schur [3].

THEOREM 1. *Let c be a complex number such that* $\operatorname{Re} c > 0$. *Then if f of degree $n \geqq 2$ is a Hurwitz polynomial, so is the polynomial f_1 of degree $n - 1$, where*

$$f_1(z) = f(z)[\bar{a}_0(z - c) - \bar{a}_1] - f^*(z)[a_0(z - c) + a_1].$$

Moreover, $\operatorname{Re}(a_1/a_0) > 0$. *Conversely, if* $\operatorname{Re}(a_1/a_0) > 0$ *and f_1 is a Hurwitz polynomial, then f is a Hurwitz polynomial.*

By repeated use of this theorem with convenient choices of c, the problem of deciding whether a polynomial is Hurwitz is reduced to the determination of the signs of a sequence of complex numbers (of the form a_1/a_0). The procedure can easily be programmed on a computer.

A special case of Schur's theorem in which $c = a_1/a_0$ is proved in Fuchs and Levin [2]. A generalization of Schur's theorem due to Benjaminowitsch [1] asserts that $f(z)$ and $f_1(z)$ have the same number of zeros in $\operatorname{Re} z > 0$, whether or not they are Hurwitz.

Our objective is to get a short proof of Theorem 1 by use of Theorem 2 below, which is very easy to prove, but seems not to have been stated heretofore. The method also applies to Benjaminowitsch's generalization, though details are not given here.

LEMMA. *Let $C = C_R$ denote the semicircular contour $z = Re^{i\theta}$, $-\pi/2 \leqq \theta \leqq \pi/2$, together with the segment $-R \leqq y \leqq R$ of the imaginary axis. Let $K = K_R$ be any continuous curve joining $z = 0$ to $z = 1$. Suppose p and q are polynomials such that, for some arbitrarily large R,*

$$|p(z) + \lambda q(z)| > 0, \qquad (\lambda, z) \in K \times C. \tag{3}$$

Then p and $p + q$ have the same number of zeros in the right half-plane.

* Received by the editors August 19, 1970, and in revised form April 7, 1971.

† Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts.

‡ Department of Mathematics, University of California, Los Angeles, California 900124.

For proof let K be given by $z = \lambda(t)$, $0 \leqq t \leqq 1$, where $\lambda = \lambda_R$ is continuous and $\lambda(0) = 0$, $\lambda(1) = 1$. If R is any value for which (3) holds, the integral

$$I(t) = \frac{1}{2\pi i} \int_C \frac{p'(z) + \lambda(t)q'(z)}{p(z) + \lambda(t)q(z)} dz$$

is a continuous function of t which is integer-valued and hence is constant. Thus $I(0) = I(1)$. But $I(0)$ is the number of zeros of p within C and $I(1)$ is the number of zeros of $p + q$ within C. Since R can be arbitrarily large, the result follows.

THEOREM 2. *Let* $A(z) = \alpha z^m + \cdots$, $B(z) = \beta z^m + \cdots$, $C(z) = \gamma z^n + \cdots$ *be polynomials of degree m, m, and n, respectively, where the leading coefficients* α, β, *and* γ *are not zero. Suppose*

$$|A(iy)| > |B(iy)|, \qquad -\infty < y < \infty, \tag{4}$$

and suppose $\alpha\gamma + \beta\bar{\gamma} \neq 0$. *Then if either of the polynomials*

$$A(z)C(z), \qquad A(z)C(z) + B(z)C^*(z)$$

is Hurwitz, so is the other.

For proof, apply the lemma with $p = AC$ and $q = BC^*$, so that

$$p(z) + \lambda q(z) = (\alpha\gamma + \lambda\beta\bar{\gamma})z^{m+n} + \cdots.$$

Let K of the lemma be any curve which lies in the disk $|z| \leqq 1$ and avoids the point $z = -\alpha\gamma/\beta\bar{\gamma}$. Then

$$|\alpha\gamma + \lambda\beta\bar{\gamma}| \geqq \delta, \qquad \lambda \in K,$$

where $\delta > 0$ is constant. This shows that $p + \lambda q$ satisfies the hypothesis of the lemma on the curved part of C_R for all sufficiently large R. Since (1) gives

$$|C(iy)| = |C^*(iy)|, \qquad -\infty < y < \infty, \tag{5}$$

any imaginary zero of C is also a zero of C^*, and hence is excluded if either of the polynomials considered in Theorem 2 is Hurwitz. Thus (5) and (4) give

$$|p(iy)| > |q(iy)|, \qquad -\infty < y < \infty.$$

Since $|\lambda| \leqq 1$ on K, this shows that $p + \lambda q$ satisfies the hypothesis of the lemma on the straight part of C. Thus Theorem 2 follows.

The following deduction of Theorem 1 from Theorem 2 is based, in part, on [1] and [3]. If $f(z)$ is Hurwitz, then $\operatorname{Re}(a_1/a_0) > 0$, since $-a_1/a_0$ is the sum of the roots. For the rest of the proof it is convenient to assume $a_0 = 1$, which is permissible because dividing f by a_0 has the same effect as dividing f_1 by $a_0\bar{a}_0$. Hence, we can take $a_0 = 1$, $\operatorname{Re} a_1 > 0$.

When $a_0 = 1$, an elementary calculation gives

$$f_1(z) = \gamma z^{n-1} + \cdots, \quad \text{where } \gamma + \bar{\gamma} = -(c + \bar{c})(a_1 + \bar{a}_1),$$

and hence f_1 has degree $n - 1$. By (1) it follows that

$$(fg)^* = f^*g^*, \qquad (f + g)^* = f^* + g^*, \tag{6}$$

where the first of these relations holds for all polynomials f and g and the second holds if f and g have the same degree. By (6),

$$f_1^*(z) = f^*(z)(z + \bar{c} + a_1) - f(z)(z + \bar{c} - \bar{a}_1).$$

This and the original equation for f_1 (with $a_0 = 1$) can be solved for f to give

$$-(c + \bar{c})(a_1 + \bar{a}_1)f(z) = (z + \bar{c} + a_1)f_1(z) + (z - c + a_1)f_1^*(z).$$

We use Theorem 2 with n replaced by $n - 1$ and with

$$A(z) = z + \bar{c} + a_1, \quad B(z) = z - c + a_1, \quad C(z) = f_1(z).$$

From $\operatorname{Re} c > 0$ and $\operatorname{Re} a_1 > 0$ follows

$$|\operatorname{Re} A(iy)| > |\operatorname{Re} B(iy)|, \qquad \operatorname{Im} A(iy) = \operatorname{Im} B(iy),$$

and hence $|A(iy)| > |B(iy)|$. Since $\alpha\gamma + \beta\bar{\gamma} = \gamma + \bar{\gamma} \neq 0$, Theorem 2 shows that $f(z)$ is Hurwitz if and only if $(z + \bar{c} + a_1)f_1(z)$ is Hurwitz. The latter is Hurwitz if and only if $f_1(z)$ is, since $\operatorname{Re}(\bar{c} + a_1) > 0$. This completes the proof.

REFERENCES

[1] S. BENJAMINOWITSCH, *Uber die Anzahl der Wurzeln einer algebraischen Gleichung in einer Halbebene und auf ihrem Rande*, Monatsh. Math. Phys., 42 (1935), pp. 279–308.

[2] B. A. FUCHS AND V. I. LEVIN, *Functions of a complex variable and some of their applications*, vol. 2, Addison-Wesley, Reading, Mass., 1961.

[3] I. SCHUR, *Uber algebraische Gleichungen, die nur Wurzeln mit negativen Realteilen besitzen*, Z. Angew. Math. Mech., 1 (1921), pp. 307–311.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

TRANSFORM METHODS FOR OBTAINING ASYMPTOTIC EXPANSIONS OF DEFINITE INTEGRALS*

FRANK STENGER†

Abstract. With the condition $\int_R |dh(t)| < \infty$, asymptotic approximations are obtained to the integral $\int_R f(t)dh(\lambda t)$ over the real line R as $\lambda \to \infty$, (a) by approximating $\hat{h}(x) = \int_R e^{ixt}\, dh(t)$ in a neighborhood of $x = 0$ and (b) by using a basis $\{\psi_k(t)\}_{k=1}^{n}$, where in contrast to the usual case $\psi_k(t)$ need not be equal to t^{k-1}.

1. Introduction. In this paper[1] we derive some new methods of obtaining asymptotic approximations of integrals of the form

$$I(f, \lambda) = \int_R f(t)\, dh(\lambda t). \tag{1.1}$$

In (1.1) R denotes the real line, λ is a large parameter, $f \in C_*(R)$, that is, f is continuous and bounded on R, and $h(t)$ is of bounded variation on R.

In most applications $dh(t) = e^{-t^2}\, dt$, or $dh(t) = \{0$ if $t \leqq 0$; $t^{\alpha-1} e^{-t}\, dt$ if $t > 0\}$, where $\alpha > 0$. In these cases the usual procedure is to expand $f(t)$ in a power series in t about $t = 0$ and to perform termwise integration. In the general case this usual procedure fails to yield an asymptotic approximation to arbitrary high order of accuracy when:

(a) f only has a finite number of derivatives at $t = 0$;

(b)

$$\mu_k = \int_R t^{k-1}\, dh(t) \tag{1.2}$$

exists only for $k = 1, 2, \cdots, n$;

(c) it is not possible to obtain μ_k explicitly.

A new approach is given for approximating $I(f, \lambda)$ by use of Fourier transforms $\hat{h}$ of dh, that is,

$$\hat{h}(x) = \int_R e^{ixt}\, dh(t). \tag{1.3}$$

Instead of proceeding in the usual manner, that is, expanding f at $t = 0$ and performing termwise integration, we approximate $\hat{h}(x)$ in a neighborhood of $x = 0$. In this way we obtain a class of asymptotic approximations, including that obtainable by expansion of f at $t = 0$ and termwise integration. We thus propose to overcome partially the difficulties (a) and (b) above by use of Fourier transforms whenever it is possible to express $\hat{h}(x)$ explicitly.

* Received by the editors June 2, 1970, and in revised form April 16, 1971.

† Department of Mathematics, University of Utah, Salt Lake City, Utah 84112.

[1] The results of this paper were first announced in [1].

Even if it may not be possible to express μ_k explicitly, it may be possible to do this for

$$v_k(\lambda) = \int_R \psi_k(t)\, dh(\lambda t) \tag{1.4}$$

when $k = 1, 2, 3, \cdots$. We describe a procedure which, under suitable conditions on the ψ_k, enables us to use the sequence $\{v_k(\lambda)\}$ to obtain an explicit asymptotic approximation for $I(f, \lambda)$.

Handelsman and Lew [2] are currently studying the use of Mellin transforms for approximating integrals of the type (1.1). In their work, the residues at the poles of the Mellin transform of $h(t)$ determine the coefficients of the asymptotic expansion of $I(f, \lambda)$. We also mention the recent work of Jones [8] which has appeared since the present work was first submitted, and which also makes it possible to handle the difficulties of the type (b) mentioned above.

2. The use of Fourier transforms. Let W denote the class of all functions $g(t)$ of bounded variation on R, that is,

$$V(g) = \int_R |dg(t)| < \infty, \tag{2.1}$$

and let $\hat{W}$ denote the isomorphic class of all transforms $\hat{g}$ of dg defined by

$$\hat{g}(x) = \int_R e^{ixt}\, dg(t). \tag{2.2}$$

Let $C_*^{(p)}(R)$ denote the class of all functions whose pth derivative is continuous and bounded on R.

Recently [5] the author showed that if

$$\frac{\hat{g}(x)}{(1 - e^{-ix})^n} = \hat{\sigma}(x) \tag{2.3}$$

for all $|x| \leqq 2$, where $\hat{g}, \hat{\sigma} \in \hat{W}$, then for every $f \in C_*^{(n)}(R) \cap C_*(R)$, we have

$$\int_R f(t)\, dg(\lambda t) = O(\lambda^{-n}) \tag{2.4}$$

as $\lambda \to \infty$. This result leads us to the following theorem.

THEOREM 2.1 (Approximation theorem). *Let $\hat{h}, \hat{g}, \hat{\sigma} \in \hat{W}$, and let*

$$\hat{h}(x) - \hat{g}(x) = x^n \hat{\sigma}(x) \tag{2.5}$$

for all x in the interval $(-s, s)$, where $s > 0$. Then

$$\int_R f(t)\, dh(\lambda t) = \int_R f(t)\, dg(\lambda t) + O(\lambda^{-n}) \tag{2.6}$$

as $n \to \infty$, for all $f \in C_*(R) \cap C_*^{(n)}(R)$.

Proof. Let us set $\hat{H}(x) = \hat{h}(sx/4)$, $\hat{G}(x) = \hat{g}(sx/4)$, $\hat{K}(x) = \hat{\sigma}(sx/4)$. Then clearly $\hat{H}$, $\hat{G}$ and $\hat{K} \in \hat{W}$, since, for example,

$$\hat{H}(x) = \hat{h}\left(\frac{sx}{4}\right) = \int_R e^{ixt}\, dh\left(\frac{4t}{s}\right) \quad \text{and} \quad \int_R \left| dh\left(\frac{4t}{s}\right)\right| = \int_R |dh(t)|, \tag{2.7}$$

and therefore,

$$\hat{H}(x) - \hat{G}(x) = x^n \left(\frac{s}{4}\right)^n \hat{K}(x) \tag{2.8}$$

for all $|x| \leqq 4$. Let us write the right-hand side of (2.8) in the form

$$x^n \left(\frac{s}{4}\right)^n \hat{K}(x) = (1 - e^{-ix})^n \left(\frac{x}{1 - e^{-ix}}\right)^n \left(\frac{s}{4}\right)^n \hat{K}(x). \tag{2.9}$$

The function $x^n/[1 - e^{-ix}]^n(s/4)^n$ is clearly infinitely differentiable on the interval $[-4, 4]$, and since any twice differentiable function of compact support is in $\hat{W}$, there exists an element $\hat{\sigma}_1 \in \hat{W}$ such that

$$\left(\frac{x}{1 - e^{-ix}}\right)^n \left(\frac{s}{4}\right)^n = \hat{\sigma}_1(x) \tag{2.10}$$

for all $|x| \leqq 4$. Furthermore, by the ring property of $\hat{W}$ we have $\hat{\sigma}_1(x)\hat{K}(x) \in \hat{W}$. Consequently, by (2.5), (2.6) and (2.7) we obtain

$$\int_R f(t)\, dh\left(\frac{4\lambda t}{s}\right) = \int_R f(t)\, dg\left(\frac{4\lambda t}{s}\right) + O\left(\frac{4\lambda}{s}\right)^{-n} \tag{2.11}$$

as $\lambda \to \infty$, which is equivalent to (2.6).

COROLLARY 2.2. *Let h, g, $\hat{h}$ and $\hat{g}$ be defined as in Theorem 2.1. If*

$$\hat{h}(x) - \hat{g}(x) = x^n[p(x) + q(|x|)], \tag{2.12}$$

where p and q are power series in x that converge in a neighborhood of $x = 0$, then

$$\int_R f(t)\, dh(\lambda t) = \int_R f(t)\, dg(\lambda t) + O(\lambda^{-n}), \qquad \lambda \to \infty, \tag{2.13}$$

for all $f \in C_*(R) \cap C_*^{(n)}(R)$.

Proof. Clearly x coincides with an element of $\hat{W}$ in some interval $(-s, s)$, $s > 0$. Furthermore, since $|x|\, e^{-x^2} \in \hat{W}$, it follows that $\hat{k}(x)|x|\, e^{-x^2} \in \hat{W}$, where $\hat{k}$ is any twice differentiable function defined on R such that $\hat{k}(x) = e^{x^2}$ on $(-s, s)$, $\hat{k}(x) = 0$ on $R - (-2s, 2s)$. This function $\hat{k}$ satisfies $\hat{k}(x)|x|\, e^{-x^2} = |x|$ on the interval $(-s, s)$. The remainder of the proof of Corollary 2.2 now follows from Theorem 2.1.

3. A more general basis. Let us set

$$\mu_k = \mu_k(\lambda) = \int_R t^{k-1}\, dh(\lambda t) \tag{3.1}$$

for $k = 1, 2, 3, \cdots$, and let $\nu_k(\lambda)$ be defined by (1.4).

DEFINITION 3.1. A set of n functions $\psi_1, \cdots, \psi_n$ is said *to have Property* A_n, if given any polynomial $P(t)$, there exist constants $c_1, \cdots, c_n$ such that

$$P(t) - \sum_{k=1}^{n} c_k \psi_k(t) = O(t^n) \tag{3.2}$$

as $t \to 0$.

THEOREM 3.2. *Let $\varphi(t)$ be any function such that*

$$\varphi(t) = \sum_{k=0}^{n-1} a_k t^k + O(t^n) \tag{3.3}$$

as $t \to 0$, where $a_k \neq 0$ for $k = 0, 1, \cdots, n-1$. If the n numbers $b_1, \cdots, b_n$ are distinct, then the sequence $\{\psi_k\}_{k=1}^n$ with $\psi_k(t) = \varphi(b_k t)$ has Property A_n.

Proof. It suffices to show that given any polynomial

$$P_{n-1}(t) = \sum_{k=0}^{n-1} p_k t^k, \tag{3.4}$$

there exist constants $c_1, \cdots, c_n$ such that

$$P_{n-1}(t) - \sum_{k=1}^{n} c_k \varphi(b_k t) = O(t^n) \tag{3.5}$$

as $t \to 0$, that is,

$$\sum_{j=0}^{n-1} p_j t^j - \sum_{k=1}^{n} \sum_{j=0}^{n-1} c_k a_j b_k^j t^j = O(t^n) \tag{3.6}$$

as $t \to 0$. This equation will be satisfied if the c_k can be chosen such that

$$\sum_{k=1}^{n} c_k b_k^j = p_j / a_j, \qquad j = 0, 1, \cdots, n-1. \tag{3.7}$$

The system (3.7) is a Vandermonde system whose determinant is not zero. Consequently $c_1, \cdots, c_n$ are uniquely determined.

THEOREM 3.3. *Let $\{\psi_k\}_{k=1}^n$ have Property A_n and let $t_1, \cdots, t_n$ be any set of n distinct points on R. Then there exists a positive number λ_0 such that whenever $\lambda > \lambda_0$, then the determinant of the $n \times n$ matrix*[2] *$[\psi_k(t_j/\lambda)]$ is not zero.*

Proof. Let the $n \times n$ matrix $[b_{ij}]$ be determined such that

$$[t^{j-1}] = [b_{kj}][\psi_k(t)] + [\varepsilon_j(t)], \tag{3.8}$$

where $\varepsilon_j(t) = O(t^n)$ as $t \to 0, j = 1, 2, \cdots, n$. Then we have

$$[t_i^{j-1}/\lambda^{j-1}] = [b_{kj}][\psi_k(t_i/\lambda)] + [\varepsilon_k(t_i/\lambda)]. \tag{3.9}$$

Since $\varepsilon_k(t) = O(t^n)$ as $t \to 0$, we have $\det[\varepsilon_k(t_i/\lambda)] = O(\lambda^{-n^2})$ as $\lambda \to \infty$. The determinant of the matrix on the left of (3.9) satisfies

$$\det[t_i^{j-1}/\lambda^{j-1}] = \lambda^{-n(n+1)/2} \det[t_i^{j-1}]. \tag{3.10}$$

[2] The notation $[a_{ij}]$ denotes a matrix with i,jth element a_{ij}; the notation $[b_i]$ denotes a vector with ith element b_i.

Hence,

$$\det([b_{kj}][\psi_k(t_i/\lambda)]) = \lambda^{-n(n+1)/2}\det[t_i^{j-1}](1 + O(\lambda^{-1})), \tag{3.11}$$

which implies that $\det[b_{kj}] \neq 0$, and $\det[\psi_k(t_i/\lambda)] \neq 0$ for all λ sufficiently large.

4. Quadrature schemes. Let numbers μ_k, $k = 1, 2, \cdots, n$, be defined by (1.2), and let us set $\mu = (\mu_1, \cdots, \mu_n)^T$, where the superscript T denotes the transpose of a vector. Let the points $t_1, \cdots, t_n \in R$ be distinct and let $w = (\omega_1, \cdots, \omega_n)^T$ be the solution of the system of equations

$$[t_j^{i-1}]w = \mu. \tag{4.1}$$

Then we have

$$\int_R P(t)\,dh(t) = \sum_{j=1}^{n} \omega_j P(t_j) \tag{4.2}$$

for every polynomial $P(t)$ of degree $n - 1$ in t. Depending upon the choice of $\{t_j\}_{j=1}^n$, (4.2) may be exact for all polynomials of degree $m - 1$, where $n \leqq m \leqq 2n$. If we let $P(f, t)$ denote the polynomial of degree $n - 1$ which interpolates[3] f at the points $t_1, \cdots, t_n$, we have

$$\int_R P(f, t)\,dh(t) = \sum_{j=1}^{n} \omega_j f(t_j). \tag{4.3}$$

The reader should consult Davis and Rabinowitz [4] for further details concerning the construction of quadrature schemes.

Now suppose that $f \in C_*(R) \cap C_*^{(n)}(R)$. Then the following result established in [5] is valid.

THEOREM 4.1. *Let $f \in C_*(R) \cap C_*^{(n)}(R)$. Then*

$$\int_R f(t)\,dh(\lambda t) - \sum_{j=1}^{n} \omega_j f(t_j/\lambda) = O(\lambda^{-n}) \quad as \quad \lambda \to \infty. \tag{4.4}$$

Now let $\{\psi_k\}_{k=1}^n$ have Property A_n and let $v = v(\lambda) = (v_1, \cdots, v_n)^T$ be determined so that

$$[\psi_j(t_i/\lambda)]v = \nu, \tag{4.5}$$

where $\nu = (\nu_1, \cdots, \nu_n)^T$ and $\nu_i(\lambda)$ is given by (1.4).

THEOREM 4.2. *Let $\{\psi_k\}_{k=1}^n$ have Property A_n and let $|\psi_k(t)| \leqq (1 + |t|^n)\varphi(t)$, where $\varphi(t)$ is a nonnegative function such that $\int_R \varphi(t)|dh(\lambda t)| = O(1)$ and $\int_R |t|^n \varphi(t)\,|dh(\lambda t)| = O(\lambda^{-n})$ as $\lambda \to \infty$. If $f \in C_*(R) \cap C_*^{(n)}(R)$, then*

$$\int_R f(t)\,dh(\lambda t) - \sum_{j=1}^{n} v_j f(t_j/\lambda) = O(\lambda^{-n}). \tag{4.6}$$

Proof. Suppose that for any polynomial P of degree $\leqq n - 1$ we have

$$\int_R P(t)\,dh(\lambda t) - \sum_{j=1}^{n} v_j P(t_j/\lambda) = O(\lambda^{-n}) \tag{4.7}$$

[3] The polynomial P satisfies $P(f, t_j) = f(t_j)$ for $j = 1, 2, \cdots, n$.

as $\lambda \to \infty$. Let $P(f_\lambda, t)$ be the polynomial of degree $n - 1$ which interpolates $f(t)$ at the points t_j/λ, $j = 1, 2, \cdots, n$. We shall also assume without loss of generality that the numbers $t_1, \cdots, t_n$ are the same as those in (4.4). If (4.7) holds, then

$$\begin{aligned} &\int_R f(t/\lambda)\,dh(t) - \sum_{j=1}^{n} v_j f(t_j/\lambda) \\ &\quad = \int_R [f(t/\lambda) - P(f_\lambda, t)]\,dh(t) - \sum_{j=1}^{n} v_j[f(t_j/\lambda) - P(f_\lambda, t_j)] + O(\lambda^{-n}). \end{aligned} \tag{4.8}$$

But since $P(f_\lambda, t_j) = f(t_j/\lambda)$, and

$$\int_R P(f_\lambda, t)\,dh(t) = \sum_{j=1}^{n} \omega_j f(t_j/\lambda), \tag{4.9}$$

where ω_j is defined as in (4.4), the left-hand side of (4.8) is bounded by the right of (4.6).

It remains to show that (4.7) holds. To this end we recall from the property of the ψ_k that there exist constants $c_1, \cdots, c_n$ such that for all $t \in R$, $\lambda > 0$,

$$|P(t/\lambda) - \sum_{j=1}^{n} c_j \psi_j(t/\lambda)| \leqq K|t/\lambda|^n \varphi(t/\lambda), \tag{4.10}$$

where K is a constant. Also, we have

$$\int_R \sum_{j=1}^{n} c_j \psi_j(t/\lambda)\,dh(t) = \sum_{k=1}^{n} v_k \sum_{j=1}^{n} c_j \psi_j(t_k/\lambda), \tag{4.11}$$

so that the left of (4.7) is equal to

$$\begin{aligned} &\int_R \left[P(t/\lambda) - \sum_{j=1}^{n} c_j \psi_j(t/\lambda) \right] dh(t) \\ &\quad - \sum_{k=1}^{n} v_k \left[P(t_k/\lambda) - \sum_{j=1}^{n} c_j \psi_j(t_k/\lambda) \right] = O(\lambda^{-n}), \end{aligned} \tag{4.12}$$

provided that $v_k = O(1)$ as $\lambda \to \infty$. Let us set $T = [t_j^{i-1}/\lambda^{i-1}]$, $\psi = [\psi_i(t_j/\lambda)]$, and $B = [b_{kj}]$, where b_{kj} is defined as in (3.8),

$$\mu = \int_R (1, t, \cdots, t^{n-1})^T\,dh(\lambda t), \quad \nu = \int_R (\psi_1(t), \cdots, \psi_n(t))^T\,dh(\lambda t).$$

Then

$$Tw = \mu, \quad \psi v = \nu, \tag{4.13}$$

and from (3.9),

$$T = B\psi + \varepsilon, \tag{4.14}$$

where $\|\varepsilon\| = O(\lambda^{-n})$, $\|\cdot\|$ denoting any suitable and compatible matrix norm corresponding to a vector norm. From (3.8) we also have

$$\mu = Bv + \eta, \quad \eta = \int_R (\varepsilon_1(t), \cdots, \varepsilon_n(t))^T \, dh(\lambda t), \tag{4.15}$$

and so $\|\eta\| = O(\lambda^{-n})$. Eliminating ψ in the second of equations (4.13) we get

$$B^{-1}(T - \varepsilon)v = B^{-1}(\mu - \eta), \tag{4.16}$$

which, in view of the first of equations (4.13) and the above bounds on ε and η, yields

$$v = w + \delta, \tag{4.17}$$

where $\|\delta\| = O(\lambda^{-1})$ (since $\|w\| = O(1)$). This completes the proof.

5. Examples. In this section we illustrate the application of Corollary 2.2 and Theorem 4.2. One application has already been made in [5]: corresponding to numbers $t_j \in R$ we found numbers ω_j such that $\hat{h}(x) - \sum_{j=1}^{n} \omega_j e^{it_j x} = O(x^n)$ as $x \to 0$.

Consider the following two integrals:

$$H(u, \lambda) = \int_R f(u + t) \, dh(\lambda t) \tag{5.1}$$

and[4]

$$K(u, \lambda) = \int_R f(u + t) \, dk(\lambda t), \tag{5.2}$$

where

$$\begin{aligned} h(t) &= \frac{1}{\pi} \int_{-\infty}^{t} \frac{du}{1 + u^2}, \\ k(t) &= \frac{1}{\pi} \int_{-\infty}^{t} \left(\frac{\sin u}{u} \right)^2 du. \end{aligned} \tag{5.3}$$

We shall apply Corollary 2.2 to determine the asymptotic behavior of one of H or K from the other,[5] as $\lambda \to \infty$.

Using the notation of (1.3) we have

$$\begin{aligned} \hat{h}(x) &= e^{-|x|}, \\ \hat{k}(x) &= \begin{cases} 0 & \text{if } |x| > 2, \\ 1 - \tfrac{1}{2}|x| & \text{if } |x| \leqq 2. \end{cases} \end{aligned} \tag{5.4}$$

Thus

$$\begin{aligned} \hat{h}(x) - 1 &= |x| p(x) + x^2 q(x), \\ \hat{k}(x) - 1 &= |x| p_1(x), \end{aligned} \tag{5.5}$$

---

[4] The function $U(\xi, \eta) = H(\xi, 1/\eta)$ is a harmonic function which has the property that $U(\xi, \eta) \to f(\xi)$ as $\eta \to 0^+$ at each point of continuity of f. The integral (5.2) is analogous to Féjer's integral in the theory of Fourier series.

[5] The functions H and K are also related in [7, pp. 28–30], through Cauchy's singular integral. However, the order relations obtained here are more accurate than those in [7].

and also

$$\hat{h}(x) - 2\hat{k}(x) + 1 - \tfrac{1}{2}x^2 = |x|^3 p_2(x) + x^3 q_2(x) \quad \text{or}$$

(5.6)

$$\hat{h}(x) - 2\hat{k}(x) + \tfrac{1}{2}(e^{ix} + e^{-ix}) = |x|^3 p_3(x) + x^3 q_3(x),$$

where the p's and q's denote power series in x that converge in a neighborhood of $x = 0$. More generally, if real numbers $a_1, a_2, \cdots, a_m$ are chosen such that $0 < a_1 < a_2 < \cdots < a_m$, then we have the system of linear equations:

$$\text{(5.7)} \qquad \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \dfrac{1}{a_1} & \dfrac{1}{a_2} & \cdots & \dfrac{1}{a_m} \\ \dfrac{1}{a_1^2} & \dfrac{1}{a_2^2} & \cdots & \dfrac{1}{a_m^2} \\ & & & \\ \dfrac{1}{a_1^{m-1}} & \dfrac{1}{a_2^{m-1}} & \cdots & \dfrac{1}{a_m^{m-1}} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The system (5.7) is obtained if we set

$$\text{(5.8)} \qquad \sum_{j=1}^{m} A_j e^{-|x|/a_j} = (1 - \tfrac{1}{2}|x|) + \varepsilon(x),$$

where $\varepsilon(x) = O(|x|^m)$ as $x \to 0$, and equate equal powers of x on each side. It is easily seen that either

$$\text{(5.9)} \qquad \varepsilon(x) = x^m p_4(x) \quad \text{or} \quad \varepsilon(x) = |x|^m q_4(x),$$

where p_4 and q_4 are power series in x that converge for all finite x.

By Corollary 2.2 we thus have from (5.5) that

$$\text{(5.5a)} \qquad \begin{aligned} H(u, \lambda) &= f(u) + O(\lambda^{-1}), \\ K(u, \lambda) &= f(u) + O(\lambda^{-1}) \end{aligned}$$

as $\lambda \to \infty$ if $f \in C^*(R) \cap C_*^{(1)}(R)$. Similarly, using Corollary 2.2 on (5.6), we get

$$\text{(5.6a)} \qquad H(u, \lambda) = 2K(u, \lambda) - \tfrac{1}{2}[f(u + 1/\lambda) + f(u - 1/\lambda)] + O(\lambda^{-3})$$

as $\lambda \to \infty$, provided that $f \in C_*(R) \cap C_*^{(3)}(R)$. Finally, Corollary 2.2 applied to (5.8) and (5.9) yields

$$\text{(5.8a)} \qquad K(u, \lambda) = \sum_{j=1}^{m} A_j H(u, a_j \lambda) + O(\lambda^{-m})$$

as $\lambda \to \infty$, provided that $f \in C_*(R) \cap C_*^{(m)}(R)$. Note that while we have not been able to express explicitly the coefficient of λ^{-1} in either of the equations (5.5a), the representation (5.6a) enables us to side-step this problem, since the coefficient of λ^{-1} in $H(u, \lambda)$ is the same (whatever it may be) as that in $2K(u, \lambda)$.

For example, if $\alpha > 0$, $\operatorname{Im} z_1, \operatorname{Im} z_2 < 0$, the integral (5.2) with $f(t) = (t - z_1)^{-\alpha} \cdot (t - z_2)^{-\alpha}$ seems difficult to evaluate. However, using residues, we can evaluate (5.1) to obtain

$$\text{(5.10)} \qquad \begin{aligned} H(u, \lambda) &= \frac{1}{\pi} \int_R \frac{\lambda \, dt}{(u + t - z_1)^\alpha (u + t - z_2)^\alpha (1 + \lambda^2 t^2)} \\ &= \frac{1}{(u + i/\lambda - z_1)^\alpha (u + i/\lambda - z_2)^\alpha}. \end{aligned}$$

By use of (5.8a) and (5.10) we can therefore obtain an approximation to

$$\text{(5.11)} \qquad K(u, \lambda) = \frac{1}{\pi} \int_R \left(\frac{\sin \lambda t}{\lambda t} \right)^2 \frac{\lambda \, dt}{(u + t - z_1)^\alpha (u + t - z_2)^\alpha},$$

for which the error is $O(\lambda^{-m})$ as $\lambda \to \infty$.

We mention that the device used to obtain (5.8a) generalizes the procedure used to derive the Romberg integration method [6]. These approximations are quite remarkable; moreover, attempts to obtain them by expansion of f about $t = 0$ and by use of termwise integration have proved unsuccessful.

5.2. An example using a more general basis. Let v, α and β be real, and let $\beta > 0$. Let us start with the well-known identity

$$\text{(5.12)} \qquad \int_0^\infty e^{-\beta t} J_v(\alpha t) \, dt = (\beta^2 + \alpha^2)^{-1/2} \left[\frac{\alpha}{(\alpha^2 + \beta^2)^{1/2} + \beta} \right]^v$$

and note by Theorem 3.2 that the sequence $\{e^{-\gamma_k t}\}_{k=1}^n$ has Property A_n (defined in Definition 3.1), provided that the γ_k are distinct. Taking $\gamma_1 = 1$, $\gamma_2 = 2$, $t_1 = 1$, $t_2 = 2$, we obtain ω_1, ω_2 by solving the system

$$\text{(5.13)} \qquad 1/\alpha \begin{pmatrix} e^{-1/\alpha} & e^{-2/\alpha} \\ e^{-2/\alpha} & e^{-4/\alpha} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \left\{ \dfrac{\alpha}{[\alpha^2 + (1 + \alpha)^2]^{1/2} + \alpha} \right\}^v & [\alpha^2 + (1 + \alpha)^2]^{-1/2} \\ \left\{ \dfrac{\alpha}{[\alpha^2 + (2 + \alpha)^2]^{1/2} + \alpha} \right\}^v & [\alpha^2 + (2 + \alpha)^2]^{-1/2} \end{pmatrix}.$$

Applying Theorem 4.2 with $\varphi(t) = e^{3t}$ we find that if $f \in C_*(R) \cap C_*^{(3)}(R)$, then

$$\text{(5.14)} \qquad \int_0^\infty \alpha \, e^{-\alpha t} J_v(\alpha t) f(t) \, dt = \omega_1 f(1/\alpha) + \omega_2 f(2/\alpha) + \varepsilon,$$

where

$$\text{(5.15)} \qquad \varepsilon = O(\alpha^{-3}) \quad \text{as } \alpha \to \infty.$$

6. Summary. In this paper we have developed two different methods of obtaining an asymptotic approximation to the integral

$$\text{(6.1)} \qquad H(f, \lambda) = \int_R f(t) \, dh(\lambda t),$$

where h is of bounded variation on R, and for purposes of this summary, f is infinitely differentiable and bounded on R.

The first method depends on knowing

$$K(f, \lambda) = \int_R f(t)\, dk(\lambda t),$$

$$\hat{h}(x) = \int_R e^{ixt}\, dh(t) \quad \text{and} \tag{6.2}$$

$$\hat{k}(x) = \int_R e^{ixt}\, dk(t),$$

where k is also of bounded variation on R. If we can furthermore find constants $0 < a_1 < a_2 \cdots < a_m$ and $A_1, A_2, \cdots, A_m$ such that

$$\hat{h}(x) - \sum_{j=1}^{m} A_j \hat{k}(x/a_j) = \begin{cases} x^n p(x) \\ \text{or} \\ |x|^n p(x), \end{cases} \tag{6.3}$$

where p is twice differentiable in a neighborhood of $x = 0$, then

$$H(f, \lambda) = \sum_{j=1}^{m} A_j K(f, a_j \lambda) + O(\lambda^{-n}) \tag{6.4}$$

as $\lambda \to \infty$.

The second method of approximating (6.1) depends on being able to express explicitly the integrals

$$v_j(\lambda) = \int_R \psi_j(t)\, dh(\lambda t), \tag{6.5}$$

where the functions ψ_j, $j = 1, 2, \cdots, m$, have the property that given any polynomial P there exist constants $c_1, c_2, \cdots, c_m$, such that

$$P(t) - \sum_{j=1}^{m} c_j \psi_j(t) = O(t^m) \tag{6.6}$$

as $t \to 0$. If $0 < t_1 < t_2 < \cdots < t_m$, we can then also solve the system of equations

$$[\psi_i(t_j/\lambda)]w = v \tag{6.7}$$

for the vector $w = (w_1, \cdots, w_m)^T$, where $v = (v_1, v_2, \cdots, v_m)^T$. We then have the approximation

$$H(f, \lambda) = \sum_{j=1}^{m} w_j f(t_j/\lambda) + O(\lambda^{-m}) \tag{6.8}$$

as $\lambda \to \infty$.

Acknowledgment. The author is grateful to Dr. R. Handelsman, and to the referees for valuable criticisms.

REFERENCES

[1] F. STENGER, Abstracts, SIAM National Meeting, Washington, D.C., June 1969.

[2] R. A. HANDELSMAN AND J. S. LEW, *Asymptotic expansion of a class of integral transforms via Mellin transforms*, Arch. Rational Mech. Anal., 35 (1970), pp. 382–396.

[3] W. RUDIN, *Fourier Analysis on Groups*, Interscience, New York, 1962.

[4] P. DAVIS AND P. RABINOWITZ, *Numerical Integration*, Blaisdell, Waltham, Mass., 1967.

[5] F. STENGER, *The asymptotic approximation of certain integrals*, this Journal, 1 (1970), pp. 392–404.

[6] W. ROMBERG, *Vereinfachte numerische integration*, Kungl. Norske Videnskab. Selskab, Trondheim Forb., 28 (1965), no. 7.

[7] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, 1937.

[8] D. S. JONES, *Generalized transforms and their asymptotic behaviour*, Philos. Trans. Roy. Soc. London Ser. A, 265 (1969), pp. 1–44.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

LIE THEORY AND GENERALIZED HYPERGEOMETRIC FUNCTIONS*

WILLARD MILLER, JR.†

Abstract. In this paper the foundations are laid for a study of generalized hypergeometric and G-functions based on the representation theory of Lie groups and algebras. It is shown that many fundamental series identities and Mellin–Barnes integrals for these functions can be derived simply and elegantly using group theory.

Introduction. In this paper we define a Lie group $G_{p,q}$ which is closely related to the differential recurrence formulas for the generalized hypergeometric functions ${}_pF_q$. Then we demonstrate that the representation theory of $G_{p,q}$ yields significant information about the properties of the ${}_pF_q$.

In §§ 1 and 3 the Lie algebraic techniques described in [1] are applied to compute basic addition theorems and generating functions for the ${}_pF_q$. Weisner's method [2] figures significantly in this approach. In § 2 Vilenkin's method of integral transforms [3] is applied to derive some Mellin–Barnes integral representations for the G-function $G^{q+1,p}_{p,q+1}$.

Here, the emphasis is on the group theoretic methods themselves, and no attempt is made to list all possible special function identities obtainable by these methods.

The techniques of this paper also apply to G-functions. In a future paper we shall discuss the insights into Mellin–Barnes integrals for general G-functions, and integrals of products of G-functions which are provided by group theory.

1. The group $G_{p,q}$. Let $\mathscr{G}_{p,q}$ be the $(2(p + q) + 1)$-dimensional complex Lie algebra with basis $\mathscr{R}_j, \mathscr{S}_j, j = 1, \cdots, p, \mathscr{L}_k, \mathscr{T}_k, k = 1, \cdots, q$, and $\mathscr{V}$, and nonzero commutation relations.

$$[\mathscr{S}_j, \mathscr{R}_j] = \mathscr{R}_j, \qquad [\mathscr{T}_k, \mathscr{L}_k] = -\mathscr{L}_k, \qquad [\mathscr{S}_j, \mathscr{V}] = [\mathscr{T}_k, \mathscr{V}] = \mathscr{V}. \tag{1.1}$$

All other commutators between basis vectors are zero. The connected, simply connected complex Lie group $G_{p,q}$ with this Lie algebra consists of elements

$$g(a_j, b_k, c; \gamma_j, \xi_k) = \exp\left(\sum_j a_j\mathscr{R}_j + \sum_k b_k\mathscr{L}_k + c\mathscr{V}\right)\exp\left(\sum_j \gamma_j\mathscr{S}_j + \sum_l \xi_k\mathscr{T}_k\right), \quad a_j, b_k, c, \gamma_j, \xi_k \in \mathbb{C}, \tag{1.2}$$

with group product

$$g(a_j, b_k, c; \gamma_j, \xi_k)g(a'_j, b'_k, c'; \gamma'_j, \xi'_k) = g(a_j + a'_je^{\gamma_j}, b_k + b'_ke^{-\xi_k}, c + c'e^{\gamma_1 + \cdots + \xi_q}; \gamma_j + \gamma'_j, \xi_k + \xi'_k). \tag{1.3}$$

---

* Received by the editors March 31, 1971, and in revised form May 20, 1971.

† Mathematics Department, University of Minnesota, Minneapolis, Minnesota 55455. This research was supported in part by the National Science Foundation under Grant GP-20871.

The identity element is $g(0,0,0;0,0) = e$ and

(1.4) $$g^{-1}(a_j, b_k, c; \gamma_j, \xi_k) = g(-a_j e^{-\gamma_j}, -b_k e^{\xi_k}, -ce^{-\gamma_1-\cdots-\xi_q}; -\gamma_j, -\xi_k).$$

A simple model of $\mathscr{G}_{p,q}$ is given by the generalized Lie derivatives

(1.5)
$$R_j = t_j, \qquad S_j = t_j\frac{\partial}{\partial t_j} + \alpha_j, \qquad \alpha_j \in \mathbb{C}, \quad j = 1, \cdots, p,$$
$$L_k = u_k^{-1}, \quad T_k = u_k\frac{\partial}{\partial u_k} + \beta_k, \quad \beta_k \in \mathbb{C}, \quad k = 1, \cdots, q,$$
$$V = t_1 \cdots t_p u_1 \cdots u_q,$$

acting on the space $\mathscr{F}_{p,q}$ of analytic functions $f(t_1, \cdots, t_p, u_1, \cdots, u_q) = f(t_j, u_k)$ which are defined for all nonzero values of t_j and u_k. A basis for $\mathscr{F}_{p,q}$ is provided by the functions

(1.6) $$f_{m_j,n_k}(t_j, u_k) = t_p^{m_1} \cdots t_p^{m_p} u_1^{n_1} \cdots u_q^{n_q},$$

where the m_j and n_k run over all integers. The action of the operators (1.5) on this basis is

(1.7)
$$R_{j'} f_{m_j,n_k} = f_{\tilde{m}_j,n_k}, \qquad \tilde{m}_j = \begin{cases} m_j & \text{if } j \neq j', \\ m_{j'} + 1 & \text{if } j = j', \end{cases}$$
$$S_{j'} f_{m_j,n_k} = (m_{j'} + \alpha_{j'}) f_{m_j,n_k},$$
$$L_{k'} f_{m_j,n_k} = f_{m_j,\tilde{n}_k}, \qquad \tilde{n}_k = \begin{cases} n_k & \text{if } k \neq k', \\ n_{k'} - 1 & \text{if } k = k', \end{cases}$$
$$T_{k'} f_{m_j,n_k} = (n_{k'} + \beta_{k'}) f_{m_j,n_k},$$
$$V f_{m_j,n_k} = f_{m_j+1,n_k+1}.$$

Expressions (1.7) define an algebraic representation $\rho(\alpha_j, \beta_k)$ of $\mathscr{G}_{p,q}$ on $\mathscr{F}_{p,q}$. Here ρ is reducible but not completely reducible. Due to the isomorphisms $\rho(\alpha_j, \beta_k) \cong \rho(\alpha_j + a_j, \beta_k + b_k)$, where α_j, b_k are arbitrary integers, we can make the restrictions $0 \leqq \operatorname{Re} \alpha_j < 1$, $0 \leqq \operatorname{Re} \beta_k < 1$. Note the identity

(1.8) $$C \equiv R_1 R_2 \cdots R_p - V L_1 L_2 \cdots L_q = 0.$$

Using standard Lie theory techniques we can extend the Lie algebra representation $\rho(\alpha_j, \beta_k)$ to a group representation of $G_{p,q}$ on $\mathscr{F}_{p,q}$ (see [1]). The induced group action is defined by operators $\mathbf{T}(g)$ such that

(1.9) $$[\mathbf{T}(g)f](t_j, u_k) = \exp\left[\sum a_j t_j + \sum b_k/u_k + ct_1 \cdots t_p u_1 \cdots u_q + \sum \alpha_j \gamma_j + \sum \beta_k \xi_k\right] f(t_j e^{\gamma_j}, u_k e^{\xi_k}),$$

$f \in \mathscr{F}_{p,q}$. These operators necessarily satisfy the group homomorphism property $\mathbf{T}(g)\mathbf{T}(g') = \mathbf{T}(gg')$.

We define the matrix elements $T(g)^{m'_j n'_k}_{m_j n_k}$ of $\mathbf{T}(g)$ with respect to the basis f_{m_j,n_k} by

$$\mathbf{T}(g)f_{m_j,n_k} = \sum_{m'_j,n'_k} T(g)^{m'_j n'_k}_{m_j n_k} f_{m'_j,n'_k} \tag{1.10}$$

or

$$\exp\left[\sum a_j t_j + \sum b_k/u_k + ct_1 \cdots u_q + \sum \alpha_j \gamma_j + \sum \beta_k \xi_k\right] \cdot (t_1 e^{\gamma_1})^{m_1} \cdots (u_q e^{\xi_q})^{n_q} = \sum_{m'_j,n'_k} T(g)^{m'_j n'_k}_{m_j n_k} t_1^{m'_1} \cdots u_q^{n'_q}. \tag{1.11}$$

Expanding the left-hand side of (1.11) in a power series and computing the coefficient of $t_1^{m'_1} \cdots u_q^{n'_q}$ we find

$$\begin{aligned} T(g)^{m'_j n'_k}_{m_j n_k} &= \exp\left(\sum_j \gamma_j(\alpha_j + m_j) + \sum_k \xi_k(\beta_k + n_k)\right) \\ &\cdot \left[\prod_{j=1,k=1}^{p}\prod^{q} \left(\frac{a_j^{m'_j - m_j}}{b_k^{n'_k - n_k}\Gamma(m_j - m'_j + 1)\Gamma(1 - n'_k + n_k)}\right)\right] \\ &\cdot {}_pF_q(m_j - m'_j; 1 - n'_k + n_k; -cb_1 \cdots b_q/(a_1 \cdots a_p)). \end{aligned} \tag{1.12}$$

See [4] or [5] for the definition of ${}_pF_q$. Note that the matrix elements are polynomials in a_j, b_k and c. (If the parameters m_j, m'_j, n_k, n'_k are allowed to take complex values, the right-hand side of (1.12) is an entire function of these parameters. The matrix element is equal to this entire function evaluated on the integers. Thus, (1.12) is zero whenever $m_j - m'_j < 0$ for some j.)

Since the operators $\mathbf{T}(g)$ define a representation of $G_{p,q}$ we immediately obtain the addition theorem

$$T(gg')^{m'_j n'_k}_{m_j n_k} = \sum_{M_j N_k - \infty}^{\infty} T(g)^{m'_j n'_k}_{M_j N_k} T(g')^{M_j N_k}_{m_j n_k} \tag{1.13}$$

valid for all $g, g' \in G_{p,q}$. Substitution of (1.3) and (1.12) into (1.13) leads to a wide variety of identities for the generalized hypergeometric functions.

Let $\mathscr{F}(G_{p,q})$ be the space of all entire functions $f(g)$ defined on the group $G_{p,q}$. The right regular representation $\boldsymbol{\sigma}$ is defined on this space by

$$[\boldsymbol{\sigma}(g')f](g) = f(gg'), \qquad g, g' \in G_{p,q}, \quad f \in \mathscr{F}(G_{p,q}). \tag{1.14}$$

It is easy to check that $\boldsymbol{\sigma}$ defines a representation of $G_{p,q}$. Moreover, for fixed m'_j, n'_k the subspace of $\mathscr{F}(G_{p,q})$ spanned by the functions

$$f_{m_j n_k}(g) = T(g)^{m'_j n'_k}_{m_j n_k}, \qquad -\infty < m_j, n_k < +\infty, \tag{1.15}$$

transforms according to the representation $\rho(\alpha_j, \beta_k)$. Indeed, (1.13)–(1.15) imply

$$[\boldsymbol{\sigma}(g')f_{m_j n_k}](g) = f_{m_j n_k}(gg') = \sum_{M_j N_k} T(g')^{M_j N_k}_{m_j n_k} f_{M_j N_k}(g). \tag{1.16}$$

Using standard techniques in Lie theory [1, Chap. 2], we can compute the Lie derivatives corresponding to the representation $\boldsymbol{\sigma}$:

$$R_j = e^{\gamma_j}\frac{\partial}{\partial a_j}, \quad S_j = \frac{\partial}{\partial \gamma_j}, \quad L_k = e^{-\xi_k}\frac{\partial}{\partial b_k}, \quad T_k = \frac{\partial}{\partial \xi_k}, \quad V = e^{\gamma_1 + \cdots + \xi_q}\frac{\partial}{\partial c}. \tag{1.17}$$

It follows that the operators (1.17) and basis functions (1.15) must satisfy relations (1.7). It is a straightforward (though tedious) computation to verify that these relations imply the following identities for the ${}_pF_q$:

$$\begin{aligned} &\left(z\frac{d}{dz} + m_{j'}\right){}_pF_q(m_j; n_k; z) = m_{j'}\,{}_pF_q(\tilde{m}_j; n_k; z), \\ &\left(z\frac{d}{dz} + n_{k'} - 1\right){}_pF_q(m_j; n_k; z) = (n_{k'} - 1)\,{}_pF_q(m_j, \tilde{n}_k; z), \\ &\frac{d}{dz}\,{}_pF_q(m_j; n_k; z) = \frac{m_1 \cdots m_p}{n_1 \cdots n_q}\,{}_pF_q(m_j + 1; n_k + 1; z). \end{aligned} \tag{1.18}$$

Here $\tilde{m}_j$ and $\tilde{n}_k$ are given by (1.7) and m_j, n_k take on all integer values such that the polynomials ${}_pF_q(m_j; n_k; z)$ are defined. These three relations are obtained from the operator identities for R_j, L_k, V, respectively. Relation (1.8) implies the differential equation

$$\left\{\left(z\frac{d}{dz} + m_1\right) \cdots \left(z\frac{d}{dz} + m_p\right) - \frac{d}{dz}\left(z\frac{d}{dz} + n_1 - 1\right) \cdots \left(z\frac{d}{dz} + n_q - 1\right)\right\} \cdot {}_pF_q(m_j; n_k; z) = 0 \tag{1.19}$$

for the generalized hypergeometric functions.

Now that we have obtained these identities for integral values of the parameters, we can easily verify from the power series definition of ${}_pF_q$ that they remain valid for complex m_j, n_k, z if $p < q + 1$, ($|z| < 1$ if $p = q + 1$). This in turn suggests another model of $\rho(\alpha_j, \beta_k)$. Namely, we set

$$\begin{aligned} &R_j = t_j\left(z\frac{\partial}{\partial z} + S_j\right), \qquad S_j = t_j\frac{\partial}{\partial t_j} + \alpha_j, \\ &L_k = u_k^{-1}\left(z\frac{\partial}{\partial z} + T_k - 1\right), \qquad T_k = u_k\frac{\partial}{\partial u_k} + \beta_k, \\ &V = t_1 \cdots t_p\, u_1 \cdots u_q\frac{\partial}{\partial z}. \end{aligned} \tag{1.20}$$

It follows from our previous identities that the operators (1.20) and the functions

$$\begin{aligned} f_{m_j n_k}(z, t_j, u_k) = &\frac{\Gamma(m_1 + \alpha_1) \cdots \Gamma(m_p + \alpha_p)}{\Gamma(n_1 + \beta_1) \cdots \Gamma(n_q + \beta_q)} \\ &\cdot {}_pF_q(m_j + \alpha_j; n_k + \beta_k; z)t_1^{m_1} \cdots t_p^{m_p}u_1^{n_1} \cdots u_q^{n_q} \end{aligned} \tag{1.21}$$

define a model of $\rho(\alpha_j, \beta_k)$ for $p \leqq q + 1$. (Here, for simplicity, we consider only the case $p < q + 1$ so that $f_{m_j n_k}(z, t_j, u_k)$ is an entire function of z. Also we require $0 < \operatorname{Re}\alpha_j < 1, 0 < \operatorname{Re}\beta_k < 1$.)

The Lie derivatives (1.20) define a local multiplier representation of $G_{p,q}$ on functions $f(z, t_j, u_k)$. This action is determined by operators

$$\begin{aligned}
&[\mathbf{Q}(g)f](z, t_j, u_k) = \exp\left(\sum \gamma_j\alpha_j + \sum \xi_k\beta_k\right)\\
&\cdot \prod_j (1 - a_j t_j)^{-\alpha_j} \prod_k (1 + b_k/u_k)^{\beta_k - 1}\\
&\cdot f\left[\frac{(z + ct_1 \cdots u_q)(1 + b_1/u_1)\cdots(1 + b_q/u_q)}{(1 - a_1t_1)\cdots(1 - a_pt_p)}, \frac{t_j e^{\gamma_j}}{1 - a_jt_j}, (u_k + b_k)\, e^{\xi_k}\right],\\
&|a_jt_j| < 1, \quad |b_k/u_k| < 1, \quad g \in G_{p,q}.
\end{aligned} \tag{1.22}$$

The matrix elements

$$\mathbf{Q}(g) f_{m_j n_k} = \sum_{m'_j, n'_k} T(g)^{m'_j n'_k}_{m_j n_k} f_{m'_j n'_k} \tag{1.23}$$

for this model are necessarily identical with (1.12). Thus, substituting (1.12), (1.21) and (1.23) and simplifying, we obtain

$$\begin{aligned}
&\prod_{j,k}\left[\frac{\Gamma(m_j + \alpha_j)}{\Gamma(n_k + \beta_k)} \frac{t_j^{m_j} u_k^{n_k}}{(1 - a_jt_j)^{m_j + \alpha_j}} (1 + b_k/u_k)^{n_k + \beta_k - 1}\right]\\
&\cdot {}_pF_q\left(m_j + \alpha_j; n_k + \beta_k; \frac{(z + ct_1 \cdots u_q)(1 + b_1/u_1)\cdots(1 + b_q/u_q)}{(1 - a_1t_1)\cdots(1 - a_pt_p)}\right)\\
&= \sum_{m'_j, n'_k = -\infty} \prod_{j,k}\left[\frac{\Gamma(m'_j + \alpha_j)}{\Gamma(n'_k + \beta_k)} \frac{a_j^{m'_j - m_j} t_j^{m'_j} u_k^{n'_k}}{b_k^{n'_k - n_k}\Gamma(m_j - m'_j + 1)\Gamma(1 - n'_k + n_k)}\right]\\
&\cdot {}_pF_q\left(m_j - m'_j; 1 - n'_k + n_k; \frac{-cb_1 \cdots b_q}{a_1 \cdots a_p}\right)\\
&\cdot {}_pF_q(m'_j + \alpha_j; n'_k + \beta_k; z), \qquad |a_jt_j| < 1, \quad |b_k/u_k| < 1.
\end{aligned} \tag{1.24}$$

If $g = \exp(a\mathscr{R}_1)$, this formula reduces to the well-known expansion

$$\begin{aligned}
&(1 - a)^{-\sigma}{}_pF_q\left(\sigma, \alpha_j; \beta_k; \frac{z}{1 - a}\right)\\
&= \sum_{h=0}^{\infty} \frac{a^h}{h!} \frac{\Gamma(\sigma + h)}{\Gamma(\sigma)} {}_pF_q(\sigma + h, \alpha_j; \beta_k; z), \qquad |a| < 1,
\end{aligned}$$

where σ, α_j, β_k are noninteger complex numbers. Similarly, if $g = \exp(b\mathscr{L}_p)$, the formula reduces to

$$\begin{aligned}
&(1 + b)^{\tau - 1}{}_pF_q(\alpha_j; \tau, \beta_k; z(1 + b))\\
&= \sum_{h=0}^{\infty} \frac{b^h}{h!} \frac{\Gamma(\tau)}{\Gamma(\tau - h)} {}_pF_q(\alpha_j; \tau - h, \beta_k; z), \qquad |b| < 1.
\end{aligned}$$

For $g = \exp(c\mathscr{V})$ we have

$$ {}_pF_q(\alpha_j;\beta_k;z+c) = \sum_{h=0}^{\infty} \frac{c^h}{h!} \prod_{j,k} \left[\frac{\Gamma(\alpha_j+h)\Gamma(\beta_k)}{\Gamma(\alpha_j)\Gamma(\beta_k+h)}\right] \cdot {}_pF_q(\alpha_j+h;\beta_k+h;z). $$

Setting $t_j = e^{i\varphi_j}$, $u_k = e^{i\theta_k}$, multiplying both sides of (1.24) by $e^{-iM_1\varphi_1}\cdots e^{-iN_q\theta_q}$ and integrating we obtain

$$ \frac{1}{(2\pi)^{p+q}}\int_0^{2\pi}\cdots\int_0^{2\pi} \prod_{j,k}\left[\frac{e^{i(m_j-M_j)\varphi_j}e^{i(n_k-N_k)\theta_k}(1+b_ke^{-i\theta_k})^{n_k+\beta_k-1}}{(1-a_je^{i\varphi_j})^{m_j+\alpha_j}}\right] $$

$$ \cdot {}_pF_q\left(m_j+\alpha_j;n_k+\beta_k;\frac{(z+ce^{i(\varphi_1+\cdots+\theta_q)})(1+b_1e^{-i\theta_1})\cdots(1+b_qe^{-i\theta_q})}{(1-a_1e^{i\varphi_1})\cdots(1-a_pe^{i\varphi_p})}\right) $$

$$ \cdot\, d\varphi_1\cdots d\varphi_p\, d\theta_1\cdots d\theta_q \tag{1.25} $$

$$ = \left[\prod_{j,k}\frac{\Gamma(M_j+\alpha_j)}{\Gamma(m_j+\alpha_j)}\frac{\Gamma(n_k+\beta_k)}{\Gamma(N_k+\beta_k)}\frac{a_j^{M_j-m_j}}{b_k^{N_k-n_k}}\frac{1}{\Gamma(m_j-M_j+1)\Gamma(1-N_k+n_k)}\right] $$

$$ \cdot {}_pF_q\left(m_j-M_j;1-N_k+n_k;\frac{-cb_1\cdots b_q}{a_1\cdots a_p}\right) $$

$$ \cdot {}_pF_q(M_j+\alpha_j;N_k+\beta_k;z), \qquad |a_j|<1, \quad |b_k|<1. $$

The method of Weisner [1], [2] can also be used to derive identities for the ${}_pF_q$ from (1.20) and (1.22). For example, we search for a simultaneous eigenfunction $h(z,t_j,u_k)$ of the commuting operators $R_1, S_j, 2 \leqq j \leqq p, T_k, 1 \leqq k \leqq q$, which satisfies $Ch = 0$ (see (1.8)):

$$ \begin{aligned} &R_ph = h, \quad S_jh = \alpha_jh, \quad 1 \leqq j \leqq p-1, \\ &T_kh = \beta_kh, \quad 1 \leqq k \leqq q, \quad Ch = 0. \end{aligned} \tag{1.26} $$

(For simplicity we set $\alpha_j = \beta_k = 0$ in (1.20).) The general solution of the first $p+q$ equations is

$$ h(z,t_j,u_k) = r(z/t_p)\exp(-t_p^{-1})t_1^{\alpha_1}\cdots t_{p-1}^{\alpha_{p-1}}u_1^{\beta_1}\cdots u_q^{\beta_q}, \tag{1.27} $$

where r is an arbitrary function. Requiring $Ch = 0$ we obtain

$$ r(x) = {}_{p-1}F_q(\alpha_j;\beta_k;x) \tag{1.28} $$

as the only solution bounded in a neighborhood of $x = 0$ for general complex α_j, β_k.

The function $[\mathbf{Q}(g)h](z,t_j,u_k)$, $g \in G_{p,q}$, is also annihilated by the operator C and can be expanded in a Laurent series in terms of $t_1,\cdots,t_p,u_1,\cdots,u_q$. The coefficients in this series are necessarily of the form ${}_pF_q$:

$$\prod_{j=1}^{p-1} (1 - a_j t_j)^{-\alpha_j} \prod_{k=1}^{q} (1 + b_k/u_k)^{\beta_k - 1} \exp(-t_p^{-1})$$

$$\cdot {}_{p-1}F_q\left(\alpha_j; \beta_k; \frac{(z/t_p + ct_1 \cdots \hat{t}_p \cdots u_q)(1 + b_1/u_1) \cdots (1 + b_q/u_q)}{(1 - a_1 t_1) \cdots (1 - a_{p-1} t_{p-1})}\right) \tag{1.29}$$

$$= \sum_{m_j, m_p = 0}^{\infty} \sum_{n_k = -\infty}^{\infty} j_{m_j, m_p, n_k}(g)$$

$$\cdot {}_pF_q(\alpha_j + m_j, -m_p; \beta_k + n_k; z) t_1^{m_1} \cdots t_{p-1}^{m_{p-1}} t_p^{-m_p} u_1^{n_1} \cdots u_q^{n_q},$$

$$|a_j t_j| < 1, \quad 1 \leqq j \leqq p - 1, \quad |b_k/u_k| < 1.$$

(Indeed, the term $J(g, z)t_1^{m_1} \cdots u_q^{n_q}$ in this expansion is annihilated by C and is a simultaneous eigenfunction of the S_j and T_k.) Setting $z = 0$ on both sides of this identity we obtain the generating function

$$\prod_{j=1}^{p-1} (1 - a_j t_j)^{-\alpha_j} \prod_{k=1}^{q} (1 + b_k/u_k)^{\beta_k - 1} \exp(-t_p^{-1})$$

$$\cdot {}_{p-1}F_q\left(\alpha_j; \beta_k; \frac{ct_1 \cdots \hat{t}_p \cdots u_q(1 + b_1/u_1) \cdots (1 + b_q/u_q)}{(1 - a_1 t_1) \cdots (1 - a_{p-1} t_{p-1})}\right) \tag{1.30}$$

$$= \sum_{m_j = 0}^{\infty} \sum_{n_k = -\infty}^{\infty} j_{m_j, m_p, n_k}(g) t_1^{m_1} \cdots t_{p-1}^{m_{p-1}} t_p^{-m_p} u_1^{m_1} \cdots u_q^{n_q}$$

for the $j_{m_j, n_k}(g)$. Here we assume $|a_j t_j| < 1, |b_k/u_k| < 1$. Comparing this expression with (1.24) we find

$$j_{m_j, m_p, n_k}(g) = \prod_{j=1}^{p-1} \prod_{k=1}^{q} \left[\frac{\Gamma(m_k + \alpha_j)\Gamma(\beta_k) a_j^{m_j} b_k^{-n_k}}{\Gamma(\alpha_j)\Gamma(n_k + \beta_k)\Gamma(1 - m_j)\Gamma(1 - n_k)}\right]$$

$${}_{p-1}F_q\left(-m_j; 1 - n_k; \frac{-cb_1 \cdots b_q}{a_1 \cdots a_{p-1}}\right) \frac{(-1)^{m_p}}{\Gamma(m_p + 1)}.$$

In the case $g = e$, (1.29) simplifies to

$$e^{\tau} {}_{p-1}F_q(\alpha_j; \beta_k; -z\tau) = \sum_{m=0}^{\infty} {}_pF_q(\alpha_j, -m; \beta_k; z) \frac{\tau^m}{m!}. \tag{1.31}$$

It is easy to obtain many similar identities by considering $[\mathbf{Q}(g)h](z, t_j, u_k)$ in regions where g is bounded away from e, e.g., $|a_j t_j| > 1$. We shall present some examples of such identities shortly.

The above analysis can be generalized through the construction of a simultaneous eigenfunction $h(z, t_j, u_k)$ of the commuting operators $S_j, 1 \leqq j \leqq p'$; $R_l, p' + 1 \leqq l \leqq p$; $T_k, 1 \leqq k \leqq q'$; $L_i, q' + 1 \leqq i \leqq q$ which satisfies $Ch = 0$:

$$\begin{aligned} &S_j h = \alpha_j h, \quad 1 \leqq j \leqq p', \quad R_l h = h, \quad p' + 1 \leqq l \leqq p, \\ &T_k h = \beta_k h, \quad 1 \leqq k \leqq q', \quad L_i h = h, \quad q' + 1 \leqq i \leqq q, \quad Ch = 0. \end{aligned} \tag{1.32}$$

(We set $\alpha_j = \alpha_l = \beta_k = 0, \beta_i = 1$ in (1.20).) Here p' can take values $0, 1, \cdots, p$ and q' can take values $0, 1, \cdots, q$. (For simplicity we require $p' \leqq q'$.) For fixed p' and q' the solution h of equations (1.32) bounded near $z = 0$ for arbitrary complex α_j, β_k is, to within a multiplicative constant:

$$
\begin{aligned}
h &= r\left(\frac{z}{t_{p'+1}\cdots t_p u_{q'+1}\cdots u_q}\right)\exp\left(u_{q'+1} + \cdots + u_q - t_{p'+1}^{-1} - \cdots - t_p^{-1}\right) \\
&\qquad \cdot t_1^{\alpha_1}\cdots t_{p'}^{\alpha_{p'}} u_1^{\beta_1}\cdots u_{q'}^{\beta_{q'}}, \qquad (1.33)\\
r(x) &= {}_{p'}F_{q'}(\alpha_j; \beta_k; x).
\end{aligned}
$$

Again the function $\mathbf{Q}(g)h$ is annihilated by C and can be expanded in a Laurent series in $t_1, \cdots, u_q$ such that the coefficients in the expansion are of the form ${}_pF_q$:

$$
\begin{aligned}
&\prod_{j=1}^{p'}(1 - a_j t_j)^{-\alpha_j}\prod_{k=1}^{q'}(1 - b_k/u_k)^{\beta_k - 1} \\
&\quad \cdot \exp\left(-t_{p'+1}^{-1} - \cdots - t_p^{-1} + u_{q'+1} + \cdots + u_q\right) \\
&\quad \cdot {}_{p'}F_{q'}\left\{\alpha_j; \beta_k; \left[\frac{z}{t_{p'+1}\cdots t_p u_{q'+1}\cdots u_q} + c t_1\cdots t_{p'} u_1\cdots u_{q'}\right]\right. \\
&\qquad \left.\cdot (1 - a_1 t_1)^{-1}\cdots(1 - a_{p'}t_{p'})^{-1}\left(1 + \frac{b_1}{u_1}\right)\cdots\left(1 + \frac{b_{q'}}{u_{q'}}\right)\right\} \qquad (1.34)\\
&= \sum_{m_j, m_l, n_k, n_i}\frac{j_{m_j, m_l, n_k, n_i}(g)}{\Gamma(n_i + 1)}\cdot {}_pF_q(\alpha_j + m_j, -m_l; \beta_k + n_k, n_i + 1; z) \\
&\qquad \cdot t_j^{m_j} t_l^{-m_l} u_k^{n_k} u_i^{n_i}, \\
&\qquad |a_j t_j| < 1, \quad |b_k/u_k| < 1.
\end{aligned}
$$

We can obtain a simple generating function for the $j_{m_j, m_l, n_k, n_i}(g)$ by setting $z = 0$ in (1.34). Comparing this expression with (1.24) we find

$$
\begin{aligned}
j_{m_j, m_l, n_k, n_i}(g) &= \left[\prod_{j=1}^{p'}\prod_{k=1}^{q'}\frac{\Gamma(m_j + \alpha_j)\Gamma(\beta_k)a_j^{m_j}b_k^{-n_k}}{\Gamma(a_j)\Gamma(n_k + \beta_k)\Gamma(1 - m_j)\Gamma(1 - n_k)}\right] \\
&\quad \cdot {}_{p'}F_{q'}\left(-m_j; 1 - n_k; \frac{-cb_1\cdots b_{q'}}{a_1\cdots a_{p'}}\right)\prod_{l=p'+1}^{p}\frac{(-1)^{m_l}}{\Gamma(m_l + 1)} \qquad (1.35)
\end{aligned}
$$

if $n_i \geqq 0$ for all i. Moreover, a more careful analysis shows that (1.35) is valid even if some of the n_i are negative.

If $p' = p, q' = q$, these expressions reduce to (1.25). If $p' = q' = 0$, they reduce to (1.11), (1.12).

We can consider many more identities of this same type by considering $\mathbf{Q}(g)h$ for g bounded away from e. For example, let $p' = p$, $q' = q$ and $g = \exp(1 \cdot \mathscr{R}_p)$. Then

$$\mathbf{Q}(g)h = [t_p/(1 - t_p)]^{\alpha_p} {}_pF_q\left(\alpha_j, \alpha_p; \beta_k; \frac{z}{1 - t_p}\right) \cdot t_1^{\alpha_1} \cdots t_{p-1}^{\alpha_{p-1}} u_1^{\beta_1} \cdots u_q^{\beta_q}. \tag{1.36}$$

If $|t_p| > 1$, then

$$[t_p/(1 - t_p)]^{\alpha_p} = (-1)^{\alpha_p}(1 - t_p^{-1})^{-\alpha_p}$$

so we can expand (1.36) as a power series in t_p^{-1}. The coefficient of t_p^{-m} in this expansion will be a multiple of ${}_pF_q(\alpha_j, -m; \beta_k; z)$:

$$(1 - \tau)^{-\alpha_p} {}_pF_q\left(\alpha_j, \alpha_p; \beta_k; \frac{-z\tau}{1 - \tau}\right) = \sum_{m=0}^{\infty} j_m \cdot {}_pF_q(\alpha_j, -m; \beta_k; z)\tau^m, \qquad |\tau| < 1, \quad \tau = t_p^{-1}. \tag{1.37}$$

Setting $z = 0$ on both sides of this equation we find

$$j_m = \frac{\Gamma(\alpha_p + m)}{\Gamma(\alpha_p)m!}.$$

2. Vilenkin's method. We apply Vilenkin's integral transform method as set forth in [3] to compute some Mellin–Barnes integral formulas for the hypergeometric functions. Let $D_{p,q}$ be the subdomain of R_{p+q} consisting of points (t_j, u_k) such that $0 < t_j, u_k < \infty$ and let $\mathscr{D}_{p,q}$ be the space of infinitely-differentiable functions on $D_{p,q}$ with compact support. Finally let $G^r_{p,q}$ be the $(2(p + q) + 1)$-dimensional real Lie group defined by (1.3) where now the parameters $a_j, b_k, c, \gamma_j, \xi_k$ are all real. Then the operators $\mathbf{T}(g)$, (1.9) define a representation $\mathbf{T}$ of $G^r_{p,q}$ on $\mathscr{D}_{p,q}$.

If $f(t_j, u_k) \in \mathscr{D}_{p,q}$, we define the Mellin transform $\mathscr{F}(\tau_j, \lambda_k)$ of f by

$$\mathscr{F}(\tau_j, \lambda_k) = \mathscr{M}f(\tau_j, \lambda_k) = \int_0^\infty \cdots \int_0^\infty f(t_j, u_k) t_1^{\tau_1 - 1} \cdots t_1^{\tau_p - 1} u_1^{\lambda_1 - 1} \cdots u_q^{\lambda_q - 1}\, dt_1 \cdots du_q. \tag{2.1}$$

It is well known that

$$f(t_j, u_k) = \mathscr{M}^{-1}\mathscr{F}(t_j, u_k) = \frac{1}{(2\pi i)^{p+q}} \int_{\rho_j - i\infty}^{\rho_j + i\infty} \int_{\delta_k - i\infty}^{\delta_k + i\infty} \mathscr{F}(\tau_j, \lambda_k) t_1^{-\tau_1} \cdots t_p^{-\tau_p} \cdot u_1^{-\lambda_1} \cdots u_q^{-\lambda_q}\, d\tau_1 \cdots d\lambda_q, \tag{2.2}$$

where the $p + q$ constants p_j, δ_k are any real numbers.

The operators $\mathbf{T}(g)$ induce a representation S on the space of Mellin transforms, defined by

$$\mathbf{S}(g) = \mathscr{M}\mathbf{T}(g)\mathscr{M}^{-1}. \tag{2.3}$$

Thus, if $\mathscr{F} = \mathscr{M}f$ is a Mellin transform, then $\mathbf{S}(g)\mathscr{F}$ is the Mellin transform of $\mathbf{T}(g)f$:

$$\begin{aligned}[\mathbf{S}(g)\mathscr{F}](\tau_j, \lambda_k) &= \frac{1}{(2\pi i)^{p+q}} \int_0^\infty \cdots \int_0^\infty dt_1 \cdots du_q \, t_1^{\tau_1 - 1} \cdots u_q^{\lambda_q - 1} \\ &\quad \cdot \mathbf{T}(g) \int_{\rho_j - i\infty}^{\rho_j + i\infty} \int_{\delta_k - i\infty}^{\delta_k + i\infty} \mathscr{F}(\nu_j, \omega_k) t_1^{-\nu_1} \cdots u_q^{-\omega_q} \, d\nu_1 \cdots d\omega_q.\end{aligned} \tag{2.4}$$

If the $2(p + q)$-fold iterated integral is absolutely convergent, we can interchange the order of integration and write

$$[\mathbf{S}(g)\mathscr{F}](\tau_j, \lambda_k) = \int_{\rho_j - i\infty}^{\rho_j + i\infty} \int_{\delta_k - i\infty}^{\delta_k + i\infty} K(g; \tau_j, \lambda_k; \nu_j, \omega_k) \cdot \mathscr{F}(\nu_j, \omega_k) \, d\nu_1 \cdots d\omega_q. \tag{2.5}$$

Thus $\mathbf{S}(g)$ is an integral operator with kernel function

$$\begin{aligned}K(g; \tau_j, \lambda_k; \nu_j, \omega_k) &= \frac{e^{-\gamma_1 \nu_1} \cdots e^{-\xi_q \omega_q}}{(2\pi i)^{p+q}} \\ &\quad \cdot \int_0^\infty \cdots \int_0^\infty \exp\big[\textstyle\sum a_j t_j + \sum b_k/u_k \\ &\quad + c t_1 \cdots t_p u_1 \cdots u_q + \sum \alpha_j \gamma_j + \sum \beta_k \xi_k\big] \\ &\quad \cdot t_1^{\tau_1 - \nu_1 - 1} \cdots u_q^{\lambda_q - \omega_q - 1} \, dt_1 \cdots du_q.\end{aligned} \tag{2.6}$$

In particular, (2.5) and (2.6) are valid if the group parameters satisfy any of the following conditions:

(a) $a_j < 0$; $c \leqq 0$, all j, k.
(b) $a_p = 0$, $a_j < 0$, $1 \leqq j \leqq p - 1$; $b_k < 0$, all k; $c < 0$.
(c) $a_j < 0$, all j; $b_q = 0$, $b_k < 0$, $1 \leqq k \leqq q - 1$; $c < 0$.
(d) $a_p < 0$; all other parameters zero.
(e) $b_q < 0$; all other parameters zero.

In the last two cases the integration is carried out over only one variable. For example, in case (d) with $g = g(0, a_p, 0, 0; 0, 0)$ we have

$$[\mathbf{S}(g)\mathscr{F}](\tau_j, \tau_p; \lambda_k) = \int_{\rho_p - i\infty}^{\rho_p + i\infty} K(g; \tau_p; \nu_p) \mathscr{F}(\tau_j, \nu_p; \lambda_k) \, d\nu_p, \tag{2.7}$$

$\operatorname{Re}(\tau_p - \rho_p) > 0$, where

$$K(g; \tau_p; \nu_p) = \frac{1}{2\pi i} \int_0^\infty e^{a_p t_p} t_p^{\tau_p - \nu_p - 1} \, dt_p = \frac{(-a_p)^{\nu_p - \tau_p}}{2\pi i} \Gamma(\tau_p - \nu_p), \tag{2.8}$$

$\operatorname{Re}(\tau_p - \nu_p) > 0$, and $\Gamma(z)$ is the gamma function [4].

Since the $\mathbf{S}(g)$ define a representation of $G^r_{p,q}$, the kernel functions satisfy the addition theorems

$$K(gg';\tau_j,\lambda_k;\tau_j',\lambda_k') = \int_{\rho_j-i\infty}^{\rho_j+i\infty}\int_{\delta_k-i\infty}^{\delta_k+i\infty} K(g;\tau_j,\lambda_k;\nu_j,\omega_k)K(g';\nu_j,\omega_k;\tau_j',\lambda_k')\,d\nu_1\cdots d\omega_q \tag{2.9}$$

provided the integral converges absolutely. Rather than compute the kernel functions (2.6) directly we shall obtain them indirectly through the use of (2.9).

Consider $g(a_j, b_k, c; 0, 0)$, where $a_j < 0, b_k < 0, c < 0$. Then $g = g_1g_2 = g_2g_1$, where

$$g_1 = g(0, a_p, 0, 0; 0, 0),$$
$$g_2 = g(a_l, 0, b_k, c; 0, 0), \qquad 1 \leqq l \leqq p-1.$$

The kernel function for g_1 is given by (2.8), and it follows easily from the definition of the gamma function that

$$\begin{aligned} K(g_2;\tau_j,\lambda_k;\nu_j,\omega_k) &= \Gamma(\tau_p-\nu_p)\,\frac{(-c)^{\nu_p-\tau_p}}{(2\pi i)^{p+q}}\prod_{l=1}^{p-1}\prod_{k=1}^{q}(-a_l)^{\nu_l-\tau_l+\tau_p-\nu_p} \\ &\quad\cdot(-b_k)^{\lambda_k-\omega_k+\nu_p-\tau_p}\Gamma(\nu_p-\tau_p+\tau_l-\nu_l)\Gamma(\tau_p-\nu_p+\omega_k-\lambda_k), \end{aligned} \tag{2.10}$$

$$\operatorname{Re}(\tau_p-\nu_p)>0,\quad \operatorname{Re}(\nu_p-\tau_p+\tau_l-\nu_l)>0,\quad \operatorname{Re}(\tau_p-\nu_p+\omega_k-\lambda_k)>0.$$

Then (2.9) yields

$$\begin{aligned} K(g;\tau_j,\lambda_k;\tau_j',\lambda_k') &= \frac{(-c)^{-\tau_p}}{(2\pi i)^{p+q+1}}(-a_p)^{\tau_p'}\int_{\gamma-i\infty}^{\gamma+i\infty}\prod_{l=1}^{p-1}\prod_{k=1}^{q}(-a_l)^{\tau_p-\eta_l}(-b_k)^{\mu_k-\tau_p}\Gamma(\tau_p-s) \\ &\quad\cdot\Gamma(s-\tau_p')\Gamma(s-\tau_p+\eta_l)\Gamma(\tau_p-s-\mu_k)\left(\frac{cb_1\cdots b_q}{a_1\cdots a_q}(-1)^{p+q+1}\right)^s ds, \end{aligned} \tag{2.11}$$

$$\operatorname{Re}\tau_p>\gamma>\operatorname{Re}\tau_p',\quad \operatorname{Re}(\tau_p-\mu_k)>\gamma>\operatorname{Re}(\tau_p-\eta_l),$$
$$\eta_l=\tau_l-\tau_l',\qquad \mu_k=\lambda_k-\lambda_k'.$$

Thus, the kernel function is a G-function,

$$\begin{aligned} K(g;\tau_j,\lambda_k;\tau_j',\lambda_k') &= \frac{(-c)^{-\tau_p}}{(2\pi i)^{p+q}}\frac{(-a_1)^{\tau_p-\eta_1}\cdots(-a_p)^{\tau_p-\eta_p}}{(-b_1)^{\tau_p-\mu_1}\cdots(-b_q)^{\tau_p-\mu_q}} \\ &\quad\cdot G^{q+1,p}_{p,q+1}\left(\frac{cb_1\cdots b_q}{a_1\cdots a_p}(-1)^{p+q+1}\middle|\begin{matrix}1-\eta_1+\tau_p,\cdots 1-\eta_p+\tau_p\\ \tau_p-\mu_1,\cdots,\tau_p-\mu_q,\tau_p\end{matrix}\right), \end{aligned} \tag{2.12}$$

a linear combination of $q+1$ hypergeometric functions ${}_pF_q$ (see [4] or [5]).

Formulas (2.9) now yield a variety of Mellin–Barnes integrals for the $G^{q+1,p}_{p,q+1}$. For example, if $g_1 = g(a_j, b_k, c; 0, 0)$, $g_2 = (a'_j, 0, 0; 0, 0)$ with $a_j < 0$, $b_k < 0, c < 0, a'_1 = a < 0, a'_j = 0$ for $j = 2, \cdots, p$, then the relation

$$g(a_j + a'_j, b_k, c; 0, 0) = g_1 g_2$$

implies

$$\tag{2.13} \begin{aligned} &(-a_1 - a)^{\tau_p - \tau_1 + \tau'_1} \\ &\quad \cdot G^{q+1,p}_{p,q+1}\left(\frac{cb_1 \cdots b_q}{(a_1 + a)a_2 \cdots a_p}(-1)^{p+q+1}\middle|\begin{matrix}1 - \eta_1 + \tau_p, \cdots, 1 - \eta_p + \tau_p \\ \tau_p - \mu_1, \cdots, \tau_p - \mu_q, \tau_p\end{matrix}\right) \\ &\quad = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} (-a_1)^{\tau_p - \tau_1 + s}(-a)^{\tau'_1 - s}\Gamma(s - \tau'_1) \\ &\quad \cdot G^{q+1,p}_{p,q+1}\left(\frac{cb_1 \cdots b_q}{a_1 a_2 \cdots a_p}(-1)^{p+q+1}\middle| \begin{matrix}1 - \tau_1 + s + \tau_p, 1 - \eta_2 + \tau_p, \cdots, 1 - \eta_p + \tau_p \\ \tau_p - \mu_1, \cdots, \tau_p - \mu_q, \tau_p\end{matrix}\right) ds, \\ &\qquad\qquad \operatorname{Re} \tau'_1 < \gamma < \operatorname{Re}(\tau_1 - \mu_k). \end{aligned}$$

We omit the routine listing of these formulas.

3. Generating functions. Here, we further demonstrate the power and simplicity of the Lie algebraic method by explicitly computing three identities of a different type than those obtained in § 1.

We choose the generators of $\mathscr{G}_{p,q}$ in the form

$$\tag{3.1} \begin{aligned} &S_j = t_j\frac{\partial}{\partial t_j}, \quad R_j = t_j\left(z\frac{\partial}{\partial z} + t_j\frac{\partial}{\partial t_j}\right), \qquad 1 \leqq j \leqq p, \\ &T_k = u_k\frac{\partial}{\partial u_k}, \quad L_k = u_k^{-1}\left(z\frac{\partial}{\partial z} + u_k\frac{\partial}{\partial u_k} - 1\right), \quad 1 < k \leqq q, \\ &V = t_1 \cdots t_p u_1 \cdots u_q\frac{\partial}{\partial z}, \end{aligned}$$

and let

$$\tag{3.2} C = R_1 R_2 \cdots R_p - VL_1 L_2 \cdots L_q.$$

For our first example we require that h be a simultaneous solution of the equations

$$\tag{3.3} \begin{aligned} &(S_1 + S_2)h = \alpha h, \quad S_j h = \alpha_j h, \quad j = 3, \cdots, p, \\ &(R_1 + R_2)h = 0, \quad T_k h = \beta_k h, \quad k = 1, \cdots, q, \\ &Ch = 0. \end{aligned}$$

(Here $[S_1 + S_2, R_1 + R_2] = R_1 + R_2$ so $S_1 + S_2$ leaves the null space of $R_1 + R_2$ invariant even though these operators do not commute.) The first $p + q$ equations imply

$$h(z, t_j, u_k) = (1 - \tau)^{-\alpha} f\left(\frac{z\tau}{(1 - \tau)^2}\right) t_2^{\alpha} t_3^{\alpha_3} \cdots t_p^{\alpha_p} u_1^{\beta_1} \cdots u_q^{\beta_q},$$

where $\tau = t_2/t_1$, and $Ch = 0$ implies

$$f\left(\frac{z\tau}{(1-\tau)^2}\right) = {}_pF_q\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \alpha_3, \cdots, \alpha_p; \beta_k; -\frac{4z\tau}{(1-\tau)^2}\right),$$

unique to within a multiplicative constant. Expanding $h(z, t_j, u_k)$ in powers of τ we find

$$(3.4)\qquad \begin{aligned}(1-\tau)^{-\alpha}{}_pF_q&\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \alpha_j; \beta_k; \frac{-4z\tau}{(1-\tau)^2}\right)\\ &= \sum_{n=0}^{\infty} c_n\, {}_pF_q(-n, \alpha+n, \alpha_j; \beta_k; z)\tau^n,\end{aligned}$$

$$p-1 \leqq q, \quad |\tau| < 1.$$

The constants c_n can be evaluated by setting $z = 0$:

$$c_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)n!};$$

see [6, p. 137].

In our second example we look for a solution h of the equations

$$(3.5)\qquad \begin{aligned}&R_1R_2h = h, \quad S_jh = \alpha_jh, \quad j = 3, \cdots, p,\\ &(S_1 - S_2)h = \alpha h, \quad T_kh = \beta_kh, \quad k = 1, \cdots, q,\\ &Ch = 0.\end{aligned}$$

Note that the operators $R_1R_2, S_j\,(3 \leqq j \leqq p), S_1 - S_2$ and T_k commute with one another. The first $p + q$ conditions are satisfied by

$$h(z, t_j, u_k) = {}_0F_1(\alpha+1; \tau)k(z\tau)t_2^{-\alpha}t_3^{\alpha_3} \cdots t_p^{\alpha_p}u_1^{\beta_1} \cdots u_q^{\beta_q},$$

where $\tau = t_1^{-1}t_2^{-1}$. The requirement $Ch = 0$ implies

$$k(x) = {}_{p-2}F_q(\alpha_j; \beta_k; x).$$

Expanding h in a power series in τ we find

$$(3.6)\qquad {}_0F_1(\alpha+1; \tau)\,{}_{p-2}F_q(\alpha_j; \beta_k; z\tau) = \sum_{n=0}^{\infty} c_n \cdot {}_pF_q(-n, -\alpha-n, \alpha_j; \beta_k; z)\tau^n.$$

To make sense of this expansion we must require $p \leqq q + 1$. The series converges for all τ if $p < q + 1$ and for $|z\tau| < 1$ if $p = q + 1$. To compute the c_n we set $z = 0$:

$$c_n = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha+n+1)}.$$

For our final example we compute a solution of the equations

$$(3.7)\qquad \begin{aligned}&(V + R_1)h = 0, \quad (S_2 - S_j)h = -\alpha_jh, \quad 3 \leqq j \leqq p,\\ &(S_2 - T_k)h = -\beta_kh, \quad 1 \leqq k \leqq q, \quad S_1h = \alpha_1h,\\ &Ch = 0.\end{aligned}$$

Here, $[S_1, V + R_1] = V + R_1$ so the null space of $V + R_1$ is invariant under S_1. The first $p + q$ equations are satisfied by

$$h(z, t_j, u_k) = (1 + z\tau)^{-\alpha_1} k(\tau) t_1^{\alpha_1} t_3^{\alpha_3} \cdots t_p^{\alpha_p} u_1^{\beta_1} \cdots u_k^{\beta_k},$$

where $\tau^{-1} = t_2 t_3 \cdots t_p u_1 \cdots u_k$. The requirement $Ch = 0$ implies

$$k(\tau) = {}_qF_{p-2}(-\beta_k + 1; -\alpha_j + 1; (-1)^{q+p}\tau).$$

Expanding h in a power series in τ we obtain

$$(1 + z\tau)^{-\alpha_1} {}_qF_{p-2}(-\beta_k + 1; -\alpha_j + 1; (-1)^{q+p}\tau)$$

$$= \sum_{n=0}^{\infty} c_n \, {}_pF_q(\alpha_1, -n, \alpha_j - n; \beta_k - n; z)\tau^n, \qquad |z\tau| < 1. \tag{3.8}$$

To compute the c_n we set $z = 0$:

$$c_n = \frac{(-1)^{n(p+q)}\Gamma(-\beta_k + n + 1)\Gamma(-\alpha_j + 1)}{n!\Gamma(-\beta_k + 1)\Gamma(-\alpha_j + n + 1)}.$$

The above examples illustrate the simplicity of our Lie algebraic method. Once the method is understood it is straightforward to derive a great variety of generating functions for the ${}_pF_q$. Furthermore, the method permits the classification of known generating functions in terms of eigenvalues of operators formed from the generators (3.1). (Indeed, an examination of the right-hand sides of the expansions (3.4), (3.6) and (3.8) and use of recurrence relations (1.18) lead easily to the eigenvalue equations (3.3), (3.5) and (3.7), respectively. Similarly, other known generating functions can be classified in terms of eigenvalue equations.)

This method reduces the search for generating functions to the computation of solutions of systems of partial differential equations. In practice such systems may be difficult to solve. (This is the case with some very complicated identities such as those found in [5, vol. 2, p. 2].) Nevertheless, once an identity is discovered it can be fitted into the Lie algebraic classification system.

REFERENCES

[1] W. MILLER, *Lie Theory and Special Functions*, Academic Press, New York, 1968.

[2] L. WEISNER, *Group-theoretic origin of certain generating functions*, Pacific J. Math., 5 (1955), pp. 1033–1039.

[3] N. VILENKIN, *Special Functions and the Theory of Group Representations*, Amer. Math. Soc. Transl., Providence, R.I., 1968.

[4] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. TRICOMI, *Higher Transcendental Functions*, vol. I, McGraw-Hill, New York, 1953.

[5] Y. LUKE, *The Special Functions and their Approximations*, vols. I, II, Academic Press, New York, 1969.

[6] E. RAINVILLE, *Special Functions*, Macmillan, New York, 1960.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ASYMPTOTIC SOLUTIONS OF A 6TH ORDER DIFFERENTIAL EQUATION WITH TWO TURNING POINTS. PART 1: DERIVATION BY METHOD OF STEEPEST DESCENT*

B. GRANOFF† AND N. BLEISTEIN‡

Abstract. Asymptotic expansions of the basis solutions of the equation

$$(D^2 - \alpha^2k^2)^3u - k^6xu = 0, \qquad k \gg 1,$$

are derived by the method of steepest descent. For $\alpha \neq 0$, there are two turning points, at $x = 0$ and $x = -\alpha^6$. All previous results pertain to the case where $\alpha = 0$, in which case there is one turning point at $x = 0$. The derivation of the asymptotic expansions of a basis set for each of the three x-intervals formed by the turning points yields a set of WKB connection formulas. Uniformly valid asymptotic expansions are given for x in the neighborhood of the turning point $x = -\alpha^6$.

1. Introduction. The ordinary differential equation

$$(D^2 - \alpha^2k^2)^3u - k^6xu = 0, \qquad k \gg 1 \quad (D = d/dx), \tag{1.1}$$

arises in the stability analysis of viscous flow between rotating cylinders; see, for example, Meksyn (1946, 1961), Chandrasekhar (1954, 1958), and Duty and Reid (1964).[1]

In this paper we shall derive the asymptotic expansions for $k \gg 1$ of the basis solutions of (1.1). Previously obtained results correspond to (1.1) with $\alpha = 0$, in which case the equation has one turning point at $x = 0$. For $\alpha \neq 0$, (1.1) has *two* turning points, at $x = 0$ and $x = -\alpha^6$. The derivation of the asymptotic expansions of a basis set for each of the three x-intervals formed by the turning points yields a set of WKB connection formulas. The results of Meksyn and of Duty and Reid correspond to our asymptotic expansions for the intervals $-\infty < x < -\alpha^6$ and $0 < x < \infty$. We further obtain asymptotic expansions which remain uniformly valid for x near $-\alpha^6$.

We briefly describe, in § 2, the derivation of the integral representation of the solutions of (1.1). In § 3, we describe the deformation of contours of integration onto paths of steepest descent (Jeffreys, 1962). The results of the application of the method of steepest descent are summarized by means of tables in § 4. The determination of these paths of steepest descent for $\alpha \neq 0$ is rather difficult, more so than for $\alpha = 0$. These paths were determined by computer. A program was developed by N. Rushfield and run on an IBM 360. In § 5 we derive the uniformly valid expansions for x near $-\alpha^6$ by using the method of Chester, Friedman and Ursell (1957) for two nearby saddle points. Asymptotic expansions uniformly valid for x near zero are possible as well with the method of Bleistein (1967).

* Received by the editors December 7, 1970, and in final revised form April 29, 1971. This work was supported in part by the National Science Foundation.

† Mathematics Department, Boston University, Boston, Massachusetts 02215.
‡ Mathematics Department, University of Denver, Denver, Colorado 80210.

[1] To identify with Meksyn replace α^2k^2 and k^6 by λ^2 and λ^6h, respectively; with Chandrasekhar, replace α^2k^2, k^6, and $-x$ by a^2, a^2T, and $1 + \alpha\zeta$, respectively; with Duty and Reid, replace α^2k^2, k^6 and x by α^2, $\alpha^2\tau$, and $z - 1$, respectively.

However, the results are given in terms of untabulated "generalized Airy functions." We consider these results as nonedifying and therefore do not include them.

2. Integral representation of the basis solutions. In order to obtain the integral representation of the solutions of (1.1) we assume that the solutions have the form

$$u(x) = \int_\Gamma v(s) \exp[ksx]\, ds. \tag{2.1}$$

The contour Γ is to be determined as well as the function $v(s)$. Substitution of (2.1) into (1.1) and integration by parts yields

$$k^6 \int_\Gamma [(s^2 - \alpha^2)^3 v(s) + k^{-1} v'(s)] \exp[ksx]\, ds - k^{-1} v(s) \exp[ksx]|_\Gamma = 0. \tag{2.2}$$

The integral in (2.2) vanishes if we set

$$v(s) = c \exp[-kp(s, \alpha)], \tag{2.3}$$

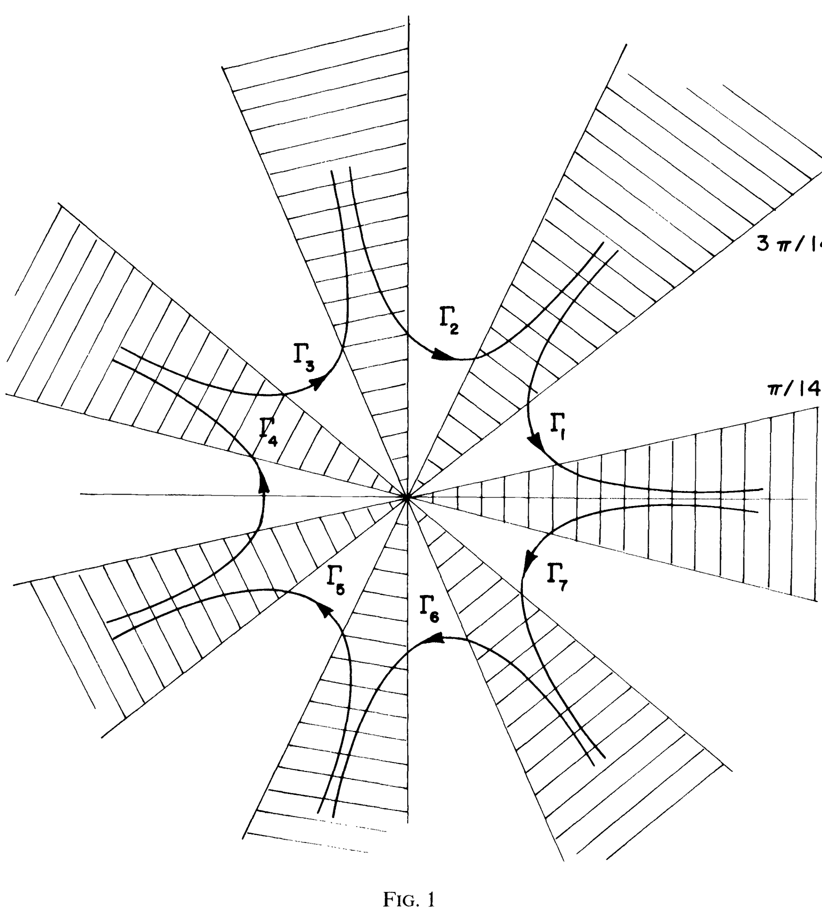

FIG. 1

where

$$p(s, \alpha) = \int_0^s (\xi^2 - \alpha^2)^3 \, d\xi = \tfrac{1}{7}s^7 - \tfrac{3}{5}s^5\alpha^2 + \alpha^4 s^3 - \alpha^6 s. \tag{2.4}$$

In order that the "endpoint contributions" of the integration by parts vanish, we require Γ to be an infinite contour with endpoints in the "valleys" of $-s^7$; i.e., in regions where $\operatorname{Re} s^7 > 0$. The seven regions which have this property are given by

$$\frac{2n\pi}{7} - \frac{\pi}{14} < \arg s < \frac{2n\pi}{7} + \frac{\pi}{14}, \qquad n = 0, 1, \cdots, 6. \tag{2.5}$$

If we introduce the contours $\Gamma_1, \Gamma_2, \cdots, \Gamma_7$ in Fig. 1, then from (2.1) and (2.3) we find that the solutions of (1.1) have the form

$$u_n(x) = \int_{\Gamma_n} \exp\{k[sx - p(s, \alpha)]\} \, ds, \qquad n = 1, 2, \cdots, 7. \tag{2.6}$$

Of course only six of these solutions are linearly independent since $\sum_{n=1}^{7} u_n(x) = 0$.

3. Asymptotic analysis. We now apply the method of steepest descent to the integrals given by (2.6). If we set

$$\varphi(s; x, \alpha) = sx - p(s; \alpha), \tag{3.1}$$

where $p(s; \alpha)$ is given by (2.4), the saddle points are defined by the equation

$$\varphi' = \frac{\partial \varphi}{\partial s} = x - (s^2 - \alpha^2)^3 = 0. \tag{3.2}$$

We first note that for $x \neq 0$ and $x \neq -\alpha^6$, (3.2) has six distinct solutions. The turning points $x = 0$ and $x = -\alpha^6$ divide the x-axis into three intervals, in each of which the expressions for the saddle points take a different form. We denote these open intervals by

$$\begin{aligned} D_-&: \quad -\infty < x < -\alpha^6, \\ D_0&: \quad -\alpha^6 < x < 0, \\ D_+&: \quad 0 < x < \infty. \end{aligned} \tag{3.3}$$

We have that

$$\varphi'(\pm s_j; x, \alpha) = 0, \qquad j = 1, 2, 3, \tag{3.4}$$

where the s_j are given in Table 1.

When $x = 0$, $s_1 = s_2 = s_3$ and the six distinct roots coalesce onto $s = \pm\alpha$, each of multiplicity three. The implications for the asymptotic analysis are that, in both instances, simple saddle points have coalesced to yield higher order saddle points. It is well known that the classical saddle-point method does not yield an expansion which remains valid as x passes through either of these critical points. We defer discussion of this problem to § 5 and consider for the present values of x belonging to one of the three domains D_-, D_0, D_+.

TABLE 1

The roots of $x - (s^2 - \alpha^2)^3 = 0$

$0 \leqq \arg\sqrt{\cdot} < \pi, \omega = \exp 2\pi i/3$

	s_1	s_2	s_3
$x \in D_-$	$i\sqrt{\lvert x\rvert^{1/3} - \alpha^2}$	$\sqrt{-\lvert x\rvert^{1/3}\omega^2 + \alpha^2}$	$\sqrt{-\lvert x\rvert^{1/3}\omega + \alpha^2}$
$x \in D_0$	$\sqrt{-\lvert x\rvert^{1/3} + \alpha^2}$	$\sqrt{-\lvert x\rvert^{1/3}\omega^2 + \alpha^2}$	$\sqrt{-\lvert x\rvert^{1/3}\omega + \alpha^2}$
$x \in D_+$	$\sqrt{\lvert x\rvert^{1/3} + \alpha^2}$	$\sqrt{\lvert x\rvert^{1/3}\omega + \alpha^2}$	$\sqrt{\lvert x\rvert^{1/3}\omega^2 + \alpha^2}$

It is always quite simple to determine the paths of steepest descent for φ locally, near the saddle points, by the use of power series. Indeed, for $\alpha = 0$, the essential features of these paths can be determined in the large. This is essentially what was done by Meksyn (1961) and by Duty and Reid. For nonzero α, we find these paths by making use of the following observations:

(i) For any x in one of the domains D_-, D_0, D_+, the essential features of the descent paths remain unchanged.

(ii) For any fixed α different from zero, say $\alpha = 1$, (i) remains valid.

Observation (i) means that the connection of each saddle point in the finite plane with the valleys at ∞ (i.e., with the sectors at ∞ in which the real part of $\varphi(s; x, \alpha)$ decays to $-\infty$) via paths of steepest descent remains essentially unchanged.

In order to verify this we first set $u(\sigma, \tau; x, \alpha) = \operatorname{Re} \varphi(s; x, \alpha)$ and $v(\sigma, \tau; x, \alpha) = \operatorname{Im} \varphi(s; x, \alpha)$, where $s = \sigma + i\tau$. For x in any closed interval bounded away from the turning point, each saddle point is simple and its position is a continuous function of x, as are the two local directions of descent at the saddle point. Furthermore, the paths of steepest descent from a saddle point are curves on which $v = \text{const.}$; except at the saddle points, these curves have $-\nabla u$ (grad with respect to σ, τ) as tangent. This vector and hence the curve $v = \text{const.}$ is a continuous function of x.

There are seven discrete valleys at infinity. A curve which depends continuously on a parameter x for all finite (σ, τ) simply cannot have its endpoint at ∞ "jump" from one valley to another. Therefore, each saddle point is connected up in a unique way to another saddle point or a valley at ∞ by paths of steepest descent. This ultimately is what really matters since we seek to replace a contour, with two endpoints in valleys at ∞, with contours which are equivalent to it by Cauchy's theorem. This we can now do for some x in the prescribed interval. Furthermore, as x varies from that value over the prescribed interval, the steepest descent contours will vary somewhat, but still, the same deformation can be accomplished; i.e., each curve Γ_j is deformed onto the same linear combination of steepest descent paths from the *same* saddle points to the *same* valleys at infinity.

The process breaks down when x attains a turning-point value. The reason is that the turning points correspond here to the coalescence of two or more saddle points into a single saddle point of higher order. A simple saddle point has two steepest descent paths leading from it; two saddle points have four paths; but when two coalesce to a saddle point of order two, this new saddle point has only three steepest descent paths. One steepest descent path is lost in the coalescence and this is a manifestation of the lack of continuity of this process as x passes through the turning point.

A program was developed by N. Rushfield and run on an IBM 360 to determine the steepest descent paths for

$$\varphi(s; -8, 1) \quad \text{(Fig. 2)},$$
$$\varphi(s; -8/27, 1) \quad \text{(Fig. 3)},$$
$$\varphi(s; 8, 1) \quad \text{(Fig. 4)}.$$

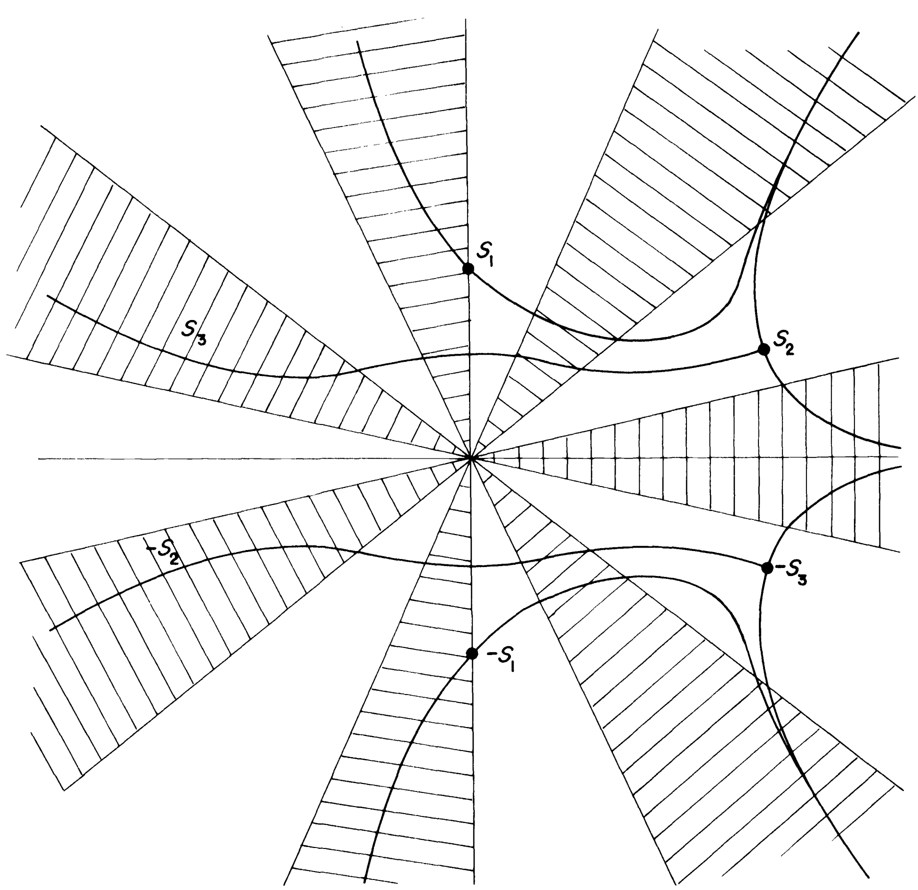

FIG. 2. *Paths of steepest descent for* $\varphi(s; -8, 1)$

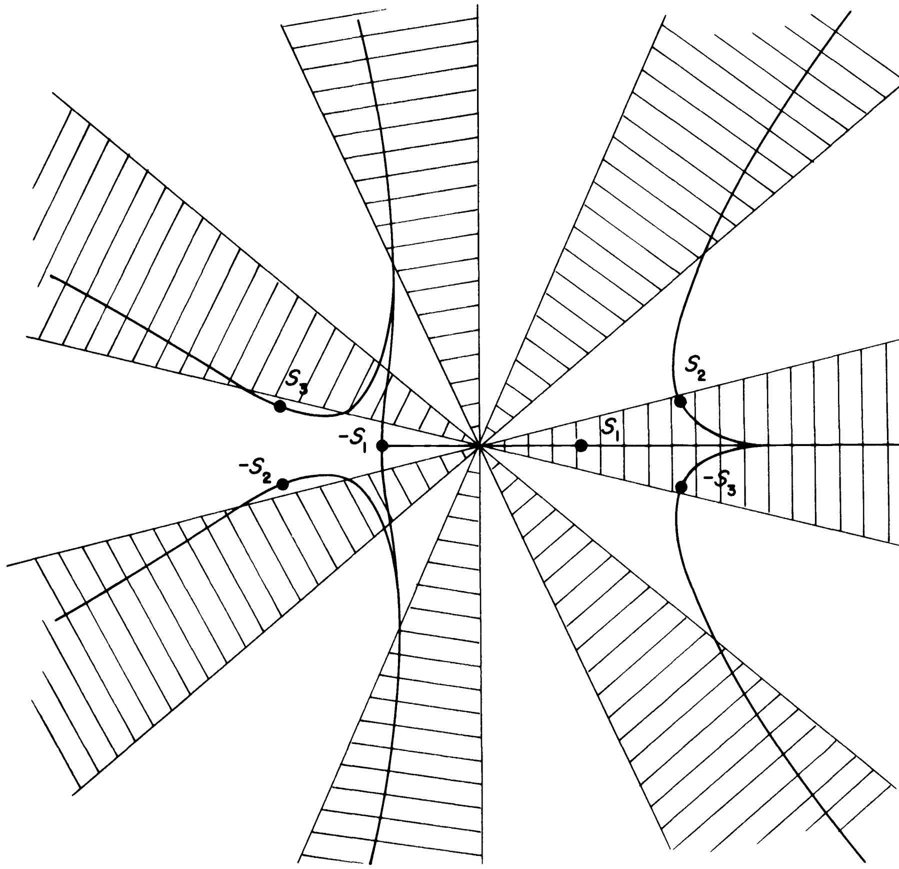

FIG. 3. *Paths of steepest descent for* $\varphi(s; -8/27, 1)$

By comparing Fig. 1 with any of these figures we can immediately identify the appropriate deformation of the contours $\Gamma_n, n = 1, 2, \cdots, 7$, onto paths of steepest descent. For example, when $x > 0$, we use Figs. 1 and 4 to find that Γ_1 is deformed onto a descent path through s_2, s_3, s_1 and $-s_1$. We note that the contributions from s_3 and $-s_1$ are exponentially smaller than those from s_2 and s_1, since the former lie on descent paths away from the latter. We shall call s_3 and $-s_1$ recessive (for Γ_1) and s_1 and s_2 dominant (for Γ_1). For each of the contours Γ_n and each of the domains D_-, D_0, D_+, we list the dominant and recessive saddle points in Table 2. The asymptotic expansion of each of the functions $u_n, n = 1, 2, \cdots, 7$, for each x-interval is given by a sum of contributions from the saddle points in the corresponding dominant column of Table 2. When there are two or more entries in a dominant position, we include contributions from each in order that six asymptotic solutions remain linearly independent.

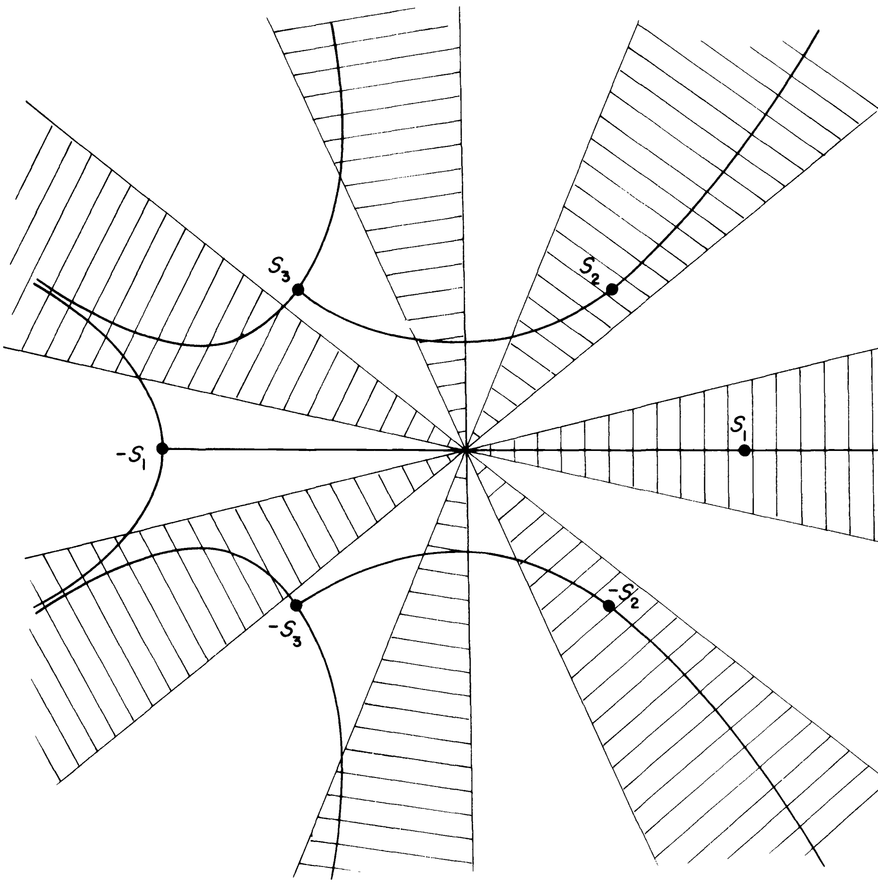

FIG. 4. *Paths of steepest descent for* $\varphi(s; 8, 1)$

TABLE 2

Saddle points contributing to the steepest descent analysis of the contours $\Gamma_n, n = 1, 2, \cdots, 7$

	$x \in D$		$x \in D_0$		$x \in D_+$	
	Dom.	Rec.	Dom.	Rec.	Dom.	Rec.
Γ_1	s_2	—	s_2	—	s_1, s_2	$-s_1, -s_2$
Γ_2	s_1	—	s_1, s_2	$-s_1$	s_2	s_3
Γ_3	s_1, s_3	s_2	s_3	—	s_3	—
Γ_4	$-s_2, s_3$	$s_2, -s_3$	$-s_1, -s_2, -s_3$	—	$-s_1$	—
Γ_5	$-s_1, -s_2$	$-s_3$	$-s_2$	—	$-s_2$	—
Γ_6	$-s_1$	—	$s_1, -s_3$	$-s_1$	$-s_3$	$-s_2$
Γ_7	$-s_3$	—	$-s_3$	—	$s_1, -s_3$	$s_3, -s_2$

4. The asymptotic solutions. In the following tables we list the leading terms of the asymptotic expansions of the solutions $u_n, n = 1, \cdots, 7$, for each of the domains D_-, D_0, D_+. For future reference we exhibit the phase $\varphi(s; x, \alpha)$ evaluated at the saddle points in each of the three domains:

$$D_- \text{ and } D_0: \quad x < -\alpha^6, \quad -\alpha^6 < x < 0:$$

$$\begin{aligned} \pm\varphi_1 &= \varphi(\pm s_1; x, \alpha) = \pm\sqrt{-|x|^{1/3} + \alpha^2} \\ &\quad \cdot[-\tfrac{6}{7}|x| + \tfrac{6}{35}|x|^{2/3}\alpha^2 + \tfrac{8}{35}|x|^{1/3}\alpha^4 + \tfrac{16}{35}\alpha^6], \end{aligned} \tag{4.1}$$

$$\begin{aligned} \pm\varphi_2 &= \varphi(\pm s_2; x, \alpha) = \pm\sqrt{-|x|^{1/3}\omega^2 + \alpha^2} \\ &\quad \cdot[-\tfrac{6}{7}|x| + \tfrac{6}{35}|x|^{2/3}\alpha^2\omega + \tfrac{8}{35}|x|^{1/3}\alpha^4\omega^2 + \tfrac{16}{35}\alpha^6], \end{aligned}$$

$$\begin{aligned} \pm\varphi_3 &= \varphi(\pm s_3; x, \alpha) = \pm\sqrt{-|x|^{1/3}\omega + \alpha^2} \\ &\quad \cdot[-\tfrac{6}{7}|x| + \tfrac{6}{35}|x|^{2/3}\alpha^2\omega^2 + \tfrac{8}{35}|x|^{1/3}\alpha^4\omega + \tfrac{16}{35}\alpha^6]. \end{aligned}$$

$$D_+: \quad x > 0:$$

$$\begin{aligned} \pm\varphi_1 &= \varphi(\pm s_1; x, \alpha) = \pm\sqrt{|x|^{1/3} + \alpha^2} \\ &\quad \cdot[\tfrac{6}{7}|x| + \tfrac{6}{35}|x|^{2/3}\alpha^2 - \tfrac{8}{35}|x|^{1/3}\alpha^4 + \tfrac{16}{35}\alpha^6], \end{aligned}$$

$$\begin{aligned} \pm\varphi_2 &= \varphi(\pm s_2; x, \alpha) = \pm\sqrt{|x|^{1/3}\omega + \alpha^2} \\ &\quad \cdot[\tfrac{6}{7}|x| + \tfrac{6}{35}|x|^{2/3}\alpha^2\omega^2 - \tfrac{8}{35}|x|^{1/3}\alpha^4\omega + \tfrac{16}{35}\alpha^6], \end{aligned} \tag{4.2}$$

$$\begin{aligned} \pm\varphi_3 &= \varphi(\pm s_3; x, \alpha) = \pm\sqrt{|x|^{1/3}\omega^2 + \alpha^2} \\ &\quad \cdot[\tfrac{6}{7}|x| + \tfrac{6}{35}|x|^{2/3}\alpha^2\omega - \tfrac{8}{35}|x|^{1/3}\alpha^4\omega^2 + \tfrac{16}{35}\alpha^6]. \end{aligned}$$

For simplicity we introduce the following notation to be used in Tables 3 and 4:

$$a_1 = \sqrt{\frac{\pi}{3k}}|x|^{-1/3}||x|^{1/3} - \alpha^2|^{-1/4}, \tag{4.3}$$

$$a_2 = \sqrt{\frac{\pi}{3k}}|x|^{-1/3}(|x|^{2/3} + \alpha^2|x|^{1/3} + \alpha^4)^{-1/8}, \tag{4.4}$$

$$\theta_- = \frac{1}{4}\arctan\frac{\sqrt{3}|x|^{1/3}}{|x|^{1/3} + 2\alpha^2}, \qquad 0 < \theta_- < \frac{\pi}{2}. \tag{4.5}$$

For use in Table 5 we introduce the additional notation:

$$a_3 = \sqrt{\frac{\pi}{3k}}|x|^{-1/3}(\alpha^2 + |x|^{1/3})^{-1/4}, \tag{4.6}$$

$$a_4 = \sqrt{\frac{\pi}{3k}}|x|^{-1/3}(|x|^{2/3} - \alpha^2|x|^{1/3} + \alpha^4)^{-1/8}, \tag{4.7}$$

$$\theta_+ = \frac{1}{4}\arctan\frac{\sqrt{3}|x|^{1/3}}{2\alpha^2 - |x|^{1/3}}, \qquad 0 \leqq \theta_+ < \pi. \tag{4.8}$$

TABLE 3

Asymptotic solutions for x in $D_-: x < -\alpha^6$

$$u_1 \sim a_2 \exp\left[k\varphi_2 - i\left(\frac{\pi}{3} + \theta_-\right)\right]$$

$$u_2 \sim a_1 \exp\left[k\varphi_1 - i\frac{\pi}{4}\right]$$

$$u_3 \sim a_2 \exp\left[k\varphi_3 - i\left(\frac{\pi}{6} - \theta_-\right)\right] - a_1 \exp\left[k\varphi_1 - i\frac{\pi}{4}\right]$$

$$u_4 \sim a_2\left\{\exp\left[-k\varphi_2 + i\left(\frac{\pi}{6} - \theta_-\right)\right] - \exp\left[k\varphi_3 - i\left(\frac{\pi}{6} - \theta_-\right)\right]\right\}$$

$$= 2ia_2 \operatorname{Im} \exp\left[-k\varphi_2 + i\left(\frac{\pi}{6} - \theta_-\right)\right]$$

$$u_5 \sim a_1 \exp\left[-k\varphi_1 + i\frac{\pi}{4}\right] - a_2 \exp\left[-k\varphi_2 + i\left(\frac{\pi}{6} - \theta_-\right)\right]$$

$$u_6 \sim a_1 \exp\left[-k\varphi_1 + i\frac{\pi}{4}\right]$$

$$u_7 \sim -a_2 \exp\left[-k\varphi_3 + i\left(\frac{\pi}{3} + \theta_-\right)\right]$$

TABLE 4

Asymptotic solutions for x in $D_0: -\alpha^6 < x < 0$

$$u_1 \sim a_2 \exp\left[k\varphi_2 - i\left(\frac{\pi}{3} + \theta_-\right)\right]$$

$$u_2 \sim a_1 \exp[k\varphi_1] - a_2 \exp\left[k\varphi_2 - i\left(\frac{\pi}{3} + \theta_-\right)\right]$$

$$u_3 \sim a_2 \exp\left[k\varphi_3 - i\left(\frac{\pi}{6} - \theta_-\right)\right]$$

$$u_4 \sim a_2\left\{\exp\left[-k\varphi_2 + i\left(\frac{\pi}{6} - \theta_-\right)\right] - \exp\left[k\varphi_3 - i\left(\frac{\pi}{6} - \theta_-\right)\right]\right\}$$

$$+ a_1 \exp\left[-k\varphi_1 + i\frac{\pi}{2}\right]$$

$$= a_1 \exp\left[-k\varphi_1 + i\frac{\pi}{2}\right] + 2ia_2 \operatorname{Im} \exp\left[-k\varphi_2 + i\left(\frac{\pi}{6} - \theta_-\right)\right]$$

$$u_5 \sim -a_2 \exp\left[-k\varphi_2 + i\left(\frac{\pi}{6} - \theta_-\right)\right]$$

$$u_6 \sim -a_2 \exp\left[-k\varphi_3 + i\left(\frac{\pi}{3} + \theta_-\right)\right] - a_1 \exp[k\varphi_1]$$

$$u_7 \sim -a_2 \exp\left[-k\varphi_3 + i\left(\frac{\pi}{3} + \theta_-\right)\right]$$

TABLE 5

Asymptotic solutions for x in D_+: $x > 0$

$$u_1 \sim a_3 \exp[k\varphi_1] - a_4 \exp\left[k\varphi_2 + i\left(\frac{\pi}{3} - \theta_+\right)\right]$$

$$u_2 \sim a_4 \exp\left[k\varphi_2 + i\left(\frac{\pi}{3} - \theta_+\right)\right]$$

$$u_3 \sim a_4 \exp\left[k\varphi_3 + i\left(\frac{\pi}{6} + \theta_+\right)\right]$$

$$u_4 \sim a_3 \exp\left[-k\varphi_1 + i\frac{\pi}{2}\right]$$

$$u_5 \sim -a_4 \exp\left[-k\varphi_2 - i\left(\frac{\pi}{6} + \theta_+\right)\right]$$

$$u_6 \sim -a_4 \exp\left[-k\varphi_3 - i\left(\frac{\pi}{3} - \theta_+\right)\right]$$

$$u_7 \sim -a_3 \exp[k\varphi_1] + a_4 \exp\left[-k\varphi_3 - i\left(\frac{\pi}{3} - \theta_+\right)\right]$$

Tables 3, 4 and 5 constitute the WKB connection formulas. When we set $\alpha = 0$, we recover Meksyn's results (Meksyn, 1961). Appropriate linear combinations of the functions $u_1, \cdots, u_7$ with $\alpha = 0$ will reproduce the results of Duty and Reid as well.

Some explanation of the results contained in these tables is necessary. First we note that, as expected, the coefficient a_1 which appears in Tables 3 and 4 becomes infinite when $x = -\alpha^6$, i.e., when $\pm s_1 = 0$. We shall deal with this in § 5. By comparing Tables 3 and 4, we also note that the expansion of u_2, for example, contains a contribution from s_2 when x is in D_0 but does not contain such a term when x is in D_-. It would seem then that the expansion has a discontinuity at $x = -\alpha^6$ for no apparent reason. Of course this discontinuity is only illusory. The only real discontinuity is in the deformation of Γ_2 onto the steepest descent paths. However, for x in D_- there do exist descent paths (not of steepest descent) for Γ_2 which allow us to pass through s_2 as well as s_1; i.e., we can add a contribution, $-a_2 \exp[k\varphi_2 - i(\pi/3 + \theta_-)]$, to the result for u_2 in Table 3, thereby retaining a contribution exponentially small compared to the term already appearing in the table. The only advantage in doing this would be to eliminate a discontinuity in exponentially small terms. In Table 4, the contribution from s_2 ultimately becomes dominant as x increases and hence it must appear in this result.

A similar observation holds for other expansions throughout the tables.

5. Uniformly valid asymptotic expansions. We have previously noted that for $x = -\alpha^6$, two saddle points coalesce. In this limit a_1 is infinite and the expansions of u_2, u_3, u_4, u_5 and u_6 given in Tables 3 and 4 are not valid. We obtain uniformly valid asymptotic expansions for x near $-\alpha^6$ by using the method of Chester, Friedman and Ursell (1957).[2] In particular, we introduce the change of variables

(5.1) $$\varphi(s; x, \alpha) = f^2(x)t - t^3/3.$$

Here $\varphi(s; x, \alpha)$ is defined by (3.1) and (2.4) and, following CFU,

(5.2) $$f(x) = \begin{cases} [\tfrac{3}{2}\varphi_1]^{1/3}, & -\alpha^6 < x < 0, \\ i|\tfrac{3}{2}\varphi_1|^{1/3}, & x < -\alpha^6. \end{cases}$$

The effect of this change of variables is to map the contours Γ_n onto the contours L_j of Fig. 5 as shown in Table 6.

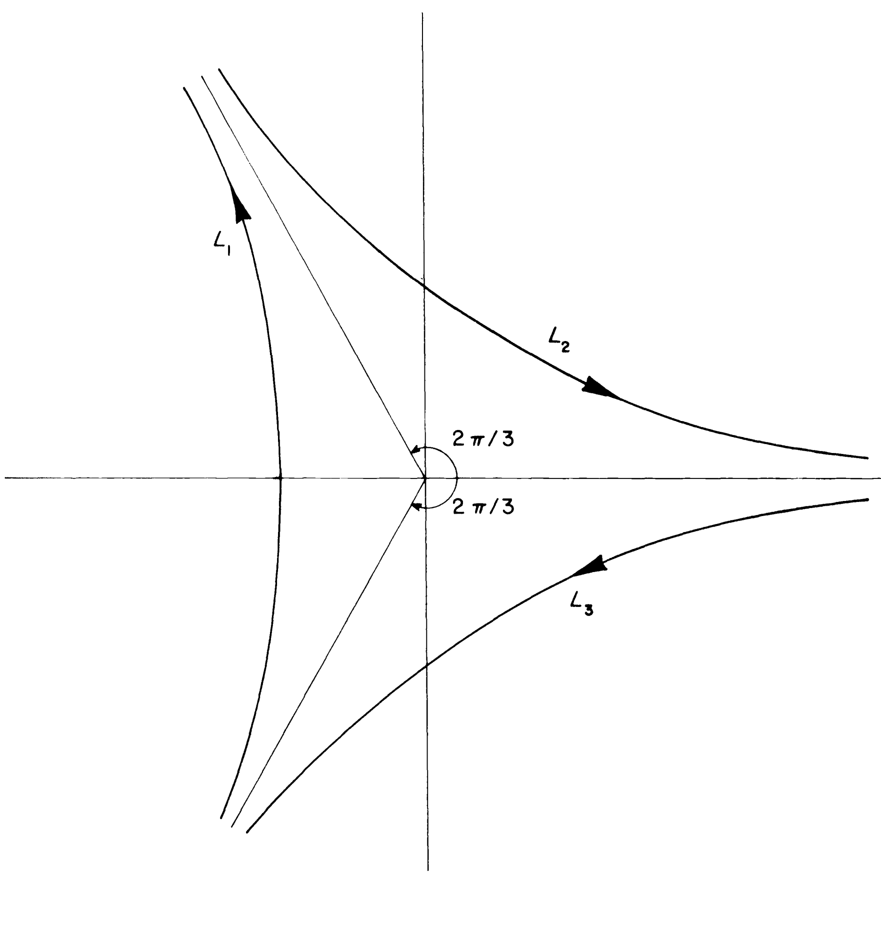

FIG. 5. *Image contours in the t-plane*

[2] We shall denote this paper by CFU hereafter.

TABLE 6

Mappings induced by (5.1)

Range of x	Contours in s-plane	Contours in t-plane
$-\alpha^6 < x < 0$	$\Gamma_1 + \Gamma_2$ $\Gamma_3 + \Gamma_4 + \Gamma_5$ $\Gamma_6 + \Gamma_7$	L_2 L_1 L_3
$x < -\alpha^6$	Γ_2 $\Gamma_3 + \Gamma_4 + \Gamma_5 - \Gamma_1 - \Gamma_7$ Γ_6	L_2 L_1 L_3

For the two x-ranges, the pre-image of any L_j is different. The effect of this anomaly is much the same as that discussed at the end of § 4.

For x negative and bounded away from zero, the method of CFU gives the following uniformly valid asymptotic expansions in terms of the Airy function (see, for example, Jeffreys (1962)):

$$u_2 \sim 2\pi g_0 k^{-1/3} \exp\left[-i\frac{\pi}{6}\right] \mathrm{Ai}[k^{2/3} f^2(x)\, e^{-2\pi i/3}] - a_2 \exp\left[k\varphi_2 - i\left(\frac{\pi}{3} + \theta_-\right)\right], \tag{5.3}$$

$$u_3 \sim -2\pi g_0 k^{-1/3} \exp\left[-i\frac{\pi}{6}\right] \mathrm{Ai}[k^{2/3} f^2(x)\, e^{-2\pi i/3}] + a_2 \exp\left[k\varphi_3 - i\left(\frac{\pi}{6} - \theta_-\right)\right], \tag{5.4}$$

$$u_4 \sim 2\pi i g_0 k^{-1/3} \mathrm{Ai}[k^{2/3} f^2(x)] + 2i \,\mathrm{Im} \exp\left[-k\varphi_2 + i\left(\frac{\pi}{6} - \theta_-\right)\right], \tag{5.5}$$

$$u_5 \sim -2\pi g_0 k^{-1/3} \exp\left[i\frac{\pi}{6}\right] \mathrm{Ai}[k^{2/3} f^2(x)\, e^{2\pi i/3}] - a_2 \exp\left[-k\varphi_2 + i\left(\frac{\pi}{6} - \theta_-\right)\right], \tag{5.6}$$

$$u_6 \sim 2\pi g_0 k^{-1/3} \exp\left[i\frac{\pi}{6}\right] \mathrm{Ai}[k^{2/3} f^2(x)\, e^{2\pi i/3}] - a_2 \exp\left[-k\varphi_3 + i\left(\frac{\pi}{3} + \theta_-\right)\right]. \tag{5.7}$$

Here

$$g_0 = \left| \frac{2f(x)}{6x^{2/3}(|x|^{1/3} - \alpha^2)^{1/2}} \right|^{1/2}. \tag{5.8}$$

We note that lim g_0 as $x \to -\alpha^6$ is finite and nonzero; indeed,

$$\lim_{x \to -\alpha^6} g_0 = (18\alpha^8)^{-1/6}. \tag{5.9}$$

To obtain uniform asymptotic expansions for x near zero we would have to apply the generalization of CFU described in Bleistein (1967). The Airy functions of CFU are replaced by "generalized Airy functions." The integral representation of these functions is much like the Airy function itself, except that in the integrand the cubic polynomial of the exponent is replaced by a quartic polynomial. Since these functions are not tabulated, there seems to be no point in carrying out such a uniform expansion.

Acknowledgment. The authors are grateful to E. Spiegel for bringing this problem to their attention, to N. Rushfield for his fine computer work, and to R. A. Handelsman for several helpful discussions on this problem.

REFERENCES

N. Bleistein (1967), *Uniform asymptotic expansions of integrals with many nearby stationary points and algebraic singularities*, J. Math. Mech., 17, pp. 553–560.

C. Chester, B. Friedman and F. Ursell (1957), *An extension of the method of steepest descents*, Proc. Cambridge Philos. Soc., 54, pp. 559–611.

S. Chandrasekhar (1954), *The stability of viscous flow between rotating cylinders*, Mathematika, 1, pp. 5–13.

——— (1958), *The stability of viscous flow between rotating cylinders*, Proc. Roy. Soc. London Ser. A, 246, pp. 301–311.

R. L. Duty and W. H. Reid (1964), *On the stability of viscous flow between rotating cylinders*, J. Fluid Mech., 20, pp. 81–94.

H. Jeffreys (1962), *Asymptotic Approximations*, Clarendon Press, Oxford.

D. Meksyn (1946), *Stability of viscous flow between rotating cylinders. II: Cylinders rotating in opposite directions*, Proc. Roy. Soc. London Ser. A, 187, pp. 480–491.

——— (1946), *Stability of viscous flow between rotating cylinders. III: Integration of a sixth order linear equation*, Ibid., 187, pp. 492–504.

——— (1961), *New Methods in Laminar Boundary-Layer Theory*, Pergamon Press, London.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ON THE ASYMPTOTIC BEHAVIOR OF FUNCTIONAL DIFFERENTIAL EQUATIONS*

T. G. HALLAM†, G. LADAS‡ AND V. LAKSHMIKANTHAM‡

Abstract. A generalized asymptotic equivalence is established between the solutions of an unperturbed linear ordinary differential equation and a nonlinear perturbed functional differential equation. These results extend and improve known results from ordinary differential equations.

1. Introduction. In this article we shall investigate some of the asymptotic relationships between the nonlinear functional differential equation

$$x'(t) = A(t)x(t) + f(t, x_t) \tag{1}$$

and the linear ordinary differential equation

$$y'(t) = A(t)y(t). \tag{2}$$

In equations (1) and (2), x, y and f are n-vectors; and $A(t)$ is a continuous $n \times n$ matrix that is defined on $J = [0, \infty)$. The symbol $\|\cdot\|$ will designate some convenient norm of a vector and its corresponding matrix norm. Let $C^n = C[[-\tau, 0], R^n]$, $\tau \geqq 0$, where the norm of an element $\phi \in C^n$ is defined by

$$\|\phi\| = \max_{-\tau \leqq s \leqq 0} \|\phi(s)\|.$$

It will be required that $f \in C[J \times C^n, R^n]$. For $x \in C[[-\tau, \infty), R^n]$, the symbol x_t is defined by the relation

$$x_t(s) = x(t + s), \qquad -\tau \leqq s \leqq 0, \qquad t \in J,$$

and is called the past history of x at t. The terminology and notation used in connection with the functional equation (1) may be found in [5, Chap. 6].

Our results establish that there is an asymptotic equivalence between equations (1) and (2) under some appropriate conditions.

The perturbation problems which we shall consider have analogues in ordinary differential equations; in fact, the motivation for our work is contained in the references [2], [3], [4]. When $\tau = 0$, system (1) is an ordinary differential equation; hence, the results obtained here are extensions of the corresponding results in the above papers. When $\tau \neq 0$, the perturbed equation (1) represents a mathematical model closer to physical reality than the corresponding perturbation without delay which was considered in [2], [3], [4].

2. Preliminary results. In this section, we shall give an apparently new comparison principle which relates the solutions of a system of functional equations

$$z'(t) = F(t, z_t) \tag{3}$$

* Received by the editors November 3, 1970, and in final revised form April 12, 1971.

† Department of Mathematics, University of Rhode Island, Kingston, Rhode Island. Now at Department of Mathematics, Florida State University, Tallahassee, Florida 32306. The work of this author was supported in part by the National Science Foundation under Grant GP-11543.

‡ Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881.

to the solutions of a scalar functional equation

$$u'(t) = g(t, u_t). \tag{4}$$

This is a more general setting than the usual comparison principle [5, p. 6] in which the comparison equation is an ordinary differential equation. In equations (3) and (4), it is assumed that $F \in C[J \times C^n, R^n]$ and $g \in C[J \times C_+, R_+]$, where $C_+ = \{u \in C | u \geqq 0\}$, $R_+ = \{r \in R | r \geqq 0\}$.

LEMMA. *Let $g(t, u)$ be monotone nondecreasing in u for each fixed $t \in J$; and*

$$\|F(t, \phi)\| \leqq g(t, \|\phi\|), \qquad (t, \phi) \in J \times C^n. \tag{5}$$

Furthermore, for a given function $u_0 \in C_+$, suppose that the maximal solution $r(t_0, u_0)$ of (4) with initial function u_0 at $t = t_0$ exists and is bounded on $[t_0, \infty)$. Then, any solution $z(t_0, \phi_0)$ of (3) with initial function ϕ_0 at $t = t_0$ satisfying $\|\phi_0\| \leqq u_0$ satisfies the inequality

$$\|z(t_0, \phi_0)(t)\| \leqq r(t_0, u_0)(t), \qquad t \geqq t_0.$$

Furthermore, there exists a constant vector ξ such that $\lim_{t \to \infty} z(t_0, \phi_0)(t) = \xi$; and, if

$$\|\phi_0(0)\| > r_\infty - u_0(0), \tag{6}$$

where $r_\infty = \lim_{t \to \infty} r(t_0, u_0)(t)$, then ξ is nonzero.

Proof. Define the function m by

$$m(t) = \|z(t_0, \phi_0)(t)\|, \qquad t \geqq t_0.$$

Then, the right-hand derivative of $m(t)$ exists and satisfies the inequalities

$$\begin{aligned} m'_+(t) &\leqq \|z'(t_0, \phi_0)(t)\| \\ &\leqq \|F(t, z_t(t_0, \phi_0))\| \\ &\leqq g(t, \|z_t(t_0, \phi_0)\|) \\ &\leqq g(t, m_t), \qquad t \geqq t_0. \end{aligned}$$

From the definition of m, it follows that $m_{t_0} \leqq u_0$. The first conclusion of the lemma now follows from Theorem 6.9.4 in [5].

An integration in (3) leads directly to

$$\xi = \phi_0(0) + \int_{t_0}^{\infty} F(s, z_s(t_0, \phi_0))\, ds. \tag{7}$$

To see that ξ is nonzero provided (6) is satisfied, we obtain, from (7), that

$$\begin{aligned} \|\xi\| &\geqq \|\phi_0(0)\| - \int_{t_0}^{\infty} \|F(s, z_s(t_0, \phi_0))\|\, ds \\ &\geqq \|\phi_0(0)\| - \int_{t_0}^{\infty} g(s, r_s(t_0, u_0))\, ds \\ &\geqq \|\phi_0(0)\| - r_\infty + u_0(0) > 0. \end{aligned}$$

3. Asymptotic equivalence. We shall use the comparison principle of the lemma to obtain an asymptotic correspondence between the solutions of (1) and (2). The following two hypotheses will be assumed throughout the remainder of the article. There exists a continuous $n \times n$ matrix, $\Delta = \Delta(t)$, defined on $[-\tau, \infty)$ such that

$$\|\Delta(t)Y(t)\| \leqq 1, \qquad t \geqq t_0, \tag{8}$$

where $Y(t)$ is the fundamental solution matrix of (2) with $Y(t_0) = I$. Let $g \in C[J \times C_+, R_+]$, $g(t, u)$ be monotone nondecreasing in u, $u \in C_+$, for each fixed $t \in J$, and

$$\|Y^{-1}(t)f(t, \phi)\| \leqq g(t, \|\Delta_t\phi\|), \qquad (t, \phi) \in J \times C^n. \tag{9}$$

THEOREM 1. *Let the conditions* (8) *and* (9) *be satisfied. Furthermore, suppose that the maximal solution* $r(t_0, u_0)$ *of* (4) *with initial function* u_0 *at* $t = t_0$ *exists and is bounded on* $[t_0, \infty)$. *Then, corresponding to each solution* $x(t_0, \phi_0)$ *of* (1) *with initial function* ϕ_0 *at* $t = t_0$ *satisfying* $\|\phi_0\| \leqq u_0$, *there exists a constant vector* ξ *such that*

$$\|\Delta(t)[x(t_0, \phi_0)(t) - Y(t)\xi]\| = o(1), \qquad t \to \infty. \tag{10}$$

Furthermore, if (6) *is satisfied, then* ξ *is nonzero.*

Proof. Let $x(t) = x(t_0, \phi_0)(t)$ and define $Y_{t_0} \equiv I_n$. The substitution $x = Yz$ transforms (1) to the system

$$z'(t) = Y^{-1}(t)f(t, Y_t z_t) \equiv F(t, z_t), \tag{11}$$

with $z_{t_0} = \phi_0$. From (8) and (9), we obtain

$$\|F(t, z_t)\| \leqq g(t, \|z_t\|).$$

An application of the lemma to (11) yields the existence of a constant vector ξ such that

$$\lim_{t \to \infty} z(t_0, \phi_0)(t) = \xi.$$

To verify that (10) holds we observe the inequality

$$\|\Delta(t)[x(t_0, \phi_0)(t) - Y(t)\xi]\| \leqq \|\Delta(t)Y(t)[z(t_0, \phi_0)(t) - \xi]\| = o(1), \qquad t \to \infty.$$

The fact that ξ is nonzero provided (6) is satisfied follows immediately from the last conclusion of the lemma.

Next, we shall consider a converse problem to the result of Theorem 1.

THEOREM 2. *Let the hypotheses of Theorem* 1 *be satisfied. Then, corresponding to each constant vector* ξ *which satisfies the inequality*

$$\|\xi\| < \lim_{t \to \infty} r(t_0, u_0)(t) = r_\infty,$$

there exists a solution $x = x(t)$ *of* (1) *which satisfies the asymptotic relationship* (10).

Proof. As shown in the proof of Theorem 1, it suffices to show that (11) has a solution $z = z(t)$ which is valid on some interval $[T_0, \infty)$ and satisfies $\lim_{t \to \infty} z(t) = \xi$.

Define the numbers η, γ by

$$\eta = [r_\infty - \|\xi\|]/2, \qquad \gamma = r_\infty - \eta.$$

Since $\gamma < r_\infty$, it follows from (4) that

$$\int^\infty g(t, \gamma)\, dt < \infty;$$

in particular, it is possible to choose $T_0 > 0$ such that

$$\int_{T_0}^\infty g(t, \gamma)\, dt < \eta.$$

Let the set F be defined by

$$F = \{z | z \in C[R, R^n]; \|z(t)\| \leqq \gamma, t \in R; z(t) \equiv \text{const. on } -\infty < t \leqq T_0\}.$$

Define the operator T on F by the equation

$$Tz(t) = \begin{cases} \xi - \int_t^\infty F(s, z_s)\, ds, & t \geqq T_0, \\ \xi - \int_{T_0}^\infty F(x, z_s)\, ds, & t < T_0. \end{cases}$$

The Tychonoff theorem will be used to establish that the mapping T has a fixed point in F. First, F is a compact convex subset of the Banach space $C[R, R^n]$ with the usual supremum norm. To see that $TF \subset F$, we note that for $t \in R$,

$$\begin{aligned} \|Tz(t)\| &\leqq \|\xi\| + \int_{T_0}^\infty \|F(s, z_s)\|\, ds \\ &\leqq \|\xi\| + \int_{T_0}^\infty g(s, \gamma)\, ds \\ &\leqq \|\xi\| + \eta = \gamma. \end{aligned}$$

Next, let the sequence $\{z^n\}_{n=1}^\infty$ converge uniformly to z on every compact subinterval of R, where z^n, z are in F, $n = 1, 2, \cdots$. Then, the sequence $\{z_t^n\}_{n=1}^\infty$ converges uniformly on compact t-subintervals of R. Suppose $\varepsilon > 0$ is given; select $T_1 > T_0$ such that

$$\int_{T_1}^\infty g(t, \gamma)\, dt < \varepsilon/4.$$

Choose $N = N(\varepsilon, T_0, T_1)$ so that if $n \geqq N_\varepsilon$,

$$\|F(t, z_t^n) - F(t, z_t)\| < \varepsilon/(2(T_1 - T_0)), \qquad t \in [T_0, T_1].$$

Then for $t \in R$, we have

$$\begin{aligned} \|Tz(t) - Tz^n(t)\| &\leqq \int_{T_0}^{T_1} \|F(s, z_s^n) - F(s, z_s)\| \, ds \\ &\quad + \int_{T_1}^{\infty} \|F(s, z_s^n)\| \, ds + \int_{T_1}^{\infty} \|F(s, z_s)\| \, ds \\ &< \varepsilon, \qquad n \geqq N. \end{aligned}$$

The above inequality shows that T is continuous on F. To see that the closure of TF is compact, we need only observe that TF is uniformly bounded and equicontinuous at each point of R.

Therefore by the Tychonoff fixed-point theorem, there exists a $z \in F$ such that

$$z(t) = \begin{cases} \xi - \int_t^{\infty} F(s, z_s) \, ds, & t \geqq T_0, \\ \xi - \int_{T_0}^{\infty} F(s, z_s) \, ds, & t \leqq T_0. \end{cases}$$

For $t \geqq T_0$, $z(t)$ satisfies (11); furthermore, $\lim_{t \to \infty} z(t) = \xi$. This completes the proof of the theorem.

Remark. The matrix Δ was not required to be nonsingular in either Theorem 1 or 2. This fact has been previously observed in the case of ordinary differential equations for Theorem 1, but not in regard to Theorem 2. The general technique used in the proof of Theorem 1 was the comparison principle; hence both Theorem 1 and its proof are extensions of one part of Theorem 1 of [4] which used an ordinary differential equation as a comparison equation. The Tychonoff fixed-point theorem, which was used in the proof of Theorem 2, has also been used for problems of this type. The ordinary differential equation analogue of Theorem 2 was obtained by this method in [3]. In that proof however the hypothesis that Δ be nonsingular was necessary.

4. An example. Many known results in ordinary differential equations have been obtained from the ordinary differential equation analogues of the above theorems by making a special choice of Δ. The references [2], [3] give several examples to illustrate this application.

We shall apply Theorem 1 to obtain a generalization of a well-known result [1, p. 114] in ordinary differential equations.[1] Consider the second order linear delay equation

$$v''(t) + p(t)v(t - \tau) = 0, \qquad t \geqq 0, \tag{12}$$

where $\tau \geqq 0$ and

$$\int^{\infty} t|p(t)| \, dt < \infty. \tag{13}$$

[1] The extension given here is probably known to many workers in the area; however, the authors are not aware of a reference which supports this statement.

Every solution $v = v(t)$ of (12) satisfies

$$\lim_{t \to \infty} \frac{v(t)}{t} = c \tag{14}$$

for some constant c. Furthermore, there exist solutions of (12) where the asymptotic constant c is nonzero.

In the usual manner, we write (12) as a system of equations of the form

$$x'(t) = Ax(t) + f(t, x_t), \tag{15}$$

where

$$x = \begin{pmatrix} v \\ v' \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f(t, \phi) = \begin{pmatrix} 0 \\ -p(t)\phi_1(-\tau) \end{pmatrix}, \quad \phi \in C^2.$$

By choosing

$$\Delta(t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(a singular matrix)},$$

then we have for $t \geqq 1$,

$$\| Y^{-1}(t) f(t, \phi) \| \leqq t|p(t)|, \qquad \|\phi(-\tau)\| \equiv g(t, \|\Delta_t \phi(-\tau)\|).$$

As the scalar comparison equation, we have

$$u'(t) = t|p(t)|u(t - \tau). \tag{16}$$

To see that all of the solutions of (16) are bounded, we note that an integration in (16) leads to

$$u(t) = u(t_0) + \int_{t_0}^{t} s|p(s)|u(s - \tau)\, ds.$$

Therefore,

$$|u(t)| \leqq K(t) \equiv |u(t_0)| + \int_{t_0}^{t} s|p(s)|\,|u(s - \tau)|\, ds.$$

Since K is monotone nondecreasing, we have

$$K'(t) \leqq t|p(t)|K(t), \qquad K(t_0) = |u(t_0)|.$$

An application of Gronwall's inequality shows that all solutions of (16) are bounded. Applying Theorem 1 to (15) we see that given any solution v of (12), there exists a constant c such that

$$v'(t) = c + o(1), \qquad t \to \infty.$$

This implies that (14) is true. An argument like that in [1, p. 115] may be used to show that c can be chosen to be nonzero.

If Δ is taken to be Y^{-1}, then the stronger hypothesis $\int^{\infty} t^2|p(t)|\, dt < \infty$ is required in the above argument to obtain the same conclusion. This illustrates an advantage of allowing Δ to be a singular matrix.

REFERENCES

[1] R. BELLMAN, *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.

[2] F. BRAUER AND J. S. W. WONG, *On the asymptotic behavior of perturbed linear systems*, J. Differential Equations, 6 (1969), pp. 142–153.

[3] T. G. HALLAM AND J. W. HEIDEL, *The asymptotic manifolds of a perturbed linear system of differential equations*, Trans. Amer. Math. Soc., 149 (1970), pp. 233–241.

[4] G. LADAS, V. LAKSHMIKANTHAM AND S. LEELA, *On the perturbability of the asymptotic manifold of a perturbed system of differential equations*, Proc. Amer. Math. Soc., 27 (1971), pp. 65–71.

[5] V. LAKSHMIKANTHAM AND S. LEELA, *Differential and Integral Inequalities, Theory and Applications*, vol. II, Academic Press, New York, 1969.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ANOTHER CHARACTERIZATION OF THE CLASSICAL ORTHOGONAL POLYNOMIALS*

W. A. AL-SALAM† AND T. S. CHIHARA‡

Abstract. The classical orthogonal polynomials of Jacobi, Laguerre and Hermite are characterized as the only orthogonal polynomials with a differentiation formula of the form

$$\pi(x)P_n'(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x), \qquad n \geqq 1,$$

where $\pi(x)$ is a polynomial. If "orthogonal polynomial" is used in the sense of "orthogonal with respect to a function of bounded variation," then the characterization remains valid if the Bessel polynomials are included in the classical family. This characterization also permits us to verify a conjecture of Karlin and Szegö.

1. The classical orthogonal polynomials of Jacobi, Laguerre and Hermite form a natural family in that, in addition to their frequent occurrences in similar problems of applied mathematics, they enjoy a number of similar properties which in turn characterize them.

For example, the classical polynomials are the only orthogonal polynomials (apart from those obtained by trivial transformations of the classical polynomials):

(i) which are the eigenfunctions of a second order Sturm–Liouville differential equation (Bochner [1]);

(ii) whose derivatives also form a sequence of orthogonal polynomials (Hahn [5]);

(iii) which have a Rodrigues-type formula. (This was first observed by Tricomi [9] whose proof however was incomplete. Complete proofs have been given recently by Ebert [3] and Cryer [2].)

Here we have been using the term "orthogonal polynomial" in the classical sense of orthogonal on the real line with respect to a nondecreasing real function. If however we use the term in the more general sense of orthogonal with respect to a function of bounded variation, then the above characterizations remain valid provided we (a) drop the usual restrictions on the parameters in the Laguerre and Jacobi polynomials and (b) include the generalized Bessel polynomials in the classical family. (This has been observed by Ebert (implicity) and Cryer for (iii) and implicitly by Bochner and Hahn in (i) and (ii), respectively.)

Another property common to the classical orthogonal polynomials (including the Bessel) is the existence of a differentiation formula of the form

$$(1) \qquad \pi(x)P_n'(x) = (\alpha_n x + \beta_n)P_n(x) + \gamma_n P_{n-1}(x),$$

where $\pi(x)$ is a polynomial. (For a unified derivation of (1), due to Tricomi, see [4, p. 167]. The Bessel polynomials are not explicitly included in this derivation but easily can be.)

* Received by the editors March 2, 1971, and in revised form May 21, 1971.

† Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada and Department of Mathematics, American University of Beirut, Beirut, Lebanon.

‡ Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

Now if $\{P_n(x)\}$ is a sequence of orthogonal polynomials (with respect to $d\varphi(x)$, where φ is of bounded variation) and if it satisfies (1), then

$$\begin{aligned} I_{mn} &\equiv \int_{-\infty}^{\infty} \pi(x)P_n'(x)P_m'(x)\,d\varphi(x) \\ &= \int_{-\infty}^{\infty} P_n(x)P_m'(x)(\alpha_n x + \beta_n)\,d\varphi(x) + \gamma_n \int_{-\infty}^{\infty} P_{n-1}(x)P_m'(x)\,d\varphi(x) \\ &= 0 \quad \text{if } m < n. \end{aligned}$$

Thus if it can be shown that $I_{nn} \neq 0$ $(n > 0)$, then $\{P_n'(x)\}$ is an orthogonal polynomial sequence with respect to $\pi(x)\,d\varphi(x)$. Then it would follow from Hahn's theorem that $P_n(x)$ is one of the classical orthogonal polynomials.

In case $\pi(x)$ is constant or linear, this is easily done but the quadratic case becomes involved. We therefore bypass this approach and give below a direct, elementary proof that (1) characterizes the classical orthogonal polynomials, thus obviating reference to Hahn's theorem.

Our characterization will also permit us to answer affirmatively a question raised by Karlin and Szegö [7]. This will be considered in § 4 together with related conjectures (the authors wish to thank the referee for calling these to our attention).

2. Let $\{P_n(x)\}$ be a sequence of monic orthogonal polynomials so that there is a recurrence formula,

$$\text{(2)} \qquad \begin{aligned} &P_{n+1}(x) = (x + B_n)P_n(x) - C_nP_{n-1}(x), \qquad n \geqq 0, \\ &P_{-1}(x) = 0, \quad P_0(x) = 1, \quad C_n \neq 0 \quad \text{for } n \geqq 1. \end{aligned}$$

Suppose further that $\{P_n(x)\}$ satisfies (1).

Comparison of coefficients in (1) shows that $\pi(x)$ is at most quadratic so write

$$\pi(x) = ax^2 + bx + c.$$

Clearly

$$\text{(3)} \qquad \alpha_n = na.$$

Now differentiate (2), multiply by $\pi(x)$ and use (1) to eliminate $\pi(x)P_k'(x)$, $k = n-1, n, n+1$. Then use (2) to eliminate $P_{n+1}(x)$. The result of all this is

$$\begin{aligned} &\{(\alpha_{n+1} - \alpha_n - a)x^2 + (\beta_{n+1} + B_n\alpha_{n+1} - \beta_n - B_n\alpha_n - b)x \\ &\qquad -c + B_n\beta_{n+1} + \gamma_{n+1} - B_n\beta_n\}P_n(x) \\ &= \{C_n[(\alpha_{n+1} - \alpha_{n-1})x + \beta_{n+1} - \beta_{n-1}] + \gamma_n(x + B_n)\}P_{n-1}(x) \\ &\qquad -C_n\gamma_{n-1}P_{n-2}(x), \qquad n \geqq 1. \end{aligned}$$

Using (3) and simplifying, we obtain

(4) $$\beta_{n+1} - \beta_n = b - aB_n,$$

(5) $$(\beta_{n+1}B_n - \beta_n B_n + \gamma_{n+1} - c)P_n(x) = \{(2aC_n + \gamma_n)x + C_n(\beta_{n+1} - \beta_{n-1}) + \gamma_n B_n\}P_{n-1}(x) - \gamma_{n-1}C_n P_{n-2}(x).$$

Examination of the derivation of (4) and (5) reveals that they remain valid for $n = 0$ if we define

$$\beta_0 = \gamma_0 = C_0 = 0$$

(and expressions with negative subscripts as finite but otherwise arbitrary).

Comparing (5) with (2), we thus find

(6) $$B_n(\beta_{n+1} - \beta_n) + \gamma_{n+1} - c = 2aC_n + \gamma_n,$$

(7) $$C_{n-1}(2aC_n + \gamma_n) = C_n\gamma_{n-1},$$

(8) $$C_n(\beta_{n+1} - \beta_{n-1}) + \gamma_n B_n = B_{n-1}(2aC_n + \gamma_n).$$

From (7) we obtain $\gamma_n/C_n = \gamma_{n-1}/C_{n-1} - 2a$, $n \geqq 2$, which gives

(9) $$\gamma_n = -(2an + d)C_n,$$

valid for $n \geqq 0$ with $d = -2a - \gamma_1/C_1$.

Substituting (9) into (6) and (8), then using (4), leads us to

(10) $$(2an + d + 2a)C_{n+1} - (2an + d - 2a)C_n = B_n(b - aB_n) - c,$$

(11) $$(2an + d + a)B_n - (2an + d - 3a)B_{n-1} = 2b, \qquad n \geqq 1.$$

We note that all formulas (4)–(10) inclusive remain valid for $n \geqq 0$ with $\beta_0 = \gamma_0 = C_0 = 0$.

3. We next consider separately the cases where $\pi(x)$ is constant, linear and quadratic.

Case I. *$\pi(x)$ is constant.* We can assume without loss of generality that $a = b = 0$ and $c = 1$. It then follows directly from (1) that $\alpha_n = \beta_n = 0$ so that $Q_n(x) = (\gamma_1\gamma_2 \cdots \gamma_n)^{-1}P_n(x)$ are Appell polynomials: $Q_n'(x) = Q_{n-1}(x)$. Now it is well known that the only Appell polynomials that are also orthogonal polynomials are essentially the Hermite polynomials.

However we can show directly that $P_n(x)$ is essentially a Hermite polynomial by using (11) to conclude that $B_n = B_0$ and then using (10) to find $C_n = -n/d$. Thus the recurrence formula becomes that satisfied by

$$P_n(x) = (2r)^{-n}H_n(rx + rB_0), \qquad r = (-d/2)^{1/2}.$$

Case II. *$\pi(x)$ is linear.* We can assume $a = c = 0$, $b = 1$. Then $\alpha_n = 0$, $\beta_n = n$ and $\gamma_n = -dC_n$.

From (11) we now obtain

$$dB_n = 2n + dB_0,$$

whence from (10),

$$d^2C_n = n(n-1) + dB_0 n.$$

The recurrence formula (2) thus shows that, in this case,

$$P_n(x) = d^{-n} n!\, L_n^{(\alpha)}(-dx), \qquad \alpha = dB_0 - 1.$$

Case III. $\pi(x)$ *is quadratic.* Without loss of generality, we take $a = 1$, $b = 0$ so that $\alpha_n = n$.

From (11) and (4) respectively we obtain

$$B_n = \frac{K}{(2n+d-1)(2n+d+1)}, \qquad K = (d^2-1)B_0, \tag{12}$$

$$\beta_n = \frac{-Kn}{(d-1)(2n+d-1)}.$$

Using these in (10) and multiplying by $2n + d$ gives

$$(2n+d+2)(2n+d)C_{n+1} - (2n+d)(2n+d-2)C_n$$
$$= -c(2n+d) - \frac{K^2(2n+d)}{(2n+d-1)^2(2n+d+1)^2}$$

which in turn yields

$$(2n+d+2)(2n+d)C_{n+1} = -c(n+1)(n+d) - \frac{K^2}{4}$$
$$\cdot [(d-1)^{-2} - (2n+d+1)^{-2}],$$

$$C_n = \frac{n(n+d-1)[K^2(d-1)^{-2} + c(2n+d-1)^2]}{(2n+d)(2n+d-1)^2(2n+d-2)}. \tag{13}$$

If $c = 0$, we then write $d = a - 1$ and $K = b(a-2)$ so that (12) and (13) become

$$B_n = \frac{b(a-2)}{(2n+a)(2n+a-2)}, \qquad C_n = \frac{-b^2 n(n+a-2)}{(2n+a-1)(2n+a-2)^2(2n+a-3)}.$$

It can now be verified from the recurrence formula for the generalized Bessel polynomial, $y_n(x, a, b)$ (Krall and Frink [8, (51)]), that

$$P_n(x) = \frac{b^n (a)_{n-1}}{2^n (a/2)_n} y_n(x, a, b).$$

Finally, if $c \neq 0$, write $d = \alpha + \beta + 1$ and $K = (\alpha^2 - \beta^2)(-c)^{1/2}$. Then

$$B_n = \frac{(\alpha^2-\beta^2)(-c)^{1/2}}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$

$$C_n = \frac{-4cn(n+\alpha+\beta)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)},$$

and this shows that we have Jacobi polynomials:

$$P_n(x) = (2s)^n \binom{2n + \alpha + \beta}{n}^{-1} P_n^{(\alpha,\beta)}(x/s), \qquad s = (-c)^{1/2}.$$

Thus the only orthogonal polynomials satisfying a relation of the form (1) are the classical polynomials together with the Bessel polynomials.

4. At the conclusion of their monumental paper on determinants whose elements are orthogonal polynomials, Karlin and Szegö [7, p. 156] suggest three conjectures concerning additional characterizations of the classical orthogonal polynomials. The first of these is the existence of a differentiation formula of the form

$$r(x)P_n'(x) = \mu_n[P_{n-1}(x) + C(x)P_n(x)]\rho(x), \qquad n \geqq 0, \tag{14}$$

where $C(x)$ is a polynomial. (The subscripts on the right side of (14) are miscopied on p. 156 of [7]—cf. their formula (24.2) on p. 100.)

Assuming that $P_0(x)$ and $P_1(x)$ are monic polynomials, we take $n = 1$ in (14) and obtain

$$r(x) = \mu_1[P_0(x) + C(x)P_1(x)]\rho(x).$$

Thus (14) becomes

$$\mu_1[1 + C(x)P_1(x)]P_n'(x) = \mu_n[P_{n-1}(x) + C(x)P_n(x)]. \tag{15}$$

Comparison of leading coefficients now shows that $\mu_n = n\mu_1$. Next consider (15) for $n = 2$ and rewrite this in the form

$$P_2'(x) - 2P_1(x) = [2P_2(x) - P_1(x)P_2'(x)]C(x).$$

The left side of this equation is of degree at most 1. On the other hand, $2P_2(x) - P_1(x)P_2'(x)$ cannot vanish identically unless $P_2(x) = kP_1^2(x)$ which is impossible if (2) is satisfied. Therefore it follows that $C(x)$ is of degree at most 1 so (15) is of the form (1) and $\{P_n(x)\}$ is classical or Bessel.

We are unable to add anything to the second conjecture in [7] but we note that the third had already been established earlier by Hahn [6].

A somewhat related question due to Askey (private communication) is: "What orthogonal polynomials have the property that their derivatives are quasi-orthogonal polynomials (here in the sense that $(P_m', P_n') = 0$ if $|m - n| > k$)?" Another related problem due to Askey is to characterize the orthogonal polynomials whose derivatives satisfy

$$\pi(x)P_n'(x) = \sum_{j=0}^{2k} \alpha_{nj}P_{n+k-j}(x), \qquad k \text{ independent of } n. \tag{16}$$

In view of the recurrence formula (2), our characterization (1) solves this problem for the case $k = 1$.

Added in proof. It should be noted that (14) does not provide a complete characterization of the classical orthogonal polynomials since (14) is not satisfied by the Jacobi polynomials except in the ultraspherical case.

REFERENCES

[1] S. BOCHNER, *Über Sturm-Liouvillesche Polynomsystem*, Math. Z., 29 (1929), pp. 730–736.

[2] C. W. CRYER, *Rodrigues' formula and the classical orthogonal polynomials*, Boll. Un. Mat. Ital., 25 (1970), pp. 1–11.

[3] R. EBERT, *Über Polynomsysteme mit Rodriguesscher Darstellung*, Cologne, 1964.

[4] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.

[5] W. HAHN, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Math. Z., 39 (1935), pp. 634–638.

[6] ———, *Über höhere Ableitungen von orthogonal Polynomen*, Ibid., 43 (1937), p. 101.

[7] S. KARLIN AND G. SZEGÖ, *On certain determinants whose elements are orthogonal polynomials*, J. Analyse Math., 8 (1961), pp. 1–157.

[8] H. L. KRALL AND O. FRINK, *A new class of orthogonal polynomials: the Bessel polynomials*, Trans. Amer. Math. Soc., 65 (1949), pp. 100–115.

[9] F. G. TRICOMI, *Serie ortogonali di funzioni*, Gheroni, Torino, 1948.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ERRATA: ON THE EVALUATION OF CERTAIN SUMS INVOLVING THE NATURAL NUMBERS RAISED TO AN ARBITRARY POWER*

KEITH B. OLDHAM†

On page 538, the relation

$$G_0 \equiv g(l - v + 1), g(l - v + 2), \cdots, g(l)$$

should be replaced by

$$G_c \equiv g(l - v + 1), g(l - v + 2), \cdots, g(l).$$

On page 539, the expression $G_{j-1}(G_c)$ on line 5 should be replaced by $C_{j-1}(G_c)$; in equation (5) the minus sign following the $\equiv$ sign should be deleted; in equation (5) *and* in the final (unnumbered) equation on the page a term $O(l^{-r-2m-1})$ should be added to the right members.

* This Journal, 1 (1970), pp. 538–546. Received by the editors April 10, 1971.

† Department of Chemistry, Trent University, Peterborough, Ontario, Canada.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

MULTIPLE SOLUTIONS OF SINGULAR PERTURBATION PROBLEMS*

DONALD S. COHEN†

Abstract. Under certain conditions on $g(x, u)$ we establish the existence and asymptotic behavior for small $\varepsilon > 0$ of *multiple* asymptotic solutions of the nonlinear boundary value problem

$$\varepsilon u'' + u' - g(x, u) = 0, \qquad 0 < x < 1,$$

$$u'(0) - au(0) = A \geqq 0, \qquad a > 0,$$

$$u'(1) + bu(1) = B > 0, \qquad b > 0.$$

Formal techniques of singular perturbation theory clearly reveal the mechanism which controls the appearance of multiple solutions. Their existence is then established rigorously by iteration schemes and the so-called "shooting method" for ordinary differential equations.

1. Introduction. We shall establish the existence and asymptotic behavior for small $\varepsilon > 0$ of multiple asymptotic solutions of the nonlinear boundary value problem

$$\varepsilon u'' + u' - g(x, u) = 0, \qquad 0 < x < 1, \tag{1.1}$$

$$u'(0) - au(0) = A \geqq 0, \qquad a > 0, \tag{1.2}$$

$$u'(1) + bu(1) = B > 0, \qquad b > 0. \tag{1.3}$$

In general, a function $u(x, \varepsilon)$ is said to be an asymptotic solution to order $O(\varepsilon^n)$ if the function satisfies the differential equation and boundary conditions to order $O(\varepsilon^n)$ as $\varepsilon \to 0$. More precisely, for this paper, we adopt the following definition.

DEFINITION. A function $u(x, \varepsilon)$ is an *asymptotic solution* of the boundary value problem (1.1)–(1.3) if $u(x, \varepsilon)$ satisfies (1.1), (1.2) and $u'(1, \varepsilon) + bu(1, \varepsilon) = B + O(\varepsilon)$ as $\varepsilon \to 0$.

Problems of this type occur in chemical reactor theory, and it has been found recently [1]–[3] that multiple *stable* steady states can occur in certain adiabatic tubular reactors. By considering the relevant physics in the various parts of the reactor, or equivalently by applying the formal techniques [4] of singular perturbation theory, the mechanism by which the multiple solutions occur is clearly revealed. We do this briefly in § 2, and this will provide us with useful insight regarding the properties of the equation and its solutions. The rest of the paper is devoted to rigorously establishing the existence and asymptotic behavior for small $\varepsilon > 0$ of the multiple asymptotic solutions of the nonlinear two-point boundary value problem (1.1)–(1.3).

* Received by the editors March 9, 1971, and in revised form May 24, 1971.

† Department of Applied Mathematics, California Institute of Technology, Pasadena, California 91109. This work was supported in part by the U.S. Army Research Office (Durham) under Contract DAHC 04-68-C-0006 and in part by the National Science Foundation under Grant GP-18471.

Our entire analysis is based on the so-called "shooting method" for ordinary differential equations. Accordingly, in § 3 we study the initial value problem

(1.4) $$\varepsilon u'' + u' - g(x,u) = 0, \qquad x > 0,$$

(1.5) $$u(0) = h \geqq 0,$$

(1.6) $$u'(0) = A + ah.$$

Note that the initial conditions (1.5), (1.6) imply that the boundary condition (1.2) is satisfied. Specific properties of $g(x, u)$ are stated, and we then prove that for all $\varepsilon > 0$ the initial value problem (1.4)–(1.6) possesses a unique solution $u(x, \varepsilon, h)$ and that $u(x, \varepsilon, h)$ and $u'(x, \varepsilon, h)$ depend continuously on h for $h \geqq 0$, and $u(x, \varepsilon, h)$ depends continuously on ε for sufficiently small $\varepsilon > 0$.

In § 4 we show that the boundary value problem (1.1)–(1.3) possesses many distinct (and we state precisely how many) asymptotic solutions. This is accomplished by demonstrating that there exist many distinct values of h such that for each of these values of h the solution $u(x, \varepsilon, h)$ of the initial value problem (1.4)–(1.6) also satisfies $u'(1, \varepsilon, h) + bu(1, \varepsilon, h) = B + O(\varepsilon)$ for sufficiently small $\varepsilon > 0$. Furthermore, we show that on the subinterval $0 < \delta \leqq x \leqq 1$ each asymptotic solution $u(x, \varepsilon)$ possesses the property that $u(x, \varepsilon) - v(x) = O(\varepsilon)$ and $u'(x, \varepsilon) - v'(x) = O(\varepsilon)$ for sufficiently small $\varepsilon > 0$, where $v(x)$ is the solution of an appropriate reduced problem (that is, the problem $v' - g(x, v) = 0$ subject to an appropriate boundary condition).

Our analysis and specific results are confined to the problem (1.1)–(1.3) for simplicity. However, our proofs and results can be extended to problems more general than (1.1). For example, it is relatively easy to extend our proofs to the case where we allow u' in (1.1) to have a positive nonlinear coefficient $f(x, u)$. Furthermore, with somewhat more work the results of the present paper taken together with those of [5] allow us to obtain quite similar results for equations of the form $\varepsilon u'' + f(x, u, u')u' - g(x, u) = 0$ for classes of f and g which occur in problems in fluid and gas dynamics.

2. Formal methods and multiple solutions. The reason for the existence of multiple solutions is clearly revealed by an application of the formal matching techniques of singular perturbation theory [4]. For $0 < \varepsilon \ll 1$ we find that there is a boundary layer of thickness $O(\varepsilon)$ near $x = 0$. Away from this boundary layer the first term of the asymptotic expansion (the outer expansion) is given by

(2.1) $$u' - g(x,u) = 0, \qquad 0 < x \leqq 1,$$

(2.2) $$u'(1) + bu(1) = B.$$

Evaluating (2.1) at $x = 1$, we find that (2.1) and (2.2) together imply that

(2.3) $$g(1, u(1)) = B - bu(1).$$

Clearly, the solutions of (2.3) provide the proper initial conditions for (2.1). Figure 1 illustrates a case where there are four roots $\alpha_i, i = 1, \cdots, 4$, of (2.3) for some nonlinearity $g = g(u)$ which is sketched.

Our formalism suggests that there are as many solutions for small $\varepsilon > 0$ as there are roots of (2.3) (later, we shall have to modify this slightly), and the first

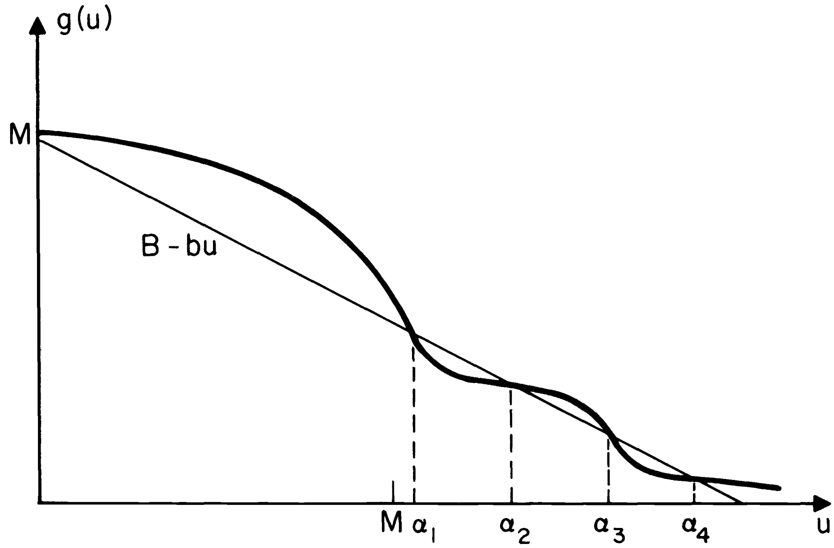

FIG. 1.

term in the outer expansion of each solution is given by

$$v' - g(x, v) = 0, \tag{2.4}$$

$$v(1) = \alpha_i. \tag{2.5}$$

In the boundary layer we introduce a new length $\tilde{x} = x/\varepsilon$ and let $u(x) \equiv u(\varepsilon\tilde{x}) \equiv w(\tilde{x})$. Then, the first term of the expansion (the inner expansion) near $x = 0$ is given by

$$w'' + w' = 0, \tag{2.6}$$

$$w'(0) - aw(0) = A, \tag{2.7}$$

$$w(\infty) = v(0). \tag{2.8}$$

The boundary condition (2.8) expresses the proper condition for matching the inner and outer solutions.

This procedure can be continued to generate succeeding terms in an asymptotic expansion, and from this procedure we could, in fact, construct an expansion which is uniformly valid on the interval $0 \leqq x \leqq 1$. Alternatively, we could employ a "two-timing" formalism to obtain the same answer. We shall not pursue this further, however, because the mechanism controlling the appearance of multiple solutions when ε is small is already clear. Quite simply, multiplicity is governed by the roots, $\alpha = \alpha_i$, of the equation

$$g(1, \alpha) = B - b\alpha. \tag{2.9}$$

Each root α_i of (2.9) gives rise to an appropriate "reduced problem" (2.4), (2.5), and as we shall see, each solution $v_i(x)$ of (2.4), (2.5) can be an asymptotic solution of (1.1)–(1.3) on any subinterval $0 < \delta \leqq x \leqq 1$ for sufficiently small $\varepsilon > 0$. (We shall also see that sufficiently small values of α_i may not generate an asymptotic solution.) We shall now proceed to give a rigorous investigation of the existence and multiplicity of asymptotic solutions of (1.1)–(1.3).

3. The shooting method. For all the work in §§ 3 and 4 the conditions imposed on g will be:

H.1 $g(x, u)$ is continuously differentiable in the region

$$R = \{(x,u)|0 \leqq x \leqq 1, u \geqq 0\}.$$

H.2: $g(x,u) \geqq 0$ on R.

H.3: $0 \leqq u_1 \leqq u_2$ implies that $g(x, u_1) \geqq g(x, u_2)$.

H.4: $g(x, u)$ satisfies a Lipschitz condition in R; that is, there exists a constant k such that for all $(x, u) \in R$,

$$|g(x,u) - g(x,v)| \leqq k|u - v|.$$

H.5: The equation $g(1,\alpha) = B - b\alpha$ possesses N roots α_i, $i = 1, \cdots, N$, such that $0 < \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_N$.

Conditions H.1 to H.4 imply that as a function of u for $u \geqq 0$ the nonlinearity $g(x, u)$ is a reasonably smooth, positive, Lipschitz continuous, nonincreasing function. Condition H.5 simply guarantees that there exists at least one root of $g(1, \alpha) = B - b\alpha$, and from the formalism of § 2 we suspect that for small positive $\varepsilon > 0$ a solution of (1.1)–(1.3) will not exist if a root α_1 does not exist. Note that the conditions H.1 to H.3 imply that $g(x, u)$ is uniformly bounded above on R. Thus, $g(x, u) \leqq M < \infty$ on R, and since g is positive and monotone nonincreasing in u, we can take

$$M = \max_{0 \leqq x \leqq 1} [g(x,0)].$$

For the rest of this paper M shall have this meaning. We wish to point out that these conditions are satisfied in many rate functions in chemical kinetics.

Write the differential equation (1.4) as $\varepsilon u'' + u' = g(x, u)$, and consider it as a first order equation in u' with initial condition $u'(0) = A + ah$. Then,

$$u'(x) = (A + ah)\, e^{-x/\varepsilon} + \frac{1}{\varepsilon}\int_0^x e^{-(x-t)/\varepsilon} g(t, u(t))\, dt. \tag{3.1}$$

Clearly, $u'(x) \geqq 0$ on $0 \leqq x \leqq 1$ if $u(x)$ exists on $0 \leqq x \leqq 1$. Integrating (3.1) and using the condition that $u(0) = h$, and performing an integration by parts, we obtain

$$u(x) = h + \varepsilon(A + ah)(1 - e^{-x/\varepsilon}) + \int_0^x [1 - e^{-(x-t)/\varepsilon}] g(t, u(t))\, dt. \tag{3.2}$$

For later convenience we shall write (3.1) and (3.2) respectively as

$$u'(x) = S[u], \qquad u(x) = T[u], \tag{3.3}$$

where the operators S and T are defined as

$$S[u] \equiv (A + ah)\, e^{-x/\varepsilon} + \frac{1}{\varepsilon}\int_0^x e^{-(x-t)/\varepsilon} g(t, u(t))\, dt, \tag{3.4}$$

$$T[u] = h + \varepsilon(A + ah)(1 - e^{-x/\varepsilon}) + \int_0^x [1 - e^{-(x-t)/\varepsilon}] g(t, u(t))\, dt. \tag{3.5}$$

The conditions H.2 and H.3 imply the following lemma which is basic for all of our results.

LEMMA 3.1. *Let y_1 and y_2 be continuously differentiable nonnegative functions of x defined on $0 \leqq x \leqq 1$. If $y_1(x) \leqq y_2(x)$, then $S[y_1] \geqq S[y_2]$ and $T[y_1] \geqq T[y_2]$.*

Define the sequences $\{u_n(x)\}$ and $\{u_n'(x)\}$ by

$$u_0(x) \equiv h, \qquad u_{n+1}(x) = T[u_n], \qquad n = 0, 1, 2, \cdots, \tag{3.6}$$

$$u_0'(x) \equiv 0, \qquad u_{n+1}'(x) = S[u_n], \qquad n = 0, 1, 2, \cdots. \tag{3.7}$$

Clearly, $u_1(x) \geqq u_0(x) \equiv h$, $u_2(x) \geqq u_0(x) \equiv h$, $u_1'(x) \geqq u_0'(x) \equiv 0$ and $u_2'(x) \geqq u_0'(x) \equiv 0$. These facts and Lemma 3.1 immediately imply the next lemma.

LEMMA 3.2.

$$\begin{aligned} &u_0 \leqq u_1, \quad u_1 \geqq u_2, \quad u_2 \leqq u_3, \quad u_3 \geqq u_4, \quad \cdots, \\ &u_0 \leqq u_2, \quad u_1 \geqq u_3, \quad u_2 \leqq u_4, \quad u_3 \geqq u_5, \quad \cdots, \\ &u_0' \leqq u_1', \quad u_1' \geqq u_2', \quad u_2' \leqq u_3', \quad u_3' \geqq u_4', \quad \cdots, \\ &u_0' \leqq u_2', \quad u_1' \geqq u_3', \quad u_2' \leqq u_4', \quad u_3' \geqq u_5', \quad \cdots; \end{aligned}$$

that is, for any positive integers k and l,

$$u_0 \leqq u_2 \leqq u_4 \leqq \cdots \leqq u_{2l} \leqq \cdots \leqq u_{2k+1} \leqq \cdots \leqq u_5 \leqq u_3 \leqq u_1, \tag{3.8}$$

and

$$u_0' \leqq u_2' \leqq u_4' \leqq \cdots \leqq u_{2l}' \leqq \cdots \leqq u_{2k+1}' \leqq \cdots \leqq u_5' \leqq u_3' \leqq u_1'. \tag{3.9}$$

That the alternating pincer movement (for fixed h) converges to the unique solution of (1.4)–(1.6) is the content of the following theorem.

THEOREM 3.3. *Let $g(x, u)$ satisfy* H.1 *to* H.4. *Then, for any $h \geqq 0$ the sequences $\{u_n(x)\}$ and $\{u_n'(x)\}$ defined by* (3.6) *and* (3.7) *converge respectively to the unique solution $u(x)$ of the initial value problem* (1.4)–(1.6) *and to its derivative $u'(x)$ on the interval* $0 \leqq x \leqq 1$.

Proof. First, we prove that for all $n \geqq 1$ we have

$$|u_n - u_{n-1}| \leqq \frac{(A + ah)k^n x^n}{n!} + \frac{M^n k^n x^n}{n!}, \tag{3.10}$$

$$|u_n' - u_{n-1}'| \leqq \frac{(A + ah)k^n x^{n-1}}{(n-1)!} + \frac{M^n k^n x^n}{(n-1)!}. \tag{3.11}$$

Here k is the Lipschitz constant of condition H.4, and M is the uniform upper bound on $g(x, u)$.

We now proceed by induction. Using the fact that $\varepsilon(1 - e^{-x/\varepsilon}) \leqq x$, we obtain

$$\begin{aligned} |u_1 - u_0| &= \varepsilon(A + ah)(1 - e^{-x/\varepsilon}) + \int_0^x [1 - e^{-(x-t)/\varepsilon}] g(t, u(t))\, dt \\ &\leqq (A + ah)x + Mx \leqq (A + ah)kx + Mkx. \end{aligned} \tag{3.12}$$

We have used the fact that $x^2/2 \leqq x$ on $0 \leqq x \leqq 1$, and we see also that we must take $k \geqq 1$. Similarly,

$$
\begin{aligned}
|u_1' - u_0'| &= (A + ah)\, e^{-x/\varepsilon} + \frac{1}{\varepsilon}\int_0^x e^{-(x-t)/\varepsilon} g(t, u(t))\, dt \\
&\leqq (A + ah) + \frac{M}{\varepsilon}\int_0^x e^{-(x-t)/\varepsilon}\, dt \qquad (3.13)\\
&\leqq (A + ah) + Mk.
\end{aligned}
$$

Hence, (3.10) and (3.11) are valid for $n = 1$. Now, assume that (3.10) and (3.11) are valid for all integers up to and including a given integer n. We must prove that they are valid for $n + 1$. Using H.4 and the induction hypotheses (3.10) and (3.11), we obtain

$$
\begin{aligned}
|u_{n+1} - u_n| &\leqq \int_0^x [1 - e^{-(x-t)/\varepsilon}]|g(t, u_n(t)) - g(t, u_{n-1}(t))|\, dt \\
&\leqq k\int_0^x |u_n(t) - u_{n-1}(t)|\, dt \\
&\leqq k\int_0^x \left[\frac{(A + ah)k^n t^n}{n!} + \frac{M^n k^n t^n}{n!}\right] dt \\
&= \frac{(A + ah)k^{n+1}x^{n+1}}{(n + 1)!} + \frac{M^{n+1}k^{n+1}x^{n+1}}{(n + 1)!}.
\end{aligned}
$$

To obtain the last line of the inequality we have used the facts that we can take $M \geqq 1$ and $x^{n+2}/(n + 2) \leqq 1$ on $0 \leqq x \leqq 1$. Similarly,

$$
|u_{n+1}' - u_n'| \leqq \frac{(A + ah)k^{n+1}x^n}{n!} + \frac{M^{n+1}k^{n+1}x^n}{n!}.
$$

Therefore, we have verified that (3.10) and (3.11) hold for all $n \geqq 1$. Now, write $u_n(x)$ as

$$
u_n(x) = u_0(x) + \sum_{j=1}^{n} [u_j(x) - u_{j-1}(x)] \qquad (3.14)
$$

with a similar formula for $u_n'(x)$. The estimates (3.10) and (3.11) immediately imply that in the limit as $n \to \infty$ the series in (3.14) converges absolutely and uniformly on the interval $0 \leqq x \leqq 1$. Consequently, the limit functions $u(x) = \lim_{n\to\infty} [u_n(x)]$ and $u'(x) = \lim_{n\to\infty} [u_n'(x)]$ exist and are continuous (since each $u_n(x)$ and $u_n'(x)$ is continuous), and it then follows in the usual manner that $u(x)$ is a solution of (1.4)–(1.6) on $0 \leqq x \leqq 1$ with derivative $u'(x)$.

We shall now prove the uniqueness of the solution $u(x)$. Suppose that $\tilde{u}(x)$ is another solution. Then, $\tilde{u}(x) \geqq u_0(x) \equiv h$, and hence,

$$
T[\tilde{u}] = \tilde{u} \leqq u_1 = T[u_0].
$$

In the same way we show that $u_{2n} \leqq \tilde{u} \leqq u_{2n+1}$. As we have just showed, the sequence $\{u_n(x)\}$ converges (i.e., the pincer closes). Then,

$$\tilde{u}(x) = \lim_{n \to \infty} [u_n(x)] = u(x).$$

This completes the proof.

We wish to note here for future use that the solution $u(x, \varepsilon, h)$ of the initial value problem (1.4)–(1.6), and its derivative, depend continuously on h for all $h \geqq 0$. This follows from the uniform convergence of the $\{u_n(x)\}$ which are clearly continuously differentiable in ε and h.

The preceding analysis was suggested by the classical paper of Hermann Weyl [6] who obtained a similar alternating process for the Blasius problem of fluid dynamics.

4. Multiple solutions and their asymptotic expansions. We shall now show that under the conditions H.1 to H.5 every root α_i of (2.9) can give rise to an asymptotic solution $u_i(x, \varepsilon)$ of the boundary value problem (1.1)–(1.3). Furthermore, we shall prove that corresponding to any α_i the asymptotic solution $u_i(x, \varepsilon)$ possesses the property that $u_i(x, \varepsilon) - v_i(x) = O(\varepsilon)$ and $u_i'(x, \varepsilon) - v_i'(x) = O(\varepsilon)$ as $\varepsilon \to 0$ uniformly on any subinterval $0 < \delta \leqq x \leqq 1$, where $v_i(x)$ is the solution of the reduced problem (2.4), (2.5).

In order to prove the existence of multiple asymptotic solutions of the boundary value problem (1.1)–(1.3) for sufficiently small $\varepsilon > 0$ we shall need the following lemmas.

LEMMA 4.1. *If* $|u'(x, \varepsilon, h)| < C$ *for sufficiently small* $\varepsilon > 0$, *where* C *is independent of* ε, *then for any* $x \in (0, 1]$ *we have*

$$\int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} g(t, u(t, \varepsilon, h))\, dt - g(x, u(x, \varepsilon, h)) = O(\varepsilon) \tag{4.1}$$

for sufficiently small $\varepsilon > 0$, *where* $u(x, \varepsilon, h)$, *for fixed* $h \geqq 0$, *is the unique solution of the initial value problem* (1.4)–(1.6).

Proof. First, note that

$$\int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon}\, dt = 1 - e^{-x/\varepsilon}.$$

Then,

$$\left| \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} g(t, u(t, \varepsilon, h))\, dt - g(x, u(x, \varepsilon, h)) \right|$$

$$= \left| \frac{1}{(1 - e^{-x/\varepsilon})} \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} [(1 - e^{-x/\varepsilon}) g(t, u(t, \varepsilon, h)) - g(x, u(x, \varepsilon, h))]\, dt \right|$$

$$\leqq \frac{1}{(1 - e^{-x/\varepsilon})} \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} |g(t, u(t, \varepsilon, h)) - g(x, u(x, \varepsilon, h))|\, dt$$

$$+ \frac{e^{-x/\varepsilon}}{(1 - e^{-x/\varepsilon})} \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} g(t, u(t, \varepsilon, h))\, dt$$

$$\leqq \frac{\max |dg/dt|}{(1 - e^{-x/\varepsilon})} \int_0^x \frac{1}{\varepsilon} e^{-(x-t)/\varepsilon} (x - t)\, dt + e^{-x/\varepsilon} \max |g|$$

$$= \frac{\max |dg/dt|}{(1 - e^{-x/\varepsilon})} [e^{-x/\varepsilon}(-x - \varepsilon) + \varepsilon] + e^{-x/\varepsilon} \max |g|.$$

Here we have used the mean value theorem and the facts that g and $dg/dt = g_t + g_u u'$ are bounded. The lemma now follows.

As an immediate consequence of applying Lemma 4.1 to (3.1) we obtain the following lemma.

LEMMA 4.2. *If* $|u'(x, \varepsilon, h)| < C$ *for sufficiently small* $\varepsilon > 0$, *where* C *is independent of* ε, *then for all* $h \geqq 0$ *the solution* $u(x, \varepsilon, h)$ *of the initial value problem* (1.4)–(1.6) *satisfies*

$$u'(x, \varepsilon, h) - g(x, u(x, \varepsilon, h)) = O(\varepsilon) \tag{4.2}$$

for sufficiently small $\varepsilon > 0$ *on any subinterval* $0 < \delta \leqq x \leqq 1$.

Now, define J as the number of roots α_i of $g(1, \alpha) = B - b\alpha$ which exceed the quantity $M + O(\varepsilon)$ for sufficiently small $\varepsilon > 0$. For example, $J = 4$ for the situation illustrated in Fig. 1, and $J = 3$ for the situation illustrated in Fig. 2.

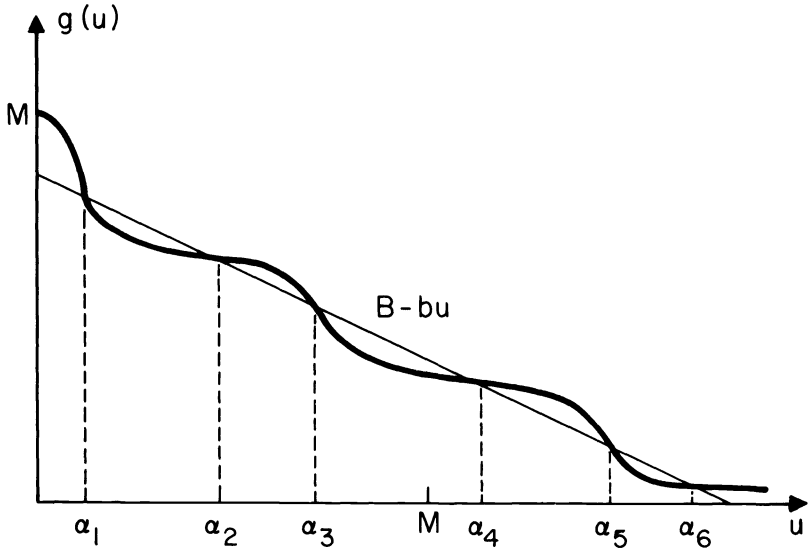

FIG. 2

We shall now prove that there exist J asymptotic solutions of the boundary value problem (1.1)–(1.3) and that on any subinterval $0 < \delta \leqq x \leqq 1$ each asymptotic solution and its derivative is asymptotic to the solution and its derivative of the reduced problem (2.4), (2.5). (Here we are assuming that $J \geqq 1$. Later we shall discuss the situation where roots α_i of $g(1, \alpha) = B - b\alpha$ exist but where $J = 0$.)

THEOREM 4.3. *Let* $g(x, u)$ *satisfy* H.1 *to* H.5. *Let* $v_i(x)$, $i = N - J + 1, \cdots, N$, *denote the solution on* $0 \leqq x \leqq 1$ *of the reduced problem*

$$v' - g(x, v) = 0, \tag{4.3}$$

$$v(1) = \alpha_i, \tag{4.4}$$

where α_i, $i = N - J + 1, \cdots, N$, *are the* J *roots of* $g(1, \alpha) = B - b\alpha$ *which exceed the quantity* $M + O(\varepsilon)$ *for sufficiently small* $\varepsilon > 0$. *Then, for all sufficiently small* $\varepsilon > 0$ *there exist* J *asymptotic solutions* $u_i(x, \varepsilon)$, $i = N - J + 1, \cdots, N$, *of* (1.1)–

(1.3) *such that* $u_i(x, \varepsilon) - v_i(x) = O(\varepsilon)$ *and* $u_i'(x, \varepsilon) - v_i'(x) = O(\varepsilon)$ *as* $\varepsilon \to 0$ *uniformly on any subinterval* $0 < \delta \leqq x \leqq 1$.

Proof. First, we prove that there exist J asymptotic solutions. Now consider the solution $u(x, \varepsilon, h)$ of the initial value problem (1.4)–(1.6). Equation (3.2) implies that for any $\varepsilon > 0$ we can choose h so large that $u(1, \varepsilon, h)$ is arbitrarily large. Furthermore, $g(x, u) \leqq M$ on R and (3.2) imply that

$$u(1, \varepsilon, h) < h + \varepsilon(A + ah)\, e^{-1/\varepsilon} + M(1 + \varepsilon + \varepsilon\, e^{-1/\varepsilon}).$$

Thus, for sufficiently small $\varepsilon > 0, u(1, \varepsilon, h) = M + h + O(\varepsilon)$. Hence, for sufficiently small $\varepsilon > 0$, $u(1, \varepsilon, h)$ varies continuously from $M + O(\varepsilon)$ to infinity as h varies from 0 to infinity. Therefore, $u(1, \varepsilon, h)$ takes on the values α_i, $i = N - J + 1, \cdots, N$, as h varies. Now, let $h_i = h_i(\varepsilon)$ denote the value of h for which $u(1, \varepsilon, h)$ takes on the value α_i; that is, $u(1, \varepsilon, h_i(\varepsilon)) = \alpha_i$ for sufficiently small $\varepsilon > 0$. Then,

$$g(1, u(1, \varepsilon, h_i(\varepsilon))) = B - bu(1, \varepsilon, h_i(\varepsilon)) \tag{4.5}$$

for sufficiently small $\varepsilon > 0$. If we can show that $|u'(1, \varepsilon, h_i(\varepsilon))| < C$ for sufficiently small $\varepsilon > 0$ where C is independent of ε, then Lemma 4.2 and (4.5) imply that $u'(1, \varepsilon, h_i) + bu(1, \varepsilon, h_i) = B + O(\varepsilon)$ for sufficiently small $\varepsilon > 0$. Therefore, if $|u'(1, \varepsilon, h_i(\varepsilon))| < C$ for sufficiently small $\varepsilon > 0$, then for each root $\alpha_i, i = N - J + 1, \cdots, N$, of $g(1, \alpha) = B - b\alpha$ there exists an $h_i = h_i(\varepsilon)$ such that corresponding to that value of h_i there exists an asymptotic solution $u_i(x, \varepsilon)$ of the boundary value (1.1)–(1.3). Therefore, if $|u'(1, \varepsilon, h_i(\varepsilon))| < C$ for sufficiently small $\varepsilon > 0$, then there exist J asymptotic solutions. To show that $|u'(1, \varepsilon, h_i(\varepsilon))| < C$ for sufficiently small $\varepsilon > 0$ note that (3.2) implies that

$$\begin{aligned} u(1, \varepsilon, h_i(\varepsilon)) = \alpha_i = h_i(\varepsilon) + \varepsilon(A + ah_i(\varepsilon))(1 - e^{-1/\varepsilon}) \\ + \int_0^1 [1 - e^{-(1-t)/\varepsilon}] g(t, u(t, \varepsilon, h_i(\varepsilon)))\, dt. \end{aligned} \tag{4.6}$$

Since all terms on the right of (4.6) are positive, then $h_i(\varepsilon) < \alpha_i$. That is, $h_i(\varepsilon) = O(1)$ as $\varepsilon \to 0$. From this together with $|g(x, u)| < M$ we conclude from (3.1) that $u'(1, \varepsilon, h_i(\varepsilon))$ is bounded independent of ε for sufficiently small $\varepsilon > 0$, and therefore there exist J asymptotic solutions. Note that in a similar way it follows that $|u'(x, \varepsilon, h_i(\varepsilon))| < c$ for all $x \in (0, 1]$.

Lemma 4.2 and the preceding paragraph imply that each asymptotic solution $u_i(x, \varepsilon)$ satisfies

$$u_i' - g(x, u_i) = O(\varepsilon),$$

$$u_i(1, \varepsilon) = \alpha_i,$$

for sufficiently small $\varepsilon > 0$ on $0 < \delta \leqq x \leqq 1$. Let $v_i(x)$ denote the solution of (4.3), (4.4) on $0 < \delta \leqq x \leqq 1$. Then, standard theorems on ordinary differential equations (for example, Theorem 5 of W. Hurewicz [7, p. 9]) immediately imply that $u_i(x, \varepsilon) - v_i(x) = O(\varepsilon)$ and $u_i'(x, \varepsilon) - v_i'(x) = O(\varepsilon)$ as $\varepsilon \to 0$ uniformly as $\varepsilon \to 0$ on $0 < \delta \leqq x \leqq 1$. This completes the proof.

It is clear that there exist functions $g(x, u)$ satisfying H.1 to H.5 such that roots α_i exist but $J = 0$. Our Theorem 4.3 does not apply here, and we can draw no conclusions as to whether or not the solutions of (4.3), (4.4) approximate solutions of (1.1)–(1.3) on $0 < \delta \leqq x \leqq 1$ for sufficiently small $\varepsilon > 0$. The formal matching techniques of singular perturbation theory [4] indicate that a root α_1 may exist but that the corresponding solution of (4.3), (4.4) is *not* an approximate solution of (1.1)–(1.3) on $0 < \delta \leqq x \leqq 1$ for sufficiently small $\varepsilon > 0$. There are other situations for which such solutions may not exist. For example, let $y = u'$ and write (1.1) as

$$\frac{dy}{du} = \frac{g(x, u) - y}{\varepsilon y}. \tag{4.7}$$

Figure 3 represents a sketch of the phase-plane trajectories corresponding to (4.7) for small $\varepsilon > 0$ for the same function g used in Fig. 1. A necessary condition for the existence of a solution of (1.1)–(1.3) is that a trajectory intersect both the

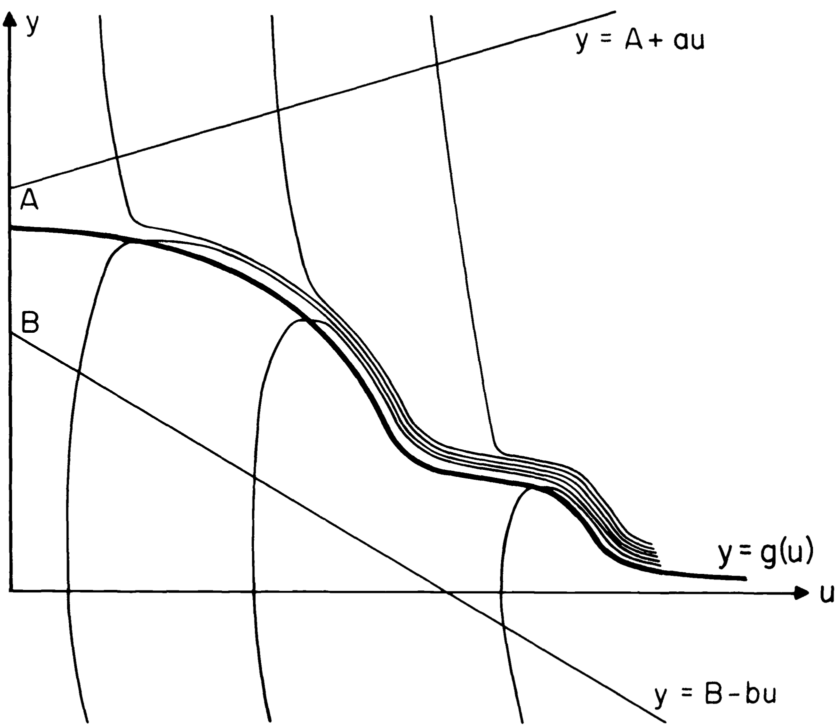

FIG. 3

lines $y = A + au$ and $y = B - bu$. The situation depicted in Fig. 3 represents a case in which no such trajectory exists. It seems reasonable to expect that under such conditions a solution of (1.1)–(1.3) will not exist. Thus, for example, if $B - bu < g(x, u)$ for all u and $A > M = \max_{0 \leqq x \leqq 1} [g(x, 0)]$, we expect that (1.1)–(1.3) has no solution for sufficiently small $\varepsilon > 0$.

Acknowledgment. The author wishes to express his appreciation to Professor Herbert B. Keller for several discussions and helpful suggestions during the course of this work.

REFERENCES

[1] D. S. COHEN, *Multiple stable solutions of nonlinear boundary value problems arising in chemical reactor theory*, SIAM J. Appl. Math., 20 (1971), pp. 1–13.

[2] R. ARIS, *On stability criteria of chemical reaction engineering*, Chem. Engrg. Sci., 24 (1968), pp. 149–169.

[3] D. LUSS, *Sufficient conditions for uniqueness of the steady state solutions in distributed parameter systems*, Ibid., 23 (1968), pp. 1249–1255.

[4] J. D. COLE, *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham, Mass., 1968.

[5] D. S. COHEN, *Singular perturbation of nonlinear two-point boundary value problems*, to appear.

[6] H. WEYL, *On the differential equations of the simplest boundary-layer problems*, Ann. of Math., 43 (1942), pp. 381–407.

[7] W. HUREWICZ, *Lectures on Ordinary Differential Equations*, M.I.T. Press and John Wiley, New York, 1958.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

DECOMPOSITION OF AN INTEGRAL OPERATOR BY USE OF MIKUSIŃSKI CALCULUS*

R. G. BUSCHMAN†

Abstract. Recently T. R. Prabhakar used fractional integrals in order to obtain explicit solutions to a convolution integral equation in which the kernel involved a confluent hypergeometric function. Decomposition of the integral operator into fractional integrals and exponential functions plays a role in the development and, following the ideas of A. Erdélyi, this decomposition is treated here in a clearer format from the standpoint of Mikusiński operators. Further, the conditions for existence and uniqueness of the solution are conveniently displayed.

In a recent paper by T. R. Prabhakar [4] fractional integrals are used in order to obtain explicit solutions of a convolution integral equation of the form

$$K_{a,b}f(t) = \int_0^t K(a; b; \lambda(t-u))f(u)\,du = g(t)$$

in which the kernel is of the form

$$K(a; b; \lambda t) = [t^{b-1}/\Gamma(b)]\,{}_1F_1(a; b; \lambda t)$$

and ${}_1F_1$ denotes the confluent hypergeometric function with $\operatorname{Re} b > 0$. Decomposition of the operator $K_{a,b}$ into a product of simpler operators plays a role in the development. Following the ideas of A. Erdélyi [2], this equation can be studied from the standpoint of Mikusiński calculus and the decompositions can be presented in a clearer format.

The following correspondences to Mikusiński operators are needed:

$$K(a; b; \lambda t) \leftrightarrow s^{a-b}(s-\lambda)^{-a}, \quad \operatorname{Re} b > 0, \quad \operatorname{Re} \lambda > 0;$$

$$[t^{\mu-1}e^{\lambda t}/\Gamma(\mu)] \leftrightarrow (s-\lambda)^{-\mu}, \quad \operatorname{Re}\mu > 0.$$

In view of the development by Erdélyi [2], these can be obtained from tables of Laplace transforms [3]. Since by Kummer's transformation,

$${}_1F_1(a; b; -\lambda t) = e^{-\lambda t}\,{}_1F_1(b-a; b; \lambda t),$$

we see that the first relation also holds for $\operatorname{Re}\lambda < 0$. For $\lambda = 0$ in the second correspondence we obtain $I^\mu \leftrightarrow s^{-\mu}$, where I^μ denotes the Riemann–Liouville fractional integral operator of order μ for $\operatorname{Re}\mu > 0$. This can be extended to $\operatorname{Re}\mu \leqq 0$ in the usual manner by choosing a positive integer n such that $\operatorname{Re}(n+\mu) > 0$ and letting $I^\mu = I^{-n}I^{\mu+n}$ in which I^{-n} denotes a differentiation operator such that the property $I^{-n}f = s^n f$ is retained.

We note that

$$\int_0^t \left[\frac{(t-u)^{a-1}e^{\lambda(t-u)}}{\Gamma(a)}\right] f(u)\,du \leftrightarrow (s-\lambda)^{-a}f.$$

* Received by the editors April 26, 1971.

† Department of Mathematics, University of Wyoming, Laramie, Wyoming 82070.

From this and from the other correspondences,

$$K_{a,b} \leftrightarrow s^{a-b}(s-\lambda)^{-a} \leftrightarrow I^{b-a}\, e^{\lambda t} I^{a}\, e^{-\lambda t},$$

and we have obtained the decomposition of $K_{a,b}$ into a product of fractional integral operators and exponential multipliers. This is the result of Lemma 9.1 of Prabhakar [4] which is used to solve the given integral equation in Theorem 9. Looking at the expression in s we can see that other decompositions are available, but this certainly seems to lead to the simplest. Another interesting result, Theorem 5 of [4], states that, if $f \in L^1$, then

$$I^{-b}K_{a,b}f(x) = f(x) + a\int_0^x {}_1F_1(a+1;2;x-t)f(t)\,dt.$$

This representation can be obtained by the following manipulations with Mikusiński operators, in which we use the formula $sf = f' + f(0)$. Consider

$$\begin{aligned} I^{-b}K_{a,b}f &\leftrightarrow ss^{a-1}(s-\lambda)^{-a}f \\ &= s\{{}_1F_1(a;1;\lambda t)\}f \\ &= [D\{{}_1F_1(a;1;\lambda t)\} + {}_1F_1(0)]f \\ &= \{a\lambda{}_1F_1(a+1;2;\lambda t)\}f + f, \end{aligned}$$

which gives us the result. We also note the correspondences which are related to Theorem 3 of [4];

$$I^{\mu}K_{a,b} \leftrightarrow s^{a-b-\mu}(s-\lambda)^{-a} \leftrightarrow K_{a,b+\mu}.$$

In terms of Mikusiński operators the integral equation becomes the algebraic equation

$$s^{a-b}(s-\lambda)^{-a}f = g$$

which has the operator solution

$$\begin{aligned} f &= s^{b-a}(s-\lambda)^{a}g \\ &= s^{b}g + [(1-\lambda/s)^{a} - 1]s^{b}g. \end{aligned}$$

The second form is useful for the investigation of the conditions under which a locally integrable solution exists. First we note that the expression in brackets is actually an integration operator so that it suffices that $s^b g$ correspond to a locally integrable function. Using ideas from [1] and [2], if we write

$$s^b g = s^{-(k-b)}(s^k g),$$

where k is the least integer such that $k > \operatorname{Re} b$, we know that if g has a locally integrable derivative of order k and $g^{(m)}(0) = 0$ for $0 \leqq m \leqq k-1$, then a locally integrable solution exists. A similar discussion holds for continuous solutions, L^1-solutions, etc. This method is directly analogous to that used by Jet Wimp [5]. The uniqueness of the solution follows from the uniqueness of the operator solution in the field of operators as in [2]; of course, this means uniqueness among the appropriate equivalence classes which are the elements of the particular space of functions under discussion.

Other similar types of equations and other decompositions of the associated integral operators can be treated in an analogous manner using the field of Mikusiński operators. The hypergeometric function of several variables Φ_2, which can be considered a generalization of ${}_1F_1$, corresponds to a rather simple expression involving the operator s, as can be seen from formula 4.24(5) of the tables [3]. It thus presents us with an example which can be treated similarly. With the easy general technique available, we omit the details of numerous special cases.

REFERENCES

[1] R. G. Buschman, *Convolution equations with generalized Laguerre polynomial kernels*, SIAM Rev., 6 (1964), pp. 166–167.

[2] A. Erdélyi, *Operational Calculus and Generalized Functions*, Holt, Rinehart and Winston, New York, 1962.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms*, vol. I, McGraw-Hill, New York, 1954.

[4] T. R. Prabhakar, *Two singular integral equations involving confluent hypergeometric functions*, Proc. Cambridge Philos. Soc., 66 (1969), pp. 71–89.

[5] Jet Wimp, *Two integral transform pairs involving hypergeometric functions*, Proc. Glasgow Math. Assoc., 7 (1965), pp. 42–44.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

EXISTENCE THEORY FOR MULTIPLE SOLUTIONS OF A SINGULAR PERTURBATION PROBLEM*

HERBERT B. KELLER†

Abstract. Nonlinear two-point boundary value problems of the form $\varepsilon y'' + y' = g(x, y)$, $0 \leqq x \leqq 1$; $y'(0) - ay(0) = A$; $f(y(1), y'(1)) = 0$ are studied. Under modest assumptions on $g(x, y)$ and $f(y, z)$ it is shown that for each simple root α_j of $F(\alpha) \equiv f(\alpha, g(1, \alpha)) = 0$, which lies in an appropriate interval, the boundary value problem has a distinct solution $y_j(x, \varepsilon)$ for all ε in $0 < \varepsilon \leqslant \varepsilon_0$. Furthermore the solutions converge uniformly on $[0, 1]$ to an appropriate solution of the reduced problem $v' = g(x, v)$, $f(v(1), v'(1)) = 0$ as $\varepsilon \downarrow 0$. As equilibrium states of a diffusion process the $y_j(x, \varepsilon)$ are stable or unstable provided $f_z(\alpha_j, g(1, \alpha_j))F'(\alpha_j) > 0$ or <0, respectively. This latter result is not demonstrated here. These problems are suggested by and have relevance to the theory of tubular chemical reactors.

1. Introduction. In [1] D. S. Cohen has shown that the singular perturbation problem

$$\varepsilon y'' + y' = g(x, y), \qquad 0 < x < 1; \tag{1.1}$$

$$y'(0) - ay(0) = A; \tag{1.2}$$

$$y'(1) + by(1) = B \tag{1.3}$$

can have several distinct "asymptotic solutions" for all sufficiently small $\varepsilon > 0$. These are functions $y(x, \varepsilon)$ which satisfy (1.1) and (1.2) exactly but only satisfy (1.3) to within $O(\varepsilon)$. We shall complete and extend Cohen's results in several ways. First the conditions on $g(x, y)$ are weakened in two essential ways: (i) by imposing smoothness requirements only on a finite y-interval rather than the half-line $y \geqq 0$; (ii) by eliminating the monotonicity condition $g_y(x, y) \leqq 0$. Under these modified conditions we show, using essentially techniques already employed in [1], that (1.1)–(1.3) can have several exact solutions $y = y_j(x, \varepsilon)$. These solutions are continuous and even continuously differentiable with respect to ε on $0 < \varepsilon \leqq \varepsilon_0$ for sufficiently small ε_0. As $\varepsilon \downarrow 0$ each $y_j(x, \varepsilon)$ converges uniformly on $0 \leqq x \leqq 1$ to an appropriate "outer solution" of singular perturbation theory. Our analysis furthermore allows us to establish all of these results when (1.3) is replaced by very general nonlinear boundary conditions of the form

$$f(y(1), y'(1)) = 0. \tag{1.4}$$

A number of other extensions of (1.1) were mentioned in [1] and of course the present analysis carries over for these problems too. In particular we point out that the present analysis applies (with but a sign change and an interchange of the treatment of the endpoints) to establish the existence of all three solutions conjectured by Cohen in [2]. Previously only the two "stable" solutions were shown to exist. In fact the proofs employed in [2] could have been applied in [1] to get the existence of an exact solution corresponding to essentially every other asymptotic solution. This has also been observed independently by S. V. Parter

* Received by the editors May 13, 1971, and in revised form June 1, 1971.

† Department of Applied Mathematics, California Institute of Technology, Pasadena, California 91109. This work was supported by the U.S. Army Research Office (Durham) under Contract DAHC 04-68-C-0007.

(private communication). However it has recently been shown by Sattinger [5] that only "stable" equilibrium states (of unsteady parabolic problems) are obtained by these monotone iteration methods. In fact the analysis in [5] can be applied almost without change to determine the stable solutions of (1.1), (1.2), (1.3). For nonlinear boundary conditions of the form (1.4) the analysis must be modified, as is already done in [4], and the stability can again be studied under some additional conditions on $f(y, z)$ that we do not require here.

We first reformulate the hypothesis of [1], and impose conditions on $g(x, y)$ only over some finite rectangle $R_Y \equiv \{x, y | 0 \leqq x \leqq 1, 0 \leqq y \leqq Y\}$ as follows:

H1: $g(x, y) \in \mathbb{C}_1(R_Y)$;

H2: $0 \leqq g(x, y) \leqq M$ on R_Y;

H3: (a) $f(y, z) \in \mathbb{C}_1([0, Y] \times [0, 2M + N])$;
(b) $F(\alpha) \equiv f(\alpha, g(1, \alpha)) = 0$ has J simple roots α_j in $M < \alpha_1 < \alpha_2 < \cdots < \alpha_J < N$;

H4: $a \geqq 0, A \geqq 0, Y \leqq 2(M + N)$.

From H1 it follows that there is some constant, say

(1.5a) $$K = \max_{R_Y} |g_y(x, y)|,$$

such that

(1.5b) $$|g(x, y) - g(x, \tilde{y})| \leqq K|y - \tilde{y}| \quad \text{for all } (x, y) \text{ and } (x, \tilde{y}) \in R_Y.$$

(This Lipschitz condition had to be assumed in [1] since R_∞ was used rather than R_Y.)

2. Initial value problems and outer solutions. To establish asymptotic solutions of (1.1), (1.2) and (1.4) as well as exact solutions we use the initial value problem

(2.1a) $$\varepsilon u'' + u' = g(x, u);$$

(2.1b) $$u(0) = h, \qquad u'(0) = A + ah.$$

By integrating (2.1a) over $[0, x]$, multiplying by the integrating factor $e^{x/\varepsilon}$, integrating again over $[0, x]$ and performing an integration by parts, we formally obtain the integral equation

(2.2a) $$u(x) = \varphi_0(x) + T_0[u(x)];$$

(2.2b) $$\varphi_0(x) \equiv h + (A + ah)\varepsilon[1 - e^{-x/\varepsilon}],$$
$$T_0[u(x)] \equiv \int_0^x [1 - e^{(t-x)/\varepsilon}] g(t, u(t))\, dt.$$

If $u(x)$ is a twice continuously differentiable solution of (2.1) for which $(x, u(x)) \in R_Y$ for $0 \leqq x \leqq 1$, then the indicated derivation of (2.2) is valid and $u(x)$ is also a solution of the integral equation on $0 \leqq x \leqq 1$. Conversely, if $u(x)$ is a continuous solution of (2.2) whose graph remains in R_Y, then it has two continuous derivatives and, by differentiation, is found to be a solution of (2.1). Thus (2.1) and (2.2) are equivalent, and we study the existence theory by means of the integral equation.

THEOREM 2.3. *Let* H1, H2 *and* H4 *hold. Then for each* (ε, h) *in*

$$0 < \varepsilon \leqq \varepsilon_0 \equiv \min(1/a, M/A); \qquad 0 \leqq h \leqq N; \tag{2.4}$$

the initial value problem (2.1) *has a unique solution* $u = u(x; \varepsilon, h)$ *which exists on* $0 \leqq x \leqq 1$ *and satisfies* $0 \leqq u(x; \varepsilon, h) \leqq Y, u'(x; \varepsilon, h) \geqq 0$. *Further* $u(x; \varepsilon, h)$ *and* $u'(x; \varepsilon, h)$ *are continuously differentiable with respect to* (ε, h) *on* (2.4) *for all* x *in* $[0, 1]$.

Proof. We consider for any fixed (ε, h) satisfying (2.4) the sequence of iterates $\{u_\nu(x; \varepsilon, h)\} \equiv \{u_\nu(x)\}$ say, defined by

$$u_0(x) = \varphi_0(x); \tag{2.5a}$$

$$u_{\nu+1}(x) = \varphi_0(x) + T_0[u_\nu(x)], \qquad \nu = 0, 1, 2, \cdots. \tag{2.5b}$$

We first show, by induction, that $0 \leqq u_\nu(x) \leqq Y$ on $[0, 1]$. From (2.2b) and (2.4) we have, recalling H4,

$$0 \leqq u_0(x) = \varphi_0(x) \leqq h(1 + a\varepsilon) + A\varepsilon \leqq 2h + M \leqq 2N + M < Y.$$

Assuming $0 \leqq u_\nu(x) \leqq Y$ on $[0, 1]$ we use H2 and (2.5b) to get

$$0 \leqq u_{\nu+1}(x) \leqq \varphi_0(x) + M \int_0^x [1 - e^{(t-x)/\varepsilon}]\, dt \leqq \varphi_0(x) + M \leqq Y,$$

concluding the first induction.

Next we show that

$$|u_\nu(x) - u_{\nu-1}(x)| \leqq \frac{M}{K} \frac{K^\nu x^\nu}{\nu!}, \qquad 0 \leqq x \leqq 1, \quad \nu = 1, 2, \cdots. \tag{2.6}$$

Clearly from (2.5), $|u_1(x) - u_0(x)| = |T_0[u_0(x)]| \leqq Mx$ for $0 \leqq x \leqq 1$, so (2.6) is established for $\nu = 1$. Using the inductive hypothesis and (1.5) we get

$$\begin{aligned} |u_{\nu+1}(x) - u_\nu(x)| &\leqq \int_0^x [1 - e^{(t-x)/\varepsilon}] K |u_\nu(t) - u_{\nu-1}(t)|\, dt \\ &\leqq K \int_0^x |u_\nu(t) - u_{\nu-1}(t)|\, dt \\ &\leqq \frac{M}{K} \frac{K^{\nu+1} x^{\nu+1}}{(\nu+1)!}, \end{aligned}$$

and so (2.6) follows.

Now in the usual way $\{u_\nu(x)\}$ is a Cauchy sequence of continuous functions on $[0, 1]$ and by the continuity of $g(x, y)$ on R_Y it follows that $\lim_{\nu\to\infty} u_\nu(x) = u(x)$ is a continuous solution of (2.2) over $[0, 1]$. Since $0 \leqq u_\nu(x) \leqq Y$ the same is true of $u(x)$. To demonstrate uniqueness we simply use the fact that $T_0[u]$ is contracting under the norm

$$\|u\|_1 = \sup_{0 \leqq x \leqq 1} e^{-\tilde{K}x} |u(x)|$$

for any $\tilde{K} > K$. Finally $u'(x) \geqq 0$ follows from differentiating in (2.2a), and the smooth dependence on (ε, h) follows from the uniform convergence of the $\{u_\nu(x)\}$ which are clearly continuously differentiable in ε and h and these derivatives converge uniformly on $0 \leqq x \leqq 1$. This completes the proof.

We next show that $u(1;\varepsilon,h)$ essentially ranges over $(M,N]$ as h ranges over $(0,N]$ for all ε sufficiently small. More precisely we have the following corollary.

COROLLARY 2.7. *Let* H1, H2 *and* H4 *hold. For any* δ *in* $0<\delta\leqq 3M$ *define*

$$\varepsilon_0(\delta)\equiv\min(1,1/a,\delta/(3A)).$$

Then for each ε *in* $0<\varepsilon\leqq\varepsilon_0(\delta)$ *as* h *ranges over* $\delta/3\leqq h\leqq N$, *the quantity* $\alpha(\varepsilon,h)\equiv u(1;\varepsilon,h)$ *ranges, at least, over* $M+\delta\leqq\alpha\leqq N$.

Proof. All (ε,h) as in the hypothesis also satisfy (2.4) and so $u(1;\varepsilon,h)$ is uniquely defined by (2.1). Integrating (2.1a) over $[0,x]$ and using H2 yields

$$\varepsilon A+(1+a\varepsilon)h\leqq\varepsilon u'(x)+u(x)\leqq\varepsilon A+(1+a\varepsilon)h+Mx.$$

However since $u(x)\geqq 0$ and $u'(x)\geqq 0$ on $[0,1]$ we have $\int_0^1 u(x)\,dx\leqq u(1)$ so that another integration, now over $[0,1]$, yields

$$h+\frac{\varepsilon}{1+\varepsilon}(A+ah)\leqq u(1;\varepsilon,h)\leqq\varepsilon A+(1+a\varepsilon)h+M.$$

The result follows from the continuity of $u(1;\varepsilon,h)$ in h by using first $h=N$ in the left-hand inequality and then $h=\delta/3$, $\varepsilon A\leqq\delta/3$ and $a\varepsilon\leqq 1$ in the right-hand inequality.

Finally we require some bounds on $u''(x)$ which are contained in the next corollary.

COROLLARY 2.8. *Let* H1, H2 *and* H4 *hold. Then there exist constants* K_1 *and* C_1, *independent of* (ε,h), *such that for all* (ε,h) *satisfying* (2.4) *the solution* $u(x;\varepsilon,h)$ *of* (2.1) *has second derivatives bounded by*

$$\varepsilon|u''(x:\varepsilon,h)|\leqq(K_1+KC_1)\varepsilon(1-e^{-x/\varepsilon})+C_1e^{-x/\varepsilon}. \tag{2.9a}$$

Further for $\varepsilon<1$,

$$|u''(1;\varepsilon,h)|\leqq K_0\equiv K_1+(1+K)C_1. \tag{2.9b}$$

Proof. By differentiation in (2.2a) we obtain the integral representation

$$u'(x;\varepsilon,h)=(A+ah)\,e^{-x/\varepsilon}+\int_0^x\frac{1}{\varepsilon}e^{(t-x)/\varepsilon}g(t,u(t))\,dt.$$

On performing a partial integration this yields, in (2.1),

$$\varepsilon u''(x;\varepsilon,h)=[g(0,h)-(A+ah)]\,e^{-x/\varepsilon}+\int_0^x e^{(t-x)/\varepsilon}\frac{dg(t,u(t))}{dt}\,dt.$$

Using the bounds

$$K_1\equiv\max_{R_Y}|g_x(x,y)|,\quad C_1\geqq\max_{0\leqq x\leqq 1}|u'(x;\varepsilon,h)|,\quad K\equiv\max_{R_Y}|g_y(x,y)|,$$

we get

$$\varepsilon|u''(x;\varepsilon,h)|\leqq(M+A+aN)\,e^{-x/\varepsilon}+\varepsilon(K_1+KC_1)(1-e^{-x/\varepsilon}).$$

Here we have recalled that $h\leqq N$ and $|g|\leqq M$. Also we find from the above integral representation that $|u'(x;\varepsilon,h)|\leqq(A+ah)+M$, and so $C_1=(M+A+aN)$ will do. Thus (2.9a) is established, and (2.9b) follows since $e^{-1/\varepsilon}<\varepsilon$ for $\varepsilon<1$.

We recall [1] that Cohen disclosed the possibility of multiple solutions of (1.1)–(1.3) by considering the reduced problem

$$v' = g(x, v), \tag{2.10a}$$

$$v'(1) + bv(1) = B; \tag{2.10b}$$

obtained by formally setting $\varepsilon = 0$ in (1.1) and dropping the boundary condition (1.2). A slight knowledge of singular perturbation theory suffices to determine which, if any, boundary condition should be retained; we refer to the text of Cole [3] for a thorough understanding. The reduced problem for (1.1), (1.2) and (1.3) is simply

$$v' = g(x, v), \tag{2.11a}$$

$$f(v(1), v'(1)) = 0. \tag{2.11b}$$

In either case we are thus led to consider initial value problems of the form

$$v' = g(x, v), \tag{2.12a}$$

$$v(1) = \alpha. \tag{2.12b}$$

The existence theory for this problem is contained in the following theorem.

THEOREM 2.13. *Let* H1 *and* H2 *hold for some* $Y > M$. *Then for each* α *in* $M \leqq \alpha \leqq Y$ *the problem* (2.12) *has a unique solution* $v = v(x;\alpha)$ *on* $0 \leqq x \leqq 1$ *satisfying* $\alpha - M(1 - x) \leqq v(x, \alpha) \leqq \alpha$.

Proof. We simply consider the equivalent integral equation

$$v(x) = \alpha - \int_x^1 g(t, u(t))\,dt$$

and the iteration scheme

$$v_0(x) = \alpha, \qquad v_{\nu+1}(x) = \alpha - \int_x^1 g(t, v_\nu(t))\,dt.$$

By induction it is easily established that, on $0 \leqq x \leqq 1$,

$$\alpha - M(1 - x) \leqq v_\nu(x) \leqq \alpha, \qquad \nu = 1, 2, \cdots,$$

$$|v_\nu(x) - v_{\nu-1}(x)| \leqq \frac{M}{K}\frac{K^\nu(1 - x)^\nu}{\nu!}, \qquad \nu = 1, 2, \cdots.$$

The results now follow in the standard fashion previously indicated.

If we use $\alpha = \alpha_j$, the roots of $f(\alpha, g(1, \alpha)) = 0$, as initial data in (2.12), we obtain $v_j(x) \equiv v(x, \alpha_j)$ which are solutions of the reduced problem (2.11). These are known as the "outer solutions" for the singular perturbation problem (1.1), (1.2), (1.4). Notice that according to H3 we may not be investigating all of the outer solutions but only those determined by the α_j which are simple roots and lie in (M, N).

3. Solutions of the boundary value problem. We now use the results of § 2 to show that the boundary value problem (1.1), (1.2) and (1.4) has several distinct solutions each converging to an appropriate solution of the reduced problem (2.11) as $\varepsilon \to 0$. The main result is contained in the following theorem.

THEOREM 3.1. *Let* $g(x, y)$ *and* $f(y, z)$ *satisfy* H1, H2, H3 *and* H4. *Then for some* $\varepsilon_1 > 0$ *the boundary value problem* (1.1), (1.2), (1.4) *has at least* J *distinct solutions,* $y = y_j(x, \varepsilon)$, *for each* ε *in* $0 < \varepsilon \leqq \varepsilon_1$. *For some constant* C_0 *independent of* ε, *these solutions satisfy*

$$|y_j(1, \varepsilon) - \alpha_j| < C_0\varepsilon, \qquad j = 1, 2, \cdots, J. \tag{3.2}$$

Proof. For all (ε, h) satisfying (2.4) we use the solution $u(x; \varepsilon, h)$ of (2.1) to define

$$G(h, \varepsilon) \equiv f(u(1; \varepsilon, h), u'(1; \varepsilon, h)). \tag{3.3a}$$

It clearly follows from Theorem 2.3 that $y = u(x; \varepsilon, h)$ is a solution of (1.1), (1.2), (1.4) if $G(h, \varepsilon) = 0$. We shall establish that $G(h, \varepsilon)$ has, for all sufficiently small ε, at least J zeros $h = h_j(\varepsilon)$. First, using $F(\alpha)$ defined in H3(b), we write

$$G(h, \varepsilon) = F(u(1; \varepsilon, h)) + \beta(h, \varepsilon), \tag{3.3b}$$

where

$$\beta(h, \varepsilon) \equiv f(u(1; \varepsilon, h), u'(1; \varepsilon, h)) - f(u(1; \varepsilon, h), g(1; u(1; \varepsilon, h))). \tag{3.3c}$$

For (ε, h) in (2.4) we have $0 \leqq u'(1; \varepsilon, h) \leqq 2M + N$, and it follows that $\beta(h, \varepsilon)$ is continuous and even continuously differentiable on this domain. The smoothness of $f(y, z)$ in H3(a) and the boundedness of $u(1; \varepsilon, h)$ and $|u''(1; \varepsilon, h)|$ from Theorem 2.3 and Corollary 2.8, respectively, imply that

$$|\beta(h, \varepsilon)| \leqq K_0 L\varepsilon, \qquad L \equiv \max_{\substack{0 \leqq y \leqq Y \\ 0 \leqq z \leqq 2M + N}} |f_z(y, z)|. \tag{3.3}$$

Since the roots α_j of $F(\alpha) = 0$ are simple we are assured that

$$m \equiv \tfrac{1}{2} \min_{1 \leqq j \leqq J} \left| \frac{dF(\alpha_j)}{d\alpha} \right| > 0. \tag{3.4a}$$

Then for some sufficiently small $\rho_0 > 0$ it follows that

$$|F(\alpha)| > m|\alpha - \alpha_j| \quad \text{on } 0 < |\alpha - \alpha_j| \leqq \rho_0, \qquad j = 1, 2, \cdots, J. \tag{3.4b}$$

Thus, since $F(\alpha)$ changes sign at each α_j, the continuous function $F(\alpha) + \beta$ must vanish at least once in each interval $|\alpha - \alpha_j| \leqq \rho$ if $|\beta| < m\rho$ for any ρ in $0 < \rho \leqq \rho_0$.

Now pick δ and ρ_1 so small that

$$M + \delta \leqq \alpha_1 - \rho_1, \quad \rho_1 \leqq \rho_0, \quad \rho_1 < \min_{2 \leqq j \leqq J} (\alpha_j - \alpha_{j-1})/2,$$

and set $\varepsilon_1 = \min \{\varepsilon_0(\delta), m\rho_1/(K_0 L)\}$. Applying Corollary 2.7 for any ε in $0 < \varepsilon \leqq \varepsilon_1$ shows with the above argument and the continuity of $\beta(h, \varepsilon)$ that as h ranges over $\delta/3 \leqq h \leqq N$, $G(h, \varepsilon) = 0$ for at least J distinct values of h, say $h = h_j(\varepsilon)$. In fact these values of h can be chosen such that for all ε in $0 < \varepsilon \leqq \varepsilon_1$,

$$\alpha_j - K_0 L\varepsilon/m \leqq u(1; \varepsilon, h_j(\varepsilon)) \leqq \alpha_j + K_0 L\varepsilon/m, \qquad j = 1, 2, \cdots, J.$$

Clearly $y_j(x, \varepsilon) \equiv u(x; \varepsilon, h_j(\varepsilon))$ for $j = 1, 2, \cdots, J$ are each solutions of (1.1), (1.2), (1.4), and they are distinct since they differ at $x = 1$. Furthermore these solutions satisfy (3.2) with $C_0 = K_0L/m$. This completes the proof.

We can show, by means of the implicit function theorem, that the $h_j(\varepsilon)$ and hence the solutions $y_j(x, \varepsilon)$ can be continuous and continuously differentiable in ε on $0 < \varepsilon \leqq \varepsilon_0$ for sufficiently small ε_0. The proof is involved so we do not present it here. However we easily show how the exact solutions are related to the "outer solutions" in the following corollary.

COROLLARY 3.5. *Let* H1–H4 *hold and* ε_1 *be as in Theorem* 3.1 *and at least as small as* $\varepsilon_1 < 1/K$. *Define the outer solutions* $v_j(x) \equiv v(x, \alpha_j)$, $j = 1, 2, \cdots, J$, *by using* (2.12) *with* $\alpha = \alpha_j$ *and Theorem* 2.13. *Then for each* ε *in* $0 < \varepsilon \leqq \varepsilon_1$,

$$|y_j(x, \varepsilon) - v_j(x)| \leqq \varepsilon\left[\left(C_0 + C_1 + \frac{K_1}{K}\right) e^{K(1-x)} + \left(\frac{C_1}{1 - \varepsilon K}\right) e^{-x/\varepsilon}\right] \tag{3.6}$$

for all $0 \leqq x \leqq 1$ *and* $j = 1, 2, \cdots, J$.

Proof. From (1.1), (2.9) and (3.2) it follows that $w_j(x, \varepsilon) \equiv y_j(x, \varepsilon) - v_j(x)$ satisfies an initial value problem of the form

$$w' = a(x, \varepsilon)w + b(x, \varepsilon), \qquad w(1) = C,$$

where

$$a(x, \varepsilon) \equiv g_y(x, \theta y_j(x, \varepsilon) + [1 - \theta]v_j(x)), \quad |a(x, \varepsilon)| \leqq K,$$

$$b(x, \varepsilon) \equiv -\varepsilon y_j''(x, \varepsilon), \quad |b(x, \varepsilon)| \leqq (K_1 + KC_1)\varepsilon + C_1 e^{-x/\varepsilon},$$

$$C \equiv y_j(1, \varepsilon) - \alpha_j, \quad |C| \leqq C_0\varepsilon.$$

The bound on $|b(x, \varepsilon)|$ follows from Corollary 2.8. Solving the equation for w yields, on taking absolute values,

$$|w(x)| \leqq e^{K(1-x)}|C| + \int_x^1 e^{K(\xi - x)}|b(\xi, \varepsilon)| \, d\xi.$$

Finally (3.6) results on inserting the bounds for $|b|$ and $|C|$ and using $\varepsilon K < 1$, thus completing the proof.

We note that the convergence implied by (3.6) is uniform in x as $\varepsilon \downarrow 0$. Thus there is no typical boundary layer jump at $x = 0$ but at most one of magnitude $O(\varepsilon)$.

REFERENCES

[1] D. S. COHEN, *Multiple solutions of singular perturbation problems*, this Journal, 3 (1972), pp. 72–82.

[2] ———, *Multiple stable solutions of nonlinear boundary value problems arising in chemical reactor theory*, SIAM J. Appl. Math., 20 (1971), pp. 1–13.

[3] J. D. COLE, *Perturbation Methods in Applied Mathematics*, Ginn-Blaisdell, Waltham, Mass., 1968.

[4] H. B. KELLER, *Elliptic boundary value problems suggested by nonlinear diffusion processes*, Arch. Rational Mech. Anal., 35 (1969), pp. 363–381.

[5] D. H. SATTINGER, *Monotone methods in nonlinear elliptic and parabolic boundary value problems*, Indian J. Math., to appear.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ASYMPTOTIC SOLUTIONS OF A 6TH ORDER DIFFERENTIAL EQUATION WITH TWO TURNING POINTS. PART II: DERIVATION BY REDUCTION TO A FIRST ORDER SYSTEM*

B. GRANOFF†

Abstract. The 6th order ordinary differential equation $(D^2 - k^2)^3u - k^6xu = 0$ possesses turning points at $x = -1$ and $x = 0$. The asymptotic behavior of a fundamental set of solutions of this equation is investigated by first transforming the scalar equation into a first order 6×6 linear system of the form $\varepsilon \mathbf{U}' = A(x)\mathbf{U}$, where $\varepsilon = k^{-1}$. In a neighborhood of the turning point $x = -1$, the system can be reduced to four uncoupled scalar equations and a 2×2 first order system. The resulting scalar equations are solved without difficulty and the solution of the 2×2 system is shown to be expressible in terms of Airy functions. This constitutes a derivation of uniformly valid asymptotic expansions of a fundamental set of solutions in a neighborhood of $x = -1$. At the turning point $x = 0$, it is shown that the 6×6 system can be reduced to two uncoupled 3×3 systems for the leading terms of the asymptotic expansions of a fundamental set. Each 3×3 system is equivalent to a third order scalar equation of the form $\varepsilon^3 v^{(3)} + \varepsilon x\mu(x)v' + xv(x)v = 0$, where $\mu(0) \neq 0$ and $v(0) \neq 0$. A comparison is made between this type of asymptotic analysis and a previous investigation which employed the method of steepest descent.

1. Introduction. In a previous paper (Granoff and Bleistein, 1972) we obtained the asymptotic expansions of a fundamental set of solutions to the equation

$$(D^2 - k^2)^3u - k^6xu = 0 \qquad (D = d/dx), \tag{1.1}$$

where k is a large positive parameter. This equation possesses two turning points, at $x = -1$ and $x = 0$. The results contained in that paper were obtained by applying the method of steepest descent to integral representations of a fundamental set of solutions. The paths of descent were determined numerically at a representative point in each of the intervals $(-\infty, -1)$, $(-1, 0)$, $(0, \infty)$. In this manner we were able to derive asymptotic expansions of a fundamental set in each interval and WKB connection formulas across the two turning points.

In the present paper we describe an analytical procedure by which we obtain the uniform asymptotic expansions of a fundamental set in a neighborhood of the turning point $x = -1$. In a neighborhood of $x = 0$ we find that (1.1) may be replaced by two uncoupled third order linear equations for the leading terms of the asymptotic expansion of a fundamental set. The procedure which we employ is described in detail in Wasow (1965).

In the next section we replace (1.1) by a first order linear system and make use of a theorem which implies that this system can be partially uncoupled by a similarity transformation in a neighborhood of each of the turning points. The details of the uncoupling at $x = -1$ are given in § 3. Four uncoupled equations and two coupled equations result. The two coupled equations are treated in § 4. There it is shown that the leading terms of the asymptotic expansion of the solutions can be given in terms of Airy functions. In § 5 we obtain the leading terms of the

* Received by the editors December 7, 1970, and in final revised form April 29, 1971.
† Mathematics Department, Boston University, Boston, Massachusetts 02215.

asymptotic expansions of a fundamental set of solutions to (1.1) in a neighborhood of $x = -1$. In § 6 it is shown that, in a neighborhood of $x = 0$, (1.1) may be replaced by two uncoupled systems consisting of three equations each. Additional analysis results in two third order equations of the form

$$\varepsilon^3 u''' + \varepsilon x g(x) u' + x h(x) u = 0, \qquad \varepsilon = k^{-1}, \tag{1.2}$$

where $g(x)$ and $h(x)$ are analytic at $x = 0$ and $g(0) \neq 0$, $h(0) \neq 0$ for the leading terms of the asymptotic expansion of a fundamental set. A comparison of the two techniques applied to (1.1) is made in § 7.

2. Reduction to a first order linear system. Expanding (1.1) fully we obtain

$$u^{(6)} - 3k^2 u^{(4)} + 3k^4 u'' - k^6(x + 1)u = 0. \tag{2.1}$$

Let us now set $\varepsilon = k^{-1}$ and

$$u_{j+1} = \varepsilon u_j', \qquad j = 1, 2, \cdots, 5.$$

Here $u_1 = u$. By (2.1), this substitution results in the first order linear system

$$\varepsilon \mathbf{U}' = A(x)\mathbf{U}, \tag{2.2}$$

where $\mathbf{U}$ is a six-component column vector and

$$A(x) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 + x & 0 & -3 & 0 & 3 & 0 \end{bmatrix}.$$

The turning points of (2.2) occur at the points x at which the eigenvalues of the matrix $A(x)$ coalesce. The characteristic equation associated with $A(x)$ is

$$\lambda^6 - 3\lambda^4 + 3\lambda^2 - (1 + x) = 0. \tag{2.3}$$

The roots are easily found and they are

$$\pm\sqrt{1 + x^{1/3}}, \quad \pm\sqrt{1 + \omega x^{1/3}}, \quad \pm\sqrt{1 + \omega^2 x^{1/3}}, \tag{2.4}$$

where $x = r\, e^{i\theta}$, $0 \leqq \theta \leqq 2\pi$, $x^{1/3} = r^{1/3}\, e^{i\theta/3}$, and $\omega = e^{2\pi i/3}$. We find that the turning points are $x = -1$, at which $\pm\sqrt{1 + x^{1/3}\omega}$ both coalesce to zero, and $x = 0$, at which $\sqrt{1 + x^{1/3}}$, $\sqrt{1 + \omega x^{1/3}}$ and $\sqrt{1 + \omega^2 x^{1/3}}$ coalesce to 1 and the remaining roots coalesce to -1.

It can be shown that in a neighborhood of each turning point it is possible to partially decouple the system of equations given by (2.2). The decomposition results from the following theorem proved in Wasow (1965).

THEOREM. *Let* $A(x, \varepsilon) \sim \sum_{j=0}^{\infty} A_j(x)\varepsilon^j$, $\varepsilon \to 0$, *uniformly for* ε *in some sector* S *of the* ε*-plane*, $0 < |\varepsilon| \leqq \varepsilon_0$, *where* $A_j(x)$, $j = 0, 1, \cdots$, *are analytic at* $x = x_0$. *Assume that the eigenvalues of* $A_0(x_0)$ *consist of two groups* $\{\lambda_1, \cdots, \lambda_p\}$ *and*

$\{\lambda_{p+1}, \cdots, \lambda_n\}$ *such that* $\lambda_j \neq \lambda_k$ *for* $j \leqq p$ *and* $k > p$. *Then there exists a matrix* $P(x, \varepsilon)$, *analytic at* $x = x_0$ *and for* $0 < |\varepsilon| \leqq \varepsilon_0$ *in a sector S of the ε-plane, possessing in S a uniform asymptotic expansion*

$$P(x, \varepsilon) \sim \sum_{j=0}^{\infty} P_j(x)\varepsilon^j, \qquad \varepsilon \to 0,$$

$\det P_0(x) \neq 0$ *for x sufficiently close to* x_0, *such that the transformation*

$$\mathbf{U} = P(x, \varepsilon)\mathbf{Y}$$

takes the system of differential equations

$$\varepsilon^h \mathbf{U}' = A(x, \varepsilon)\mathbf{U}$$

into

$$\varepsilon^h \mathbf{Y}' = B(x, \varepsilon)\mathbf{Y},$$

where $B(x, \varepsilon)$ *has the block diagonal form*

$$B(x, \varepsilon) = \begin{bmatrix} B_{11}(x, \varepsilon) & 0 \\ 0 & B_{22}(x, \varepsilon) \end{bmatrix}.$$

The matrices $B_{jj}(x, \varepsilon)$ *have asymptotic power series for* $\varepsilon \to 0$ *in S and the eigenvalues of* $\lim_{\varepsilon \to 0} B_{jj}(x_0, \varepsilon)$ *are* $\{\lambda_1, \cdots, \lambda_p\}$ *for* $j = 1$ *and* $\{\lambda_{p+1}, \cdots, \lambda_n\}$ *for* $j = 2$.

Upon investigation of the eigenvalues associated with the matrix $A(x)$ given in (2.4), we find that this theorem implies that, in a neighborhood of the turning point $x = -1$, we can replace (2.2) by a system consisting of four first order uncoupled equations and two equations which remain coupled. The uncoupled equations can be solved for the leading terms of the asymptotic expansion of their solutions. Because of the manner in which the two eigenvalues coalesce to zero at $x = -1$, we can show that the two coupled equations can be reduced to a problem involving Airy's equation. The analysis leading to these results is contained in §§ 3 and 4.

In a neighborhood of the turning point $x = 0$, the above theorem implies that (2.2) may be replaced by two uncoupled systems each consisting of three equations. This problem is treated in § 6.

3. Block diagonalization of $A(x)$ **about** $x = -1$. The first step in decoupling (2.2) about $x = -1$ is to construct a matrix $P(x)$, analytic and nonsingular at $x = -1$, such that $P^{-1}(x)A(x)P(x)$ is in the appropriate block diagonal form. This may be accomplished by considering the cyclic invariant subspaces of E^6 relative to the matrix $A(x)$ in a neighborhood of $x = -1$. The theory of cyclic invariant subspaces is described in Gantmacher (1959). We are led to consider the matrix

$$(3.1) \qquad P(x) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ \lambda_1 & -\lambda_1 & \lambda_2 & -\lambda_2 & 0 & 1 \\ \lambda_1^2 & \lambda_1^2 & \lambda_2^2 & \lambda_2^2 & \lambda_3^2 & 0 \\ \lambda_1^3 & -\lambda_1^3 & \lambda_2^3 & -\lambda_2^3 & 0 & \lambda_3^2 \\ \lambda_1^4 & \lambda_1^4 & \lambda_2^4 & \lambda_2^4 & \lambda_3^4 & 0 \\ \lambda_1^5 & -\lambda_1^5 & \lambda_2^5 & -\lambda_2^5 & 0 & \lambda_3^4 \end{bmatrix},$$

where $\lambda_1 = \sqrt{1 - \omega|x|^{1/3}}$, $\lambda_2 = \sqrt{1 - \omega^2|x|^{1/3}}$, and $\lambda_3^2 = 1 - |x|^{1/3}$. It is easily seen that $P(x)$ is analytic at $x = -1$ and possesses an analytic inverse at $x = -1$. By direct computation we find that

$$P^{-1}(x)A(x)P(x) = \mathscr{A}(x)$$

$$(3.2) \qquad = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \lambda_3^2 & 0 \end{bmatrix}.$$

From (2.2) we have that

$$(3.3) \qquad \varepsilon P^{-1}(x)\mathbf{U}' = \mathscr{A}(x)P^{-1}(x)\mathbf{U}.$$

Setting

$$(3.4) \qquad \mathbf{U} = P(x)\mathbf{Y}$$

we obtain

$$(3.5) \qquad \varepsilon\mathbf{Y}' = C(x, \varepsilon)\mathbf{Y},$$

where

$$C(x, \varepsilon) = \mathscr{A}(x) - \varepsilon P^{-1}(x)P'(x).$$

It is shown in Wasow (1965) that by means of a transformation of the form

$$(3.6) \qquad \mathbf{Y} = Q(x, \varepsilon)\mathbf{V},$$

where $Q(x, \varepsilon) = I + \sum_{j=1}^{\infty} Q_j(x)\varepsilon^j$, it is possible to transform (3.5) into

$$(3.7) \qquad \varepsilon\mathbf{V}' = \left[\mathscr{A}(x) + \sum_{j=1}^{\infty} \mathscr{A}_j(x)\varepsilon^j\right]\mathbf{V},$$

where the matrix $\sum_{j=1}^{\infty} \mathscr{A}_j(x)\varepsilon^j$ has the same block diagonal form as the matrix $\mathscr{A}(x)$.

Thus (3.7) reduces to the system of four uncoupled equations:

$$(3.8) \qquad \begin{aligned} \varepsilon v_1' &= \left[\lambda_1(x) + \sum_{j=1}^{\infty} \mu_{1j}(x)\varepsilon^j\right]v_1, \\ \varepsilon v_2' &= \left[-\lambda_1(x) + \sum_{j=1}^{\infty} \mu_{2j}(x)\varepsilon^j\right]v_2, \\ \varepsilon v_3' &= \left[\lambda_2(x) + \sum_{j=1}^{\infty} \mu_{3j}(x)\varepsilon^j\right]v_3, \\ \varepsilon v_4' &= \left[-\lambda_2(x) + \sum_{j=1}^{\infty} \mu_{4j}(x)\varepsilon^j\right]v_4, \end{aligned}$$

and the system of two coupled equations:

$$\varepsilon v_5' = v_6 + \sum_{j=1}^{\infty} [\mu_{5j}(x)v_5 + \nu_{5j}(x)v_6]\varepsilon^j, \tag{3.9}$$
$$\varepsilon v_6' = \lambda_3^2(x)v_5 + \sum_{j=1}^{\infty} [\nu_{6j}(x)v_5 + \mu_{6j}(x)v_6]\varepsilon^j.$$

4. Solution of the coupled equations. We now proceed to treat the coupled equations (3.9). We rewrite them here as

$$\varepsilon \mathbf{W}' = \left[B_0(x) + \sum_{j=1}^{\infty} B_j(x)\varepsilon^j \right] \mathbf{W}, \tag{4.1}$$

where

$$\mathbf{W} = \begin{bmatrix} v_5 \\ v_6 \end{bmatrix} \quad \text{and} \quad B_0(x) = \begin{bmatrix} 0 & 1 \\ 1 - |x|^{1/3} & 0 \end{bmatrix}.$$

Our object is to obtain the leading term in ε of a fundamental matrix to system (4.1).

Following the procedure in Wasow (1965) we find that if we set

$$t(x) = \left[\frac{3}{2} \int_{-1}^{x} \sqrt{1 - |\xi|^{1/3}}\, d\xi \right]^{2/3} \tag{4.2}$$

and

$$R(x) = \begin{bmatrix} 1 & 0 \\ 0 & dt/dx \end{bmatrix}, \tag{4.3}$$

then, by means of the transformation

$$\mathbf{W} = R(x)\mathbf{W}^*, \tag{4.4}$$

we obtain

$$\varepsilon \frac{d\mathbf{W}^*}{dt} = \left[B_0^*(t) + \sum_{j=1}^{\infty} B_j^*(t)\varepsilon^j \right] \mathbf{W}^*. \tag{4.5}$$

Here

$$B_0^*(t) = \begin{bmatrix} 0 & 1 \\ t & 0 \end{bmatrix}.$$

Observe that $t = t(x)$, defined by (4.2), is analytic at $x = -1$.

We shall now employ the following theorem proved in Wasow (1965).

THEOREM. *There exists an asymptotic power series* $\sum_{j=0}^{\infty} R_j(t)\varepsilon^j$ *whose coefficients are analytic in a region* $|t| \leqq t_1$, *and with* $\det R_0(0) = 1$, *such that the formal transformation*

$$\mathbf{W}^* = \sum_{j=0}^{\infty} R_j(t)\varepsilon^j \mathbf{Z} \tag{4.6}$$

changes the differential equation (4.5) *into*

$$\varepsilon \mathbf{Z}' = B_0^*(t)\mathbf{Z}. \tag{4.7}$$

Furthermore it is proved in Wasow (1965) that the formal series $\sum_{j=0}^{\infty} R_j(t)\varepsilon^j$ in the above theorem is the asymptotic expansion of functions $R(t, \varepsilon)$ such that the transformation $\mathbf{W}^* = R(t, \varepsilon)\mathbf{Z}$ actually transforms (4.5) into (4.7).

A fundamental matrix of (4.7) is

$$\mathbf{Z} = \begin{bmatrix} \mathrm{Ai}(t\varepsilon^{-2/3}) & \mathrm{Ai}(\omega t\varepsilon^{-2/3}) \\ \varepsilon^{1/3}\mathrm{Ai}'(t\varepsilon^{-2/3}) & \varepsilon^{1/3}\omega\mathrm{Ai}'(\omega t\varepsilon^{-2/3}) \end{bmatrix}, \tag{4.8}$$

where $\mathrm{Ai}(x)$ is Airy's integral and $\omega = \exp 2\pi i/3$. Therefore, the leading term of the fundamental matrix of (4.5) is given by

$$\mathbf{W}^* \sim R_0(t)\mathbf{Z} = \begin{bmatrix} w_{11}^* & w_{12}^* \\ w_{21}^* & w_{22}^* \end{bmatrix}, \tag{4.9}$$

where

$$\begin{aligned} w_{11}^* &= r_{11}(t)\,\mathrm{Ai}\,(t\varepsilon^{-2/3}) + r_{12}(t)\varepsilon^{1/3}\,\mathrm{Ai}'\,(t\varepsilon^{-2/3}), \\ w_{12}^* &= r_{11}(t)\,\mathrm{Ai}\,(\omega t\varepsilon^{-2/3}) + r_{12}(t)\varepsilon^{1/3}\omega\,\mathrm{Ai}'\,(\omega t\varepsilon^{-2/3}), \\ w_{21}^* &= r_{21}(t)\,\mathrm{Ai}\,(t\varepsilon^{-2/3}) + r_{22}(t)\varepsilon^{1/3}\,\mathrm{Ai}'(t\varepsilon^{-2/3}), \\ w_{22}^* &= r_{21}(t)\,\mathrm{Ai}\,(\omega t\varepsilon^{-2/3}) + r_{22}(t)\varepsilon^{1/3}\omega\,\mathrm{Ai}'\,(\omega t\varepsilon^{-2/3}). \end{aligned}$$

From (4.4) we obtain the leading term of a fundamental matrix of (4.1). It is

$$\mathbf{W} \sim R(x)\mathbf{W}^* = \begin{bmatrix} w_{11}^* & w_{12}^* \\ w_{21}^* \dfrac{dt}{dx} & w_{22}^* \dfrac{dt}{dx} \end{bmatrix}. \tag{4.10}$$

In (4.10) there appear four unknown functions, r_{11}, r_{12}, r_{21}, and r_{22}. We shall find it necessary to determine only the functions r_{11} and r_{12} in order to obtain the leading terms of a fundamental set of solutions of the original scalar equation (1.1). This will be considered in the next section.

5. Derivation of a fundamental set of solutions for (1.1) at $x = -1$. From (3.8) and (4.10) we find that the leading term of the asymptotic expansion of a fundamental matrix of (3.7) is

$$V = \begin{bmatrix} v_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{11}^* & w_{12}^* \\ 0 & 0 & 0 & 0 & w_{21}^* \dfrac{dt}{dx} & w_{22}^* \dfrac{dt}{dx} \end{bmatrix}, \tag{5.1}$$

where

$$v_1 \sim \exp\left[\varepsilon^{-1}\int_{-1}^{x} \lambda_1(\xi)\,d\xi + \mu_1(x)\right],$$

$$v_2 \sim \exp\left[-\varepsilon^{-1}\int_{-1}^{x} \lambda_1(\xi)\,d\xi + \mu_2(x)\right],$$

$$v_3 \sim \exp\left[\varepsilon^{-1}\int_{-1}^{x} \lambda_2(\xi)\,d\xi + \mu_3(x)\right],$$

$$v_4 \sim \exp\left[-\varepsilon^{-1}\int_{-1}^{x} \lambda_2(\xi)\,d\xi + \mu_4(x)\right],$$

and w_{11}^*, w_{12}^*, w_{21}^*, w_{22}^* are given by (4.9). The function $t = t(x)$ is given by (4.2). The functions $\mu_j(x)$, $j = 1, \cdots, 4$, are still unknown. They will be determined shortly by resorting directly to the original scalar equation (1.1).

From (3.4) and (3.6), we find that the leading terms of the asymptotic expansion of a fundamental matrix of (2.2) are given by

$$\mathbf{U} \sim P(x)Q(x, 0)\mathbf{V} = P(x)\mathbf{V}. \tag{5.2}$$

In order to obtain the leading terms of the asymptotic expansions of a fundamental set of solutions to the scalar equation (1.1), we need consider only the entries in the first row of the fundamental matrix $\mathbf{U}$ given by (5.2). By direct computation we find that these terms are

$$u_j = v_j, \qquad j = 1, \cdots, 4, \tag{5.3}$$

where v_j, $j = 1, \cdots, 4$, are given by (5.1) and

$$u_5 = w_{11}^*, \quad u_6 = w_{12}^*, \tag{5.4}$$

where w_{11}^*, w_{12}^* are given by (4.9).

We now proceed to find those unknown functions $\mu_j(x)$, $j = 1, \cdots, 4$, appearing in (5.1), and $g(x) = r_{11}[t(x)]$, $h(x) = r_{12}[t(x)]$, appearing in (4.9). First let us substitute the expression

$$u = \exp\left[\varepsilon^{-1}\int_{-1}^{x} \lambda(\xi)\,d\xi + \mu(x)\right]$$

into (1.1). Observe that this is the typical form of u_j, $j = 1, \cdots, 4$. This substitution yields the equation

$$\begin{aligned}&[\lambda^6 - 3\lambda^4 + 3\lambda^2 - (x + 1)] + \varepsilon[(6\lambda^5 - 12\lambda^4 + 6\lambda)\mu' \\ &\qquad + (15\lambda^4 - 18\lambda^2 + 3)\lambda'] + O(\varepsilon^2) = 0.\end{aligned} \tag{5.5}$$

If $\lambda = \pm\lambda_1$ or $\pm\lambda_2$, where λ_1 and λ_2 are given in (3.1), then the first bracketed term in (5.5) vanishes. Equating the coefficient of ε to zero, we obtain

$$(6\lambda^5 - 12\lambda^4 + 6\lambda)\mu' + (15\lambda^4 - 18\lambda^2 + 3)\lambda' = 0, \tag{5.6}$$

where $\lambda = \pm\lambda_1$ or $\pm\lambda_2$. We observe that implicit differentiation of the characteristic equation given by (2.3) results in

$$(6\lambda^5 - 12\lambda^3 + 6\lambda)\lambda' - 1 = 0. \tag{5.7}$$

Since $\lambda = \pm\lambda_1$ or $\pm\lambda_2$ are simple roots of (2.3) in a neighborhood of $x = -1$, we may solve (5.7) for λ' and then substitute it into (5.6). The result is

$$\mu' = -(15\lambda^4 - 18\lambda^2 + 3)(\lambda')^2. \tag{5.8}$$

The solution is

$$\mu(x) = -\int_{-1}^{x} (15\lambda^4 - 18\lambda^2 + 3)(\lambda')^2 \, d\xi. \tag{5.9}$$

If $\lambda = \pm\lambda_1$, then

$$\begin{aligned} \mu(x) &= \frac{1}{12}\int_{-1}^{x} \frac{5\omega|\xi|^{1/3} - 4}{\omega|\xi|^{1/3} - 1}|\xi|^{-1}\, d\xi \\ &= \ln |x|^{-1/3}(1 - \omega|x|^{1/3})^{-1/4}(1 - \omega)^{1/4}. \end{aligned} \tag{5.10}$$

If $\lambda = \pm\lambda_2$, then

$$\mu(x) = \ln |x|^{-1/3}(1 - \omega^2|x|^{1/3})^{-1/4}(1 - \omega^2)^{1/4}. \tag{5.11}$$

Hence we obtain

$$\begin{aligned} u_1 &\sim (1 - \omega)^{1/4}|x|^{-1/3}(1 - \omega|x|^{1/3})^{-1/4} \exp \varepsilon^{-1}\int_{-1}^{x} \lambda_1(\xi)\, d\xi, \\ u_2 &\sim (1 - \omega)^{1/4}|x|^{-1/3}(1 - \omega|x|^{1/3})^{-1/4} \exp -\varepsilon^{-1}\int_{-1}^{x} \lambda_1(\xi)\, d\xi, \\ u_3 &\sim (1 - \omega^2)^{1/4}|x|^{-1/3}(1 - \omega^2|x|^{1/3})^{-1/4} \exp \varepsilon^{-1}\int_{-1}^{x} \lambda_2(\xi)\, d\xi, \\ u_4 &\sim (1 - \omega^2)^{1/4}|x|^{-1/3}(1 - \omega^2|x|^{1/3})^{-1/4} \exp -\varepsilon^{-1}\int_{-1}^{x} \lambda_2(\xi)\, d\xi. \end{aligned} \tag{5.12}$$

By substituting w_{11}^*, given by (4.9), into (1.1) and retaining only leading powers of ε, we find that $g(x) = r_{11}[t(x)]$ must satisfy the equation

$$g' + \left[\frac{t''}{2t'} + \frac{1}{2}\frac{(t^2t'^4 - 2tt'^2 + 1)'}{t^2t'^4 - 2tt'^2 + 1}\right]g = 0 \tag{5.13}$$

and that $h(x) = r_{12}[t(x)]$ satisfies

$$h' + \left[\frac{t'}{2t} + \frac{t''}{2t'} + \frac{1}{2}\frac{(t^2t'^4 - 2tt'^2 + 1)'}{t^2t'^4 - 2tt'^2 + 1}\right]h = 0. \tag{5.14}$$

The solutions are given by

$$g(x) = c_1|x|^{-1/3}t^{1/4}(x)[1 - |x|^{1/3}]^{-1/4} \qquad (c_1 = \text{const.}), \tag{5.15}$$

where $t^{1/4}(x)[1 - |x|^{1/3}]^{-1/4}$ is analytic at $x = -1$, and

$$h(x) = d_1\left[\frac{3}{2}\int_{-1}^{x} \sqrt{1 - |\xi|^2}\, d\xi\right]^{-1/3}|x|^{-1/3}t^{1/4}(x)[1 - |x|^{1/3}]^{-1/4} \qquad (d_1 = \text{const.}). \tag{5.16}$$

Observe that if $d_1 \neq 0$ in (5.16), $h(x) \to \infty$ as $x \to -1$. Hence $h(x)$ is analytic at $x = -1$ if and only if $d_1 = 0$.

Therefore the leading terms for u_5 and u_6 are

$$u_5 \sim |x|^{-1/3} t^{1/4}(x)[1 - |x|^{1/3}]^{-1/4} \operatorname{Ai}[\varepsilon^{-2/3} t(x)] \tag{5.17}$$

and

$$u_6 \sim |x|^{-1/3} t^{1/4}(x)[1 - |x|^{1/3}]^{-1/4} \operatorname{Ai}[\varepsilon^{-2/3} \omega t(x)], \tag{5.18}$$

where

$$t^{1/4}(x) = \left[\frac{3}{2}\int_{-1}^{x} \sqrt{1 - |\xi|^{1/3}}\, d\xi\right]^{1/6}.$$

From (4.2) we see that

$$t^{1/4}(x)(1 - |x|^{1/3})^{-1/4} = (\tfrac{24}{35} + \tfrac{36}{35}|x|^{1/3} + \tfrac{9}{7}|x|^{2/3})^{1/6}. \tag{5.19}$$

6. Block diagonalization of $A(x)$ about $x = 0$. In a manner similar to the one described in § 3, we may obtain a matrix $Q(x)$ which, together with its inverse, is analytic at $x = 0$ and such that $Q^{-1}(x)A(x)Q(x)$ is in the desired block diagonal form. A matrix which has the required properties is

$$Q(x) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & q_1 & 0 & 0 & -q_1 \\ 0 & q_1 & q_2 & 0 & -q_1 & q_2 \\ q_1 & q_2 & q_3 & -q_1 & q_2 & -q_3 \\ q_2 & q_3 & q_4 & q_2 & -q_3 & q_4 \\ q_3 & q_4 & q_5 & -q_3 & q_4 & -q_5 \end{bmatrix}, \tag{6.1}$$

if we set

$$\begin{aligned} a_1 &= -\sqrt{1 + x^{1/3}} - \sqrt{1 + \omega x^{1/3}} - \sqrt{1 + \omega^2 x^{1/3}}, \\ a_2 &= \sqrt{1 - \omega^2 x^{1/3} + x^{2/3}} + \sqrt{1 - x^{1/3} + x^{2/3}} + \sqrt{1 - x^{1/3} + \omega^2 x^{2/3}}, \\ a_3 &= -\sqrt{1 + x}, \end{aligned} \tag{6.2}$$

where the elements of $Q(x)$ are given by the formulas

$$\begin{aligned} q_1 &= -a_3, \\ q_2 &= a_1 a_3, \\ q_3 &= -a_3(3 + a_2), \\ q_4 &= 1 + x + 3a_1 a_3, \\ q_5 &= -6a_3 - 3a_2 a_3. \end{aligned} \tag{6.3}$$

By direct computation we find that

$$(6.4)\qquad Q^{-1}(x)A(x)Q(x)=\begin{bmatrix}0&0&-a_3&0&0&0\\1&0&-a_2&0&0&0\\0&1&-a_1&0&0&0\\0&0&0&0&0&a_3\\0&0&0&1&0&-a_2\\0&0&0&0&1&a_1\end{bmatrix}.$$

From (2.2) we obtain

$$(6.5)\qquad \varepsilon\mathbf{Y}'=\mathscr{A}(x)\mathbf{Y}-\varepsilon Q^{-1}(x)Q'(x)\mathbf{Y},$$

where $\mathbf{U}=Q(x)\mathbf{Y}$ and $\mathscr{A}(x)=Q^{-1}(x)A(x)Q(x)$. From comments made in § 3, (6.5) can be further transformed by $\mathbf{V}=(I+\sum R_j(x)\varepsilon^j)\mathbf{Y}$ in such a manner that we obtain a system of the form

$$(6.6)\qquad \varepsilon\mathbf{V}'=\left[\mathscr{A}_0(x)+\sum_{j=1}^{\infty}\mathscr{A}_j(x)\varepsilon^j\right]\mathbf{V},$$

where $\mathscr{A}_0(x)=\mathscr{A}(x)$ and the matrix $\sum_{j=1}^{\infty}\mathscr{A}_j(x)\varepsilon^j$ has the same block diagonal form as matrix $\mathscr{A}(x)$. Therefore, (6.6) is equivalent to the two uncoupled systems

$$(6.7)\qquad \begin{aligned}\varepsilon\mathbf{V}_1'&=\begin{bmatrix}0&0&-a_3\\1&0&-a_2\\0&1&-a_1\end{bmatrix}\mathbf{V}_1+\left(\sum_{j=1}^{\infty}A_{1j}(x)\varepsilon^j\right)\mathbf{V}_1,\\ \varepsilon\mathbf{V}_2'&=\begin{bmatrix}0&0&a_3\\1&0&-a_2\\0&1&a_1\end{bmatrix}\mathbf{V}_2+\left(\sum_{j=1}^{\infty}A_{2j}(x)\varepsilon^j\right)\mathbf{V}_2.\end{aligned}$$

Let us restrict our attention to the first system appearing in (6.7). If we set

$$\mathbf{V}_1=T\mathbf{W}_1\exp\varepsilon^{-1}x,$$

where

$$T=\begin{bmatrix}1&-1&1\\-2&1&0\\1&0&0\end{bmatrix},$$

then we obtain for $\mathbf{W}_1$,

$$(6.8)\qquad \varepsilon\mathbf{W}_1'=\begin{bmatrix}\alpha(x)&1&0\\\beta(x)&0&1\\\gamma(x)&0&0\end{bmatrix}\mathbf{W}_1+O(\varepsilon)\mathbf{W}_1.$$

Here

$$(6.9)\qquad \begin{aligned}\alpha(x)&=-3-a_1(x),\\ \beta(x)&=-3-2a_1(x)-a_2(x),\\ \gamma(x)&=-1-a_1(x)-a_2(x)-a_3(x).\end{aligned}$$

Now set $\mathbf{W}_1 = S(x)\mathbf{U}_1$, where

$$S(x) = \begin{bmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ -\beta & -\alpha & 1 \end{bmatrix}.$$

Then

$$\varepsilon \mathbf{U}_1' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \gamma & \beta & \alpha \end{bmatrix} \mathbf{U}_1 + O(\varepsilon)\mathbf{U}_1. \tag{6.10}$$

It may be easily verified that the leading term in the asymptotic expansion of the first component of the three-vector $\mathbf{U}_1$ must satisfy the third order equation given by

$$\varepsilon^3 u''' - \varepsilon^2 \alpha(x) u'' - \varepsilon \beta(x) u' - \gamma(x) u = 0. \tag{6.11}$$

The change of dependent variable

$$v = u \exp\left[-\frac{\varepsilon^{-1}}{3} \int_0^x \alpha(\xi)\, d\xi \right]$$

results in the equation

$$\varepsilon^3 v''' + \varepsilon \mu(x) v' + \nu(x) v = 0 \tag{6.12}$$

for the leading term of the asymptotic expansion of v. Here

$$\begin{aligned} \mu(x) &= -(\tfrac{1}{3}\alpha^2 + \beta), \\ \nu(x) &= -(\tfrac{2}{27}\alpha^3 + \alpha\beta + \gamma). \end{aligned} \tag{6.13}$$

It can be shown that $\mu(0) = \nu(0) = 0$ and that $\mu'(0) \neq 0$ and $\nu'(0) \neq 0$.

An equation similar to (6.12) can be obtained when treating the second system in (6.7).

Some special third order differential equations with a turning point have been investigated by Langer (1955). However, asymptotic expansions of a fundamental set of solutions of third order equations of the form of (6.12), which possess a turning point at $x = 0$, have not yet been obtained. In a forthcoming paper we shall consider this problem.

7. Comparison. The results given by (5.12) and (5.17), (5.18) are most easily compared to the results of Granoff and Bleistein (1972) with the aid of the following equations:

$$\begin{aligned} \int_{-1}^{x} \lambda(\xi)\, d\xi = {} & \lambda(x)(-\tfrac{6}{7}|x| + \tfrac{6}{35}|x|^{2/3}\omega^2 + \tfrac{8}{25}|x|^{1/3}\omega + \tfrac{16}{15}) \\ & - \lambda(-1)(\tfrac{6}{35}\omega^2 + \tfrac{8}{35}\omega - \tfrac{14}{35}), \end{aligned} \tag{7.1}$$

where $\lambda(x) = \lambda_1(x)$ or $\lambda_2(x)$, and

$$
(7.2)\qquad
\begin{aligned}
(1 - \omega|x|^{1/3})^{-1/4} &= |1 + |x|^{1/3} + |x|^{2/3}|^{-1/8} \exp\left[\frac{i}{4}\arctan\frac{3|x|^{1/3}}{2 + |x|^{1/3}}\right],\\
(1 - \omega^2|x|^{1/3})^{-1/4} &= |1 + |x|^{1/3} + |x|^{2/3}|^{-1/8} \exp\left[\frac{-i}{4}\arctan\frac{3|x|^{1/3}}{2 + |x|^{1/3}}\right].
\end{aligned}
$$

Reverting to the notation used in Granoff and Bleistein (1972), we obtain

$$
(7.3)\qquad
\begin{aligned}
u_1 &\sim c_1 a_2(x) \exp[k\varphi_3(x) + i\theta_-(x)],\\
u_2 &\sim c_2 a_2(x) \exp[-k\varphi_3(x) + i\theta_-(x)],\\
u_3 &\sim c_3 a_2(x) \exp[k\varphi_2(x) - i\theta_-(x)],\\
u_4 &\sim c_4 a_2(x) \exp[-k\varphi_2(x) - i\theta_-(x)],\\
u_5 &\sim c_5 g_0(x) \mathrm{Ai}(k^{2/3} f^2(x)),\\
u_6 &\sim c_6 g_0(x) \mathrm{Ai}(k^{2/3}\omega f^2(x)).
\end{aligned}
$$

Here $a_2(x)$, $\varphi_2(x)$, $\varphi_3(x)$, $\theta_-(x)$, $g_0(x)$, and $f(x)$ are given by (4.1)–(4.5) and (5.2), (5.8) in the abovementioned paper and the c_j, $j = 1, \cdots, 6$, are constants.

In comparing the two procedures used in order to obtain the uniformly valid asymptotic expansions of a fundamental set of equation (1.1) in the neighborhood of the turning point $x = -1$, it is clear that the one described in the present paper is the simpler to employ. This technique gives the required result in a straightforward analytical manner without the necessity to resort to a numerical analysis, which was required in the previous investigation. The analysis described in this paper is recommended for those equations which may be uncoupled into systems of equations of order not greater than two.

The situation in the neighborhood of the turning point $x = 0$ is not as simple. The present procedure leads to a problem which has not been fully investigated, whereas the treatment given in Granoff and Bleistein (1972) does yield the WKB connection formulas across this turning point.

Acknowledgment. The author is grateful to D. Ludwig and D. W. Blackett for a number of helpful suggestions.

REFERENCES

F. R. GANTMACHER (1959), *The Theory of Matrices*, Chelsea, New York.

B. GRANOFF AND N. BLEISTEIN (1972), *Asymptotic solutions of a 6th order differential equation with two turning points. Part 1: Derivation by method of steepest descent*, this Journal, 3 (1972).

R. E. LANGER (1955), *The solutions of the differential equation* $v''' + \lambda^2 zv' + 3\mu\alpha^2 v = 0$, Duke Math. J., 22, pp. 525–542.

W. WASOW (1965), *Asymptotic Expansions for Ordinary Differential Equations*, Interscience, New York.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

GENERAL BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE*

JOHN E. LAGNESE†

Abstract. An L_p theory ($1 < p < \infty$) of existence and regularity of solutions of the partial differential equation $(1 - \gamma\mathscr{M}(t))(\partial u/\partial t) - \mathscr{L}(t)u = f$ satisfying general boundary conditions is given. For each t, $\mathscr{M}(t)$ is a linear elliptic partial differential operator in the space variables, $\mathscr{L}(t)$ is a linear differential operator whose order does not exceed that of $\mathscr{M}(t)$ and γ is a nonzero complex constant.

1. Introduction. The following note is concerned with existence and regularity of solutions of the equation

$$(1.1)\qquad (1 - \gamma\mathscr{M}(x, t; D))\frac{\partial u}{\partial t}(x, t) - \mathscr{L}(x, t; D)u(x, t) = f(x, t)$$

in a cylindrical domain $\Lambda = \Omega \times (s, T), \Omega \subset R^n$, which satisfy the boundary conditions

$$(1.2)\qquad C_k(x, t; D)u(x, t) = 0, \qquad k = 1, 2, \cdots, s,$$

$$(1.3)\qquad B_j(x, t; D)\frac{\partial u}{\partial t}(x, t) = 0, \qquad j = 1, 2, \cdots, m,$$

on the lateral portion of Λ. In (1.1), γ is a nonzero complex constant and $\mathscr{M}$ and $\mathscr{L}$ are linear partial differential operators in x of respective orders $2m$ and $\bar{l} \leqq 2m$ with complex-valued coefficients defined in Λ, and $\mathscr{M}$ is elliptic. $\{C_k\}_{k=1}^s$ and $\{B_j\}_{j=1}^m$ are given sets of linear differential operators in x with coefficients defined on the lateral part of Λ. In what follows, $\{C_k\}_{k=1}^s$ can be, for example, a subsystem $\{B_{j_k}\}_{k=1}^s$ of $\{B_j\}_{j=1}^m$ having the property that the order of B_{j_k} does not exceed $l - 1$, where $\bar{l} \leqq l \leqq 2m$, and the coefficients of B_{j_k} do not depend on t.

We now outline our main results. Assume first of all the coefficients in the differential operators appearing in (1.1)–(1.3) are independent of t, and let M (resp., L) be the realization in $L_p(\Omega)$ ($1 < p < \infty$) of the operator $\mathscr{M}$ (resp., $\mathscr{L}$) under the boundary conditions (1.3) (resp., (1.2)). In § 2 we prove, for example, that the initial value problem

$$(1.4)\qquad (1 - \gamma M)\frac{du}{dt} - Lu = f, \qquad u(0) = u_0$$

has a unique solution for all complex γ with the exception of a discrete sequence $\{\gamma_i\}$ consisting of the characteristic numbers of M. Regularity of the solution is also studied and it is proved, in particular, that solutions of (1.1)–(1.3) are C^∞ in all variables provided all the given data in the problem is C^∞. In § 3 we study (1.4) in the case when γ is a characteristic number of M. Assuming that $L = M$ and $p = 2$, we give necessary and sufficient conditions in order that (1.4) have a solution and study also the question of uniqueness of solutions. Our results in this connection generalize some results of R. E. Showalter [9], where it is assumed that

* Received by the editors November 17, 1970, and in revised form March 25, 1971.
† Department of Mathematics, Georgetown University, Washington, D.C. 20007.

$L = M = M^*$. The extension to a nonself-adjoint operator M is nontrivial and makes considerable use of the theory of bases in a Hilbert space. In § 4 we extend the results of § 2 to the case where M and L depend on t.

The literature on equations of the form (1.1), or more generally, linear partial differential equations having mixed space and time derivatives in their highest order terms (equations of Sobolev type; see [12]), is extensive. We refer to the references listed in [9]–[11] for an adequate bibliography on the subject and for information on the physical origins of (1.1). The papers most relevant to the present note are those of R. E. Showalter and T. W. Ting [9], [11] who treat (1.1)–(1.3) in $L_2(\Omega)$ by Hilbert space methods, assuming the coefficients to be independent of t. Their methods are, however, essentially bound to the validity of Gårding-type inequalities and, consequently, their results apply only to a small class of boundary value problems. (Cf. M.I. Visik [16, especially Thm. 5″].)

2. The stationary case. Let Ω be a bounded open set in R^n with smooth boundary and let x denote a variable point in Ω. Write $D_i = \partial/\partial x_i$, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n}$ for any multi-integer $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $\alpha_i \geqq 0$, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The operator $\mathscr{M}$ is the elliptic operator

$$\mathscr{M}(x; D) = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha$$

with complex-valued coefficients a_α defined in $\bar{\Omega}$. Thus the characteristic polynomial associated with the principal part $\mathscr{M}'$ of $\mathscr{M}$ satisfies $\mathscr{M}'(x; \xi) \neq 0$ for all real vectors $(\xi_1, \xi_2, \cdots, \xi_n) \neq 0$ and $x \in \bar{\Omega}$. If $n = 2$, we shall also assume that $\mathscr{M}$ satisfies the following condition.

Roots condition. For every pair of linearly independent real vectors ξ, η and $x \in \bar{\Omega}$, the polynomial in z, $\mathscr{M}'(x; \xi + z\eta)$, has its roots equally divided between the upper and lower half-planes.

As is well known, this condition is a consequence of the ellipticity of $\mathscr{M}$ if $n > 2$ or if $n = 2$ and the coefficients of $\mathscr{M}'$ are real.

In addition to $\mathscr{M}$, we give m differential boundary operators

$$B_j u = \sum_{|\alpha| \leqq m_j} b_\alpha^j(x) D^\alpha, \qquad m_j \leqq 2m - 1, \quad j = 1, \cdots, m,$$

with complex-valued coefficients b_α^j defined on $\partial\Omega$. We shall always assume $\{B_j\}_{j=1}^m$ is a *normal system.* This means that $m_j \neq m_k$ if $j \neq k$ and that $\partial\Omega$ is noncharacteristic to B_j at each point. We further require the following condition.

Complementing condition. At each point x on $\partial\Omega$, let ν be the normal vector and $\xi \neq 0$ be any real vector parallel to $\partial\Omega$ at x. Denote by $z_k^+(\xi)$ the m roots with positive imaginary part of the polynomial $\mathscr{M}'(x; \xi + z\nu)$. Then the polynomials $B_j'(x; \xi + z\nu)$ are linearly independent modulo the polynomial $\prod_{k=1}^m (z - z_k^+(\xi))$.

We next state our smoothness assumptions which will depend on a nonnegative index q.

(A_q) Ω is a bounded domain of class C^{2m+q}. The coefficients in $\mathscr{M}$ are of class $C^q(\bar{\Omega})$ and those in B_j of class $C^{2m+q-m_j}(\partial\Omega)$.

Following Agmon [1], we call the boundary value problem $(\mathscr{M}, \{B_j\}_1^m, \Omega)$ a *regular elliptic boundary value problem* if the conditions of the present section are satisfied. In what follows all elliptic boundary value problems will be assumed regular.

For $1 < p < \infty$, let $W_{k,p}(\Omega)$ denote the Banach space consisting of the subclass of functions in $L_p(\Omega)$ whose distributional derivatives of orders $\leqq k$ belong to $L_p(\Omega)$, with the norm

$$\|u\|_{k,p,\Omega} = \left(\sum_{|\alpha| \leqq k} \int_\Omega |D^\alpha u|^p \, dx \right)^{1/p}.$$

If $2m \leqq k \leqq 2m + q$ and $u \in W_{k,p}(\Omega)$, $B_j u$ has a well-defined trace on $\partial\Omega$ as a function in the space $W_{k-m_j-1/p,p}(\partial\Omega)$ (see, for example, [7, Chap. 1]). We define $W_{k,p}(\Omega; \{B_j\})$ as the closed subspace of $W_{k,p}(\Omega)$ consisting of those functions which satisfy

$$B_j u = 0 \quad \text{on } \partial\Omega, \qquad 1 \leqq j \leqq m.$$

In what follows, we shall omit the subscripts p and Ω and write $W_k(\Omega)$, $W_k(\Omega; \{B_j\})$ and $\|\cdot\|_k$ in place of $W_{k,p}(\Omega)$, $W_{k,p}(\Omega; \{B_j\})$ and $\|\cdot\|_{k,p,\Omega}$ respectively.

Under the assumptions of the present section the following well-known a priori estimates hold for all $u \in W_{2m+q}(\Omega)$:

$$\|u\|_{2m+q} \leqq C(\|\mathscr{M}u\|_q + \sum_{j=1}^m \|B_j u\|_{2m+q-m_j-1/p} + \|u\|_0), \tag{2.1}$$

where C does not depend on u.

We denote by M the unbounded linear operator in $W_q(\Omega)$ defined as follows:

(i) $D(M) = W_{2m+q}(\Omega; \{B_j\})$.

(ii) For $u \in D(M)$, $Mu = \mathscr{M}(x; D)u$.

It is clear that M is a closed operator in $W_q(\Omega)$ and it follows from (2.1) that M has a finite-dimensional null space and closed range. If the spectrum of M is not the whole complex plane, that is, if $(\lambda - M)^{-1}$ exists for some $\lambda = \lambda_0$, it follows (since $(\lambda_0 - M)^{-1}$ is compact) that $(\lambda - M)^{-1}$ exists for all λ except a discrete sequence $\{\lambda_n\}$ of eigenvalues of M, each of finite multiplicity, having no finite accumulation point. In general, however, one cannot exclude the possibility that the spectrum of M is the whole complex plane. We therefore assume the following condition.

Spectrum condition. The spectrum of M is not the entire complex plane.

If the spectrum condition is fulfilled and λ is in the resolvent set of M, a special form of the estimates (2.1) is valid:

$$\|u\|_{2m+q} \leqq C\{\|(\lambda - \mathscr{M})u\|_q + \sum_{j=1}^m \|B_j u\|_{2m+q-m_j-1/p}\}, \tag{2.2}$$

where C may depend on λ but not on u. A sufficient condition for the spectrum condition to hold is given by the following result of Agmon [1].

THEOREM 2.1. *Suppose for some $\theta, 0 \leqq \theta < 2\pi$, the following two conditions are satisfied:*

(i) $(-1)^m \dfrac{\mathscr{M}'(x;\xi)}{|\mathscr{M}'(x;\xi)|} \neq e^{i\theta}$ *for all real vectors $\xi \neq 0$ and $x \in \bar{\Omega}$.*

(ii) *At any point x of $\partial\Omega$ let ν be the normal vector and $\xi \neq 0$ be any real vector parallel to $\partial\Omega$ at x. Denote by $z_k^+(\xi, \lambda)$ the m roots with positive imaginary part of the polynomial in z, $(-1)^m \mathscr{M}'(x; \xi + z\nu) - \lambda$, where λ is any number on the ray* $\arg \lambda = \theta$. *Then the polynomials $B_j'(x; \xi + z\nu)$ are linearly independent modulo the polynomial $\prod_{k=1}^m (z - z_k^+(\xi, \lambda))$.*

Then the spectrum of M is discrete. Moreover, there is a sector $\Sigma: |\arg \lambda - \theta| < \delta, |\lambda| \geqq R$, of the complex plane such that, for all $\lambda \in \Sigma$ and $u \in D(M)$,

$$\sum_{k=0}^{2m+q} |\lambda|^{k/2m} \|u\|_{2m+q-k} \leqq C \sum_{k=0}^{q} |\lambda|^{k/2m} \|\lambda u - Mu\|_{q-k}, \tag{2.3}$$

where C does not depend on λ or on u.

In particular, Σ is contained in the resolvent set of M. The inequalities (2.3) were established only for $q = 0$ in [1], but their derivation for $q > 0$ may be carried out in essentially the same manner. If $p = 2$, Theorem 2.1 is contained in [2]. In this case the conditions (i) and (ii) are also necessary for the validity of (2.3).

Next we state our assumptions on $\mathscr{L}$. Actually, in what follows we need not assume $\mathscr{L}$ is a differential operator at all. We therefore make certain assumptions about some linear operator L which will be satisfied, in particular, if L is a suitable realization in $W_q(\Omega)$ of a boundary value problem.

(B_q) $D(L)$ is a closed subspace of $W_{l+q}(\Omega)$ for some $l \leqq 2m$. L is a bounded linear operator from $V = D(L)$ into $W_q(\Omega)$ and $V \supset D(M)$.

Example 2.1. Suppose $\mathscr{L}(x; D)$ is a partial differential operator in Ω of order $\bar{l} \leqq 2m$ with $C^q(\bar{\Omega})$ coefficients. Let $\bar{l} \leqq l \leqq 2m$ and $\{B_{j_k}\}_{k=1}^s$ be a subsystem of $\{B_j\}_{j=1}^m$ such that $m_{j_k} \leqq l - 1$. Let V be a closed subspace of $W_{l+q}(\Omega)$ such that $W_{l+q}(\Omega; \{B_{j_k}\}) \subset V \subset W_{l+q}(\Omega)$ and define L as follows: $D(L) = V$ and for $u \in D(L)$, $Lu = \mathscr{L}(x; D)u$. Then L satisfies hypothesis (B_q).

We now proceed to discuss solutions of the equation

$$(1 - \gamma M)u'(t) - Lu(t) = f(t), \tag{2.4}$$

where $f(t)$ is a given $W_q(\Omega)$-valued function.

Definition 2.1. A solution of (2.4) on an interval I is a strongly continuously differentiable function $u: I \to V$ such that $u'(t) \in D(M)$ and (2.4) holds for all t in I.

If L is the realization in $W_q(\Omega)$ of the boundary value problem $(\mathscr{L}(x; D), V)$ given in Example 2.1, and if u is a solution of (2.4) on I, then u will be called a *solution* of the boundary value problem

$$(1 - \gamma \mathscr{M}(x; D)) \frac{\partial u}{\partial t} - \mathscr{L}(x; D)u = f(x, t), \qquad (x, t) \in \Omega \times I, \tag{2.4'}$$

$$u \in V, \qquad t \in I, \tag{2.5}$$

$$B_j(x; D) \frac{\partial u}{\partial t} = 0, \quad (x, t) \in \partial\Omega \times I, \qquad j = 1, \cdots, m. \tag{2.6}$$

Example 2.2. By choosing $l = 2m$ and $V = W_{2m+q}(\Omega; \{B_j\})$ in Example 2.1, the boundary conditions (2.5) become $B_j(x; D)u = 0, j = 1, 2, \cdots, m$. The boundary conditions (2.6) are therefore a consequence of (2.5) in this case. At the opposite extreme we take $l = \bar{l}$ and $V = W_{l+q}(\Omega)$. Then no boundary conditions are imposed by (2.5). Since V will be the space of initial conditions for solutions of (2.4), in certain situations it may be desirable to have V as large as possible.

We denote by $\chi(M)$ the discrete set of characteristic numbers of M. Thus $\gamma \in \chi(M)$ if and only if γ^{-1} is in the spectrum of M.

THEOREM 2.2. *For any given* $u_0 \in V$, $\gamma \notin \chi(M)$ *and* $t_0 \in [s, T]$, *equation* (2.4) *has at most one solution* u *on* (s, T) *such that*

$$\lim_{t \to t_0} u(t) = u_0. \tag{2.7}$$

THEOREM 2.3. *Let* f *be a continuous function from an interval* $[s, T]$ *into* $W_q(\Omega)$. *For any given* $u_0 \in V$, $\gamma \notin \chi(M)$ *and* $t_0 \in [s, T]$, *equation* (2.4) *has a unique solution* u *on* $[s, T]$ *satisfying* (2.7).

Theorems 2.2 and 2.3 are a consequence of the following lemma.

LEMMA 2.1. *For any* $\gamma \notin \chi(M)$, $(1 - \gamma M)^{-1}L$ *is a bounded linear operator on* V.

Proof. Theorem 2.1 and hypothesis (B_q) shows that $(1 - \gamma M)^{-1}L$ is a well-defined linear operator from V into itself for each $\gamma \notin \chi(M)$. Moreover, for each $u \in V$ we have from the estimates (2.2):

$$\|(1 - \gamma M)^{-1}Lu\|_{2m+q} \leqq \frac{C}{|\gamma|}\|Lu\|_q \leqq \frac{C}{|\gamma|}\|u\|_{l+q}.$$

The lemma follows from this last inequality since $l \leqq 2m$.

We now form the group $\{e^{tA} : -\infty < t < +\infty\}$ of bounded linear operators in V, where $A = (1 - \gamma M)^{-1}L$. It is well known that e^{tA} is infinitely differentiable with respect to t on $(-\infty, \infty)$ in the uniform operator topology and

$$\left(\frac{d}{dt}\right)^n e^{tA} = A^n e^{tA}.$$

Moreover, if $\gamma \notin \chi(M)$, equation (2.4) is equivalent to

$$u'(t) = (1 - \gamma M)^{-1}Lu(t) + (1 - \gamma M)^{-1}f(t), \tag{2.8}$$

which in turn is equivalent to

$$u(t) = e^{(t-t_0)A}u_0 + \int_{t_0}^{t} e^{(t-\sigma)A}(1 - \gamma M)^{-1}f(\sigma)\,d\sigma,$$

whenever $f(\sigma)$ is continuous on $[s, T]$ and (2.7) holds. Theorems 2.2 and 2.3 follow easily.

We next examine the regularity of solutions of (2.4). Suppose that $f(t)$ has derivatives to order k on (s, T) as a function in $W_q(\Omega)$ and $u(t)$ is a solution of (2.4) on (s, T) for some fixed $\gamma \notin \chi(M)$. Since $(1 - \gamma M)^{-1}$ (resp., $(1 - \gamma M)^{-1}L$) is a bounded linear operator from $W_q(\Omega)$ (resp., V) into $D(M)$, it follows from (2.8), by forming difference quotients, that $u'(t)$ has derivatives to order $k + 1$ in $D(M)$ and that

$$u^{(j+1)}(t) = (1 - \gamma M)^{-1}Lu^{(j)}(t) + (1 - \gamma M)^{-1}f^{(j)}(t), \qquad j = 0, 1, \cdots, k.$$

In particular, $u'(t)$ is continuous on (s, T) as a function in $D(M)$. Writing

$$u(t+h) - u(t) = \int_t^{t+h} u'(t)\,dt, \qquad s < t < t+h < T, \tag{2.9}$$

where the integral is a Riemann integral in $D(M)$, we conclude that $h^{-1}[u(t+h) - u(t)]$ lies in $D(M)$ and converges to $u'(t)$ in $D(M)$. We summarize the above discussion in the following theorem.

THEOREM 2.4. *For any fixed $\gamma \notin \chi(M)$, let $u(t)$ be a solution of (2.4) on (s, T). If $f(t)$ has continuous derivatives on (s, T) to order k as a function in $W_q(\Omega)$, then $u(t)$ has continuous derivatives on (s, T) to order $k+1$ as a function in $W_{2m+q}(\Omega; \{B_j\})$.*

We note from (2.9) that if $u(t)$ is a solution of (2.4) on (s, T) satisfying (2.7) and if $u_0 \in D(M)$, then $u(t)$ is also in $D(M)$ for $s \leqq t < T$. In particular, this implies that the group $\{e^{tA}: -\infty < t < +\infty\}$ leaves invariant $W_{2m+q}(\Omega; \{B_j\})$. This same conclusion could also be reached by noting that $(1 - \gamma M)^{-1}L$ is a bounded linear operator on $W_{2m+q}(\Omega; \{B_j\})$.

Let $C^{r,k}(\bar{\Omega} \times (s, T))$ denote the class of functions $u(x, t)$ defined and having continuous partial derivatives $D_x^\alpha D_t^j u(x, t)$ on $\bar{\Omega} \times (s, T)$ for $|\alpha| \leqq r, 0 \leqq j \leqq k$. By employing Theorem 2.4 in conjunction with the Sobolev imbedding theorem, we obtain the following result concerning differentiability in the classical sense of solutions of (2.4).

THEOREM 2.5. *With the hypotheses of Theorem 2.4, let $u(t) = u(\cdot, t)$ be a solution of (2.4) on (s, T) and suppose $2m + q > n/p$. Then $u \in C^{2m+q-[n/p]-1,k+1}(\bar{\Omega} \times (s, T))$ after correction on a set of measure zero.*

COROLLARY. *Let $u(x, t)$ be a solution of (2.4′) satisfying the boundary conditions*

$$B_j(x; D)u(x, t) = 0, \qquad (x, t) \in \partial\Omega \times I, \qquad j = 1, \cdots, m$$

(that is, $V = W_{2m+q}(\Omega; \{B_j\})$). Suppose Ω is of class C^∞, the coefficients of $\mathscr{L}$ and $\mathscr{M}$ are of class $C^\infty(\bar{\Omega})$ and the coefficients of B_j are of class $C^\infty(\partial\Omega)$. Then $u \in C^\infty(\bar{\Omega} \times I)$ whenever $f \in C^\infty(\bar{\Omega} \times I)$.

3. The singular points. In this section we suppose γ is a characteristic number of M. Since the spectrum of M is discrete we may suppose without loss that zero is in the resolvent set of M, so that γ is an eigenvalue of M^{-1}. We make the simplifying assumptions that $L = M$, $p = 2$ and $q = 0$. Thus we are concerned with the problem

$$(1 - \gamma M)u'(t) - Mu(t) = f(t), \qquad u(0) = u_0 \in V, \tag{3.1}$$

for a function $u \in C'(R, V)$, where $R = (-\infty, \infty)$, $V = W_{2m,2}(\Omega; \{B_j\})$ and f is a continuous function from R to $L_2(\Omega)$. (We could, of course, treat (3.1) on any interval.) By setting $v(t) = Mu(t)$, (3.1) is clearly equivalent to

$$(M^{-1} - \gamma)v'(t) - v(t) = f(t), \qquad v(0) = Mu_0 \in L_2(\Omega), \tag{3.2}$$

for a function $v \in C'(R, L_2(\Omega))$, where we consider M^{-1} as a compact operator in $L_2(\Omega)$. We shall use $(\cdot, \cdot)$ to denote the inner product in $L_2(\Omega): (u, v) = \int_\Omega u\bar{v}\,dx$.

In this section we first derive necessary conditions, in terms of $f(t)$, u_0 and the root vectors of M corresponding to γ^{-1}, in order that (3.2) have a solution. We then show that solutions of (3.2) are uniquely determined by $f(t)$ and u_0 if the system of root subspaces of M form a basis for $L_2(\Omega)$. Finally we prove that the necessary conditions are also sufficient in order that (3.2) have a solution provided the system of root subspaces form an unconditional basis for $L_2(\Omega)$ and only a finite number of the root subspaces contain root vectors which are not eigenvectors.

It may be helpful at this point to recall a few facts concerning root vectors. For a more complete discussion particularly relevant to this paper we refer the reader to V. B. Lidskii [6, Chap. 2].

If λ is an eigenvalue of M, a nonzero vector ϕ is called a *root vector of M corresponding to* λ if for some positive integer k, $(\lambda - M)^k\phi = 0$. (One assumes, of course, that $\phi^{(1)} = (\lambda - M)\phi$, $\phi^{(2)} = (\lambda - M)\phi^{(1)}$, and so on, all belong to $D(M)$.) The set of all root vectors of M corresponding to λ spans a (root) subspace of $L_2(\Omega)$ which we denote by $R(\lambda)$. Clearly ϕ is a root vector of M corresponding to λ if and only if ϕ is a root vector of the compact operator M^{-1} corresponding to the eigenvalue $\mu = \lambda^{-1}$; the root subspace $R(\lambda) = R(\mu)$ is therefore finite-dimensional. One can choose a basis in $R(\mu)$ consisting of Jordan chains (a Jordan basis) of eigenvectors and root vectors of the operator $M^{-1}|_{R(\mu)}$. Each such chain $\phi^{(1)}$, $\phi^{(2)}, \cdots, \phi^{(p)}$ is a Jordan chain for M^{-1} and is transformed by M^{-1} according to the formulas

$$M^{-1}\phi^{(1)} = \mu\phi^{(1)}, \quad M^{-1}\phi^{(2)} = \mu\phi^{(2)} + \phi^{(1)}, \quad \cdots, \quad M^{-1}\phi^{(p)} = \mu\phi^{(p)} + \phi^{(p-1)}. \tag{3.3}$$

Consider the sequence $\{\gamma_i\}$ of all distinct eigenvalues of M^{-1} and choose a Jordan basis in each root subspace. In this way we obtain a certain sequence $\{\phi_k\}$ in $L_2(\Omega)$. This sequence determines in a unique way another sequence $\{\psi_j\}$, consisting of eigenvectors and root vectors of $(M^{-1})^*$, which together with $\{\phi_k\}$ forms a biorthonormal system, that is, $(\phi_k, \psi_j) = \delta_{kj}$. More precisely, if $\phi^{(1)}, \cdots, \phi^{(p)}$ is a Jordan chain in the system $\{\phi_k\}$ which transforms according to (3.3), then the vectors $\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(p)}$, having the same indices in the system $\{\psi_i\}$ as the vectors $\phi^{(i)}$ have in the system $\{\phi_k\}$, transform according to

$$\begin{aligned}(M^{-1})^*\psi^{(p)} = \bar{\mu}\psi^{(p)}, \quad (M^{-1})^*\psi^{(p-1)} = \bar{\mu}\psi^{(p-1)} + \psi^{(p)}, \quad \cdots,\\ (M^{-1})^*\psi^{(1)} = \bar{\mu}\psi^{(1)} + \psi^{(2)}.\end{aligned} \tag{3.4}$$

Therefore the matrix $[(M^{-1}\phi^{(i)}, \psi^{(j)})]_1^p$ has Jordan form, having μ at each diagonal element and, if $p > 1$, 1 at each subdiagonal element and zeros elsewhere. Thus if $\{\phi_k\}_{n_i+1}^{n_{i+1}}$ $(n_1 = 0)$ is the Jordan basis for $R(\gamma_i)$ and $\{\psi_k\}_{n_i+1}^{n_{i+1}}$ the corresponding dual Jordan basis in $R^*(\bar{\gamma}_i)$ (the root subspace of $(M^{-1})^*$ corresponding to the eigenvalue $\bar{\gamma}_i$), the matrix

$$[(M^{-1}\phi_k, \psi_j)]_{n_i+1}^{n_{i+1}}$$

has Jordan form, having γ_i at each diagonal element, either 1 or 0 at each subdiagonal element and zeros elsewhere. In particular, the above matrix is diagonal if $R(\gamma_i)$ contains only eigenvectors.

Finally, we note that if a vector f in $L_2(\Omega)$ can be written as a strongly convergent series

$$f = \sum_{k=1}^{\infty} c_k \phi_k,$$

then by virtue of the existence of the dual system $\{\psi_j\}$ the coefficients c_k are uniquely determined: $c_k = (f, \psi_k)$.

We return to the problem (3.1) and derive necessary conditions for it to have a solution. Let $\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(p)}$ be a Jordan chain in the root subspace $R^*(\bar{\gamma})$ which transforms according to (3.4) with $\mu = \gamma$. If we form the inner product of (3.2) with $\psi^{(i)}$, we obtain

$$\frac{d}{dt}(v(t), \psi^{(i+1)}) - (v(t), \psi^{(i)}) = (f(t), \psi^{(i)}), \qquad i = 1, \cdots, p-1, \tag{3.5}$$

$$-(v(t), \psi^{(p)}) = (f(t), \psi^{(p)}). \tag{3.6}$$

We therefore see that $(f(t), \psi^{(p)})$ must be continuously differentiable and, setting $t = 0$ in (3.6), that

$$-(Mu_0, \psi^{(p)}) = (f(0), \psi^{(p)}).$$

Similarly, from (3.5) and (3.6) we find that

$$(f(t), \psi^{(p-1)}) + \frac{d}{dt}(f(t), \psi^{(p)})$$

must be continuously differentiable and

$$-(Mu_0, \psi^{(p-1)}) = (f(0), \psi^{(p-1)}) + \frac{d}{dt}(f(t), \psi^{(p)})\bigg|_{t=0}.$$

Proceeding step-by-step we obtain the following theorem.

THEOREM 3.1. *In order that* (3.1) *have a solution, the following conditions are necessary: For each Jordan chain* $\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(p)}$ *in the Jordan basis for* $R^*(\bar{\gamma})$, *transforming according to* (3.4) *with* $\mu = \gamma$, *the functions defined recursively by*

$$G^{(p)}(t) = (f(t), \psi^{(p)}), \tag{3.7}$$

$$G^{(p-i)}(t) = \frac{d}{dt} G^{(p-i+1)}(t) + (f(t), \psi^{(p-i)}), \qquad i = 1, \cdots, p-1, \tag{3.8}$$

are continuously differentiable and

$$(Mu_0, \psi^{(i)}) = -G_i(0), \qquad i = 1, 2, \cdots, p. \tag{3.9}$$

We remark separately on the case when M, which is a closed, densely defined operator in $L_2(\Omega)$, is normal: $MM^* = M^*M$. Then M^{-1} is a compact normal operator in $L_2(\Omega)$, its root subspaces therefore contain only eigenvectors and the eigenspaces $E(\gamma) = R(\gamma)$ and $E^*(\bar{\gamma}) = R^*(\bar{\gamma})$ coincide. Consequently Theorem 3.1 implies the following theorem.

THEOREM 3.2. *Suppose M is a normal operator in* $L_2(\Omega)$. *In order that* (3.1) *have a solution it is necessary and sufficient that* $(f(t), \phi)$ *be continuously differentiable* *for each* $\phi \in E(\gamma)$ *and* $Mu_0 + f(0)$ *be orthogonal in* $L_2(\Omega)$ *to* $E(\gamma)$. *The solution is unique whenever it exists.*

The statements concerning sufficiency of the conditions and uniqueness of solution are a consequence of Theorems 3.3 and 3.4 below. If $M = M^*$, Theorem 3.2 was essentially obtained in [9].

We proceed to the question of uniqueness of solution. Let $\{R(\gamma_i)\}$ be the system of root subspaces of M arranged in some order. This system is called a *basis* for $L_2(\Omega)$ if each $f \in L_2(\Omega)$ can be written as a strongly convergent series

$$f = \sum_{i=1}^{\infty} f_i, \qquad f_i \in R(\gamma_i). \tag{3.10}$$

This implies, in particular, that the set of characteristic numbers is infinite. If $\{\phi_k\}_{n_i+1}^{n_{i+1}}$ $(n_1 = 0)$ is a Jordan basis in $R(\gamma_i)$ and

$$f_i = \sum_{k=n_i+1}^{n_{i+1}} c_k \phi_k,$$

then $c_k = (f_i, \psi_k)$, where $\{\psi_k\}_{n_i+1}^{n_{i+1}}$ is the dual Jordan basis in $R^*(\bar{\gamma})$. Since ψ_k is orthogonal to all the root subspaces $R(\gamma_j)$ with $j \neq i$ we have $(f, \psi_k) = (f_i, \psi_k) = c_k$. The representation (3.10) is therefore unique.

The system $\{R(\gamma_i)\}$ is an *unconditional basis* for $L_2(\Omega)$ if it remains a basis under any permutation of its constituent subspaces. It is known (see, for example, I. C. Gohberg and M. G. Krein [4, pp. 335–336]) that this is equivalent to the following statement: There is a bounded, invertible linear operator on $L_2(\Omega)$ which transforms $\{R(\gamma_i)\}$ into an orthogonal basis for $L_2(\Omega)$.

THEOREM 3.3. *If some arrangement of the system* $\{R(\gamma_i)\}$ *of root subspaces of* M *forms a basis for* $L_2(\Omega)$, *the problem* (3.2) *has at most one solution.*

Proof. Let the characteristic numbers of M be indexed so that $\{R(\gamma_i)\}_1^\infty$ is a basis for $L_2(\Omega)$. We may assume without loss that $\gamma_1 = \gamma$. If $v(t)$ is a solution of (3.2), then

$$\begin{aligned} v(t) &= \sum_{i=1}^{\infty} \left(\sum_{k=n_i+1}^{n_{i+1}} v_k(t)\phi_k \right), \\ v(0) &= \sum_{i=1}^{\infty} \left(\sum_{k=n_i+1}^{n_{i+1}} \mathring{v}_k \phi_k \right), \end{aligned} \tag{3.11}$$

where $\{\phi_k\}_{n_i+1}^{n_{i+1}}$ is a Jordan basis for $R(\gamma_i)$ with corresponding dual Jordan basis $\{\psi_k\}_{n_i+1}^{n_{i+1}}$ for $R^*(\bar{\gamma}_i)$ and

$$v_k(t) = (v(t), \psi_k), \qquad \mathring{v}_k = (Mu_0, \psi_k).$$

A similar series representation is valid for $v'(t)$:

$$v'(t) = \sum_{i=1}^{\infty} \left(\sum_{k=n_i+1}^{n_{i+1}} w_k(t)\psi_k \right), \qquad w_k(t) = (v'(t), \psi_k) = v_k'(t). \tag{3.12}$$

We have to show that the functions $\{v_k(t)\}_1^\infty$ are uniquely determined by $f(t)$ and the initial condition Mu_0. Substituting (3.11) and (3.12) in (3.2) gives

$$\sum_{i=1}^{\infty} \sum_{k=n_i+1}^{n_{i+1}} [v_k'(t)(M^{-1} - \gamma)\phi_k - v_k(t)\phi_k] = \sum_{i=1}^{\infty} \sum_{k=n_i+1}^{n_{i+1}} f_k(t)\phi_k, \tag{3.13}$$

where $f_k(t) = (f(t), \psi_k)$. Each subspace $R(\gamma_i)$ is invariant under $M^{-1} - \gamma$. Consider first $R(\gamma_1) = R(\gamma)$ and let $\phi^{(1)}, \phi^{(2)}, \cdots, \phi^{(p)}$ be a Jordan chain in its Jordan basis. If $v^{(1)}(t), v^{(2)}(t), \cdots, v^{(p)}(t)$ are the corresponding coefficients in the expansion (3.11), we have from (3.5) and (3.6),

$$v^{(i)}(t) = -G^{(i)}(t), \qquad i = 1, 2, \cdots, p, \tag{3.14}$$

where the functions $G^{(i)}(t)$ are defined by (3.7) and (3.8). The coefficients $v^{(1)}, \cdots, v^{(p)}$ are therefore uniquely determined by $f(t)$. Since the basis for $R(\gamma_1)$ consists of a finite number of such Jordan chains, it follows that all the coefficients $\{v_k\}_1^{n_2}$ are uniquely determined by $f(t)$.

Consider next a subspace $R(\gamma_i)$ with $i \neq 1$. This subspace is invariant under $M^{-1} - \gamma$ so that

$$(M^{-1} - \gamma)\phi_k = \sum_{j=n_i+1}^{n_{i+1}} c_{kj}\phi_j, \qquad k = n_i + 1, \cdots, n_{i+1},$$

where

$$c_{kj} = (M^{-1}\phi_k, \psi_j) - \gamma\delta_{kj}.$$

Hence from (3.13),

$$\sum_{i=1}^{\infty} \sum_{j=n_i+1}^{n_{i+1}} \left\{ \sum_{k=n_i+1}^{n_{i+1}} [(M^{-1}\phi_k, \psi_j) - \gamma\delta_{kj}]v_k'(t) - v_j(t) - f_j(t) \right\} \phi_j = 0,$$

and therefore,

$$\sum_{k=n_i+1}^{n_{i+1}} [(M^{-1}\phi_k, \psi_j) - \gamma\delta_{kj}]v_k'(t) - v_j(t) - f_j(t) = 0, \quad j = n_i + 1, \cdots, n_{i+1}. \tag{3.15}$$

The matrix $[(M^{-1}\phi_k, \psi_j) - \gamma\delta_{kj}]_{n_i+1}^{n_{i+1}}$ has Jordan form, having $\gamma_i - \gamma$ at each diagonal element, 1 or zero at each subdiagonal element and zeros elsewhere. The matrix is therefore nonsingular which in turn implies that the system (3.15) has a unique solution $\{v_k(t)\}_{n_i+1}^{n_{i+1}}$ subject to the initial conditions

$$v_k(0) = (Mu_0, \psi_k). \tag{3.16}$$

The coefficients in the expansion (3.11) are therefore uniquely determined by u_0 and $f(t)$, proving the theorem.

Before discussing existence of solutions to (3.1) we comment on the basis condition of Theorem 3.3. If M is a self-adjoint or normal operator, then of course the root subspaces form an unconditional basis for $L_2(\Omega)$. This same result is true for various classes of nonnormal operators (see, for example, [4, Chap. 6] and V. N. Vizitei and A. S. Markus [15]). This is true, in particular, if M is a dissipative operator (that is, $\operatorname{Im}(Mu, u) \leqq 0$ for $u \in D(M)$) such that the closed linear hull of its root subspaces equals $L_2(\Omega)$ and whose characteristic numbers $\{\gamma_i\}_1^\infty$ satisfy

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{\infty} m_i m_j \frac{\operatorname{Im}\gamma_i \operatorname{Im}\gamma_j}{|\gamma_i - \bar{\gamma}_j|^2} < \infty$$

and

$$\limsup_{\substack{i,j\to\infty\\ i\neq j}} (m_i - 1)\frac{\operatorname{Im}\gamma_i}{|\gamma_i - \bar{\gamma}_j|^2} < \frac{1}{2\sqrt{2}},$$

where m_i is the smallest positive integer for which $(M^{-1} - \gamma_i)^{m_i}R(\gamma_i) = 0$. (See A. S. Markus [8] and [4, Chap. 6].) If $m_i = 1$ for all $i \geqq i_0$, that is, if all but a finite number of the $R(\gamma_i)$ contain only eigenvectors, these conditions reduce to

$$\sum_{\substack{i,j=1\\ i\neq j}}^{\infty} \frac{\operatorname{Im}\gamma_i \operatorname{Im}\gamma_j}{|\gamma_j - \bar{\gamma}_k|^2} < \infty .$$

The condition that the closed linear hull of the root subspaces equals $L_2(\Omega)$ is known to hold under mild conditions on the regular elliptic problem $(\mathscr{M}, \{B_j\}, \Omega)$ ([1, Thm. 3.2]). This result holds in particular if $(\mathscr{M}, \{B_j\}, \Omega)$ differs from a self-adjoint problem $(\tilde{\mathscr{M}}, \{\tilde{B}_j\}, \Omega)$ only in lower order terms, that is, $\mathscr{M}' = \tilde{\mathscr{M}}'$ and $B_j' = \tilde{B}_j'$. If in addition $2m > n$ and the order of $\mathscr{M} - \tilde{\mathscr{M}}$ is less than $2m - n$, the following stronger result is valid: The root subspaces form an *unconditional basis with parentheses* for $L_2(\Omega)$ ([15, Thm. 5.3]). This means there is a sequence $\{k_j\}_1^\infty$ of natural numbers $(k_1 = 0)$ such that the system of subspaces $\{R_j\}_1^\infty$ defined by

$$R_j = R(\gamma_{k_j+1}) \oplus \cdots \oplus R(\gamma_{k_{j+1}}), \qquad j = 1, 2, \cdots,$$

is an unconditional basis for $L_2(\Omega)$.

We now turn to the question of existence of solution of (3.1). Assuming that the necessary conditions of Theorem 3.1 and the basis condition of Theorem 3.3 are satisfied, we form the series (3.11) in which the coefficients $\{v_k\}_{n_i+1}^{n_{i+1}}$ are defined by (3.15) and (3.16) for $i > 1$ and, for $i = 1$, by (3.14) for each Jordan chain $\phi^{(1)}, \cdots, \phi^{(p)}$ in the Jordan basis for $R(\gamma_1)$. Then (3.11) will obviously be a solution of (3.2) provided the two series (3.11) and (3.12) converge uniformly on bounded t-intervals.

THEOREM 3.4. *Suppose the necessary conditions of Theorem* 3.1 *are satisfied and that*

(i) $\{R(\gamma_i)\}$ *is an unconditional basis for* $L_2(\Omega)$;

(ii) *all but a finite number of the* $R(\gamma_i)$ *contain only eigenvectors.*

Then the problem (3.1) *has a unique solution.*

Proof. We suppose $\gamma_1 = \gamma$ and arrange the other γ_i in order of decreasing magnitude, so that $\{\gamma_i\}$ converges to zero. By hypothesis (ii), there is a positive integer i_0 such that if $i \geqq i_0$ the matrix $[(M^{-1}\phi_k, \psi_j) - \gamma\delta_{kj}]_{n_i+1}^{n_{i+1}}$ has diagonal form, having $\gamma_i - \gamma$ at each diagonal element. Therefore the system (3.15) uncouples and becomes

$$(\gamma_i - \gamma)v_k'(t) - v_k(t) = f_k(t), \qquad k = n_i + 1, \cdots, n_{i+1}.$$

Thus for $i \geqq i_0$,

$$v_k(t) = (Mu_0, \psi_k)\exp[(\gamma_i - \gamma)^{-1}t] + \frac{1}{\gamma_i - \gamma}\int_0^t \exp[(\gamma_i - \gamma)^{-1}(t - s)](f(s), \psi_k)\, ds,$$

$$k = n_i + 1, \cdots, n_{i+1},$$

so that

$$V_i(t) \equiv \sum_{k=n_i+1}^{n_{i+1}} v_k(t)\phi_k = \exp[(\gamma_i - \gamma)^{-1}t]U_i + \frac{1}{\gamma_i - \gamma}\int_0^t \exp[(\gamma_i - \gamma)^{-1}(t-s)]F_i(s)\,ds,$$

where

$$U_i = \sum_{k=n_i+1}^{n_{i+1}} (Mu_0, \psi_k)\phi_k, \qquad F_i(t) = \sum_{k=n_i+1}^{n_{i+1}} (f(t), \psi_k)\phi_k.$$

Therefore,

$$\|V_i(t)\|_0 \leqq C(t)\left[\|U_i\|_0 + \int_0^t \|F_i(s)\|_0\,ds\right], \qquad i \geqq i_0, \tag{3.17}$$

where $C(t)$ is bounded on bounded intervals (here we use the fact that $\gamma_i \to 0$).

We have

$$Mu_0 = \sum_{i=1}^{\infty} U_i, \qquad f(t) = \sum_{i=1}^{\infty} F_i(t). \tag{3.18}$$

By hypothesis (i), there is a bounded, invertible linear operator A mapping $L_2(\Omega)$ onto $L_2(\Omega)$ such that $\{AR(\gamma_i)\}_1^\infty$ is an orthogonal basis for $L_2(\Omega)$. It follows from (3.18) that

$$\|A(Mu_0)\|_0^2 = \sum_{i=1}^{\infty} \|AU_i\|_0^2, \qquad \|Af(t)\|_0^2 = \sum_{i=1}^{\infty} \|AF_i(t)\|_0^2,$$

and since A has a bounded inverse,

$$\sum_{i=1}^{\infty} \|U_i\|_0^2 < \infty, \qquad \sum_{i=1}^{\infty} \|F_i(t)\|_0^2 < \infty. \tag{3.19}$$

Since each $F_i(t)$ is continuous we can apply Dini's theorem to deduce that the second series in (3.19) converges uniformly on compact intervals. It then follows from (3.17) that $\sum V_i(t)$ converges for each t, uniformly on bounded intervals.

Since

$$V_i'(t) = \frac{1}{\gamma_i - \gamma}[V_i(t) + F_i(t)], \qquad i \geqq i_0,$$

and $\gamma_i - \gamma \to -\gamma$, $\sum V_i'(t)$ likewise converges uniformly on bounded t-intervals. The proof is therefore complete.

4. The nonstationary case. We suppose the coefficients in the differential operators $\mathscr{M}(x,t;D)$ and $\{B_j(x,t;D)\}_1^m$ are defined in the infinite cylinder $\bar{\Omega} \times (-\infty, \infty)$ and that $(\mathscr{M}, \{B_j\}, \Omega)$ is a regular elliptic boundary value problem satisfying the spectrum condition for each fixed t. We also make the following smoothness assumptions.

(A_{qr}) Ω is a bounded domain of class C^{2m+q}. The coefficients in $\mathscr{M}$ are of class $C^{q,r}(\bar{\Omega} \times (-\infty, \infty))$ and those in B_j of class $C^{2m+q-m_j,r}(\partial\Omega \times (-\infty, \infty))$.

As in § 2, let $M(t)$ be the realization in $W_q(\Omega)$ of the boundary value problem $(\mathscr{M}(x, t; D), \{B_j(x, t; D)\}_{j=1}^m)$. We further introduce a family $\{L(t): -\infty < t < +\infty\}$ of linear operators which satisfy the following hypotheses.

(B_{qr}) $D(L(t)) = V$ is a closed subspace of $W_{l+q}(\Omega)$, $l \leqq 2m$, which is independent of t, and $V \supset D(M(t))$. $L(t)$ is a bounded linear operator from V into $W_q(\Omega)$ having continuous t-derivatives to order r in the uniform operator topology.

Example 4.1. Let $\mathscr{L}(x, t; D)$ be a partial differential operator in x of the form

$$\mathscr{L}(x, t; D) = \sum_{|\alpha| \leqq \bar{l}} l_\alpha(x, t) D^\alpha$$

with coefficients of class $C^{q,r}(\bar{\Omega} \times (-\infty, \infty))$. Let $\bar{l} \leqq l \leqq 2m$ and $\{B_{j_k}\}_{k=1}^s$ be a subsystem of $\{B_j\}_{j=1}^m$ such that $m_{j_k} \leqq l - 1$ and the coefficients in B_{j_k} are independent of t. Let V be a closed subspace of $W_{l+q}(\Omega)$ such that $W_{l+q}(\Omega; \{B_{j_k}\}) \subset V \subset W_{l+q}(\Omega)$. Define $L(t)$ as follows: $D(L(t)) = V$ and for $u \in V = D(L(t))$, $L(t)u = \mathscr{L}(x, t; D)u$. Then $\{L(t): -\infty < t < +\infty\}$ satisfies (B_{qr}). In fact, the derivative $L^{(j)}(t)$ of $L(t)$ is the operator defined by $L^{(j)}(t)u = \mathscr{L}^{(j)}(x, t; D)u$ for $u \in V$, where $\mathscr{L}^{(j)}(x, t; D)$ is the differential operator obtained by differentiating the coefficients of $\mathscr{L}$ j-times with respect to t.

We now discuss solutions of the equation

$$(4.1) \qquad (1 - \gamma M(t))u'(t) - L(t)u(t) = F(t).$$

DEFINITION 4.1. A solution of (4.1) on an interval I is a strongly continuously differentiable function $u: I \to V$ such that $u'(t) \in D(M(t))$ and (4.1) holds for all t in I.

In the particular case when $L(t)$ is defined as in Example 4.1, a solution of (4.1) will be called a solution of the boundary value problem

$$(4.2) \qquad (1 - \gamma\mathscr{M}(x, t; D))\frac{\partial u}{\partial t} - \mathscr{L}(x, t; D)u = f(x, t), \qquad (x, t) \in \Omega \times I,$$

$$(4.3) \qquad u \in V,$$

$$(4.4) \qquad B_j(x, t; D)\frac{\partial u}{\partial t} = 0, \qquad (x, t) \in \partial\Omega \times I, \qquad j = 1, \cdots, m.$$

For any t-interval we denote by χ_I the set

$$\chi_I = \bigcup_{t \in I} \chi(M(t)).$$

Although each $\chi(M(t))$ is discrete, we cannot in general rule out the possibility that χ_I is the entire complex plane. However, if we suppose that conditions (i) and (ii) of Theorem 2.1 hold for each t and consider a bounded interval I, there is a sector $\Sigma_I: |\arg \lambda - \theta| < \delta$, $|\lambda| \geqq R$, in which the estimates (2.3) hold uniformly in $t \in I$, so that χ_I must lie entirely outside of Σ_I in this case.

THEOREM 4.1. *Let $I = [s, T]$. For any given $u_0 \in V$, $t_0 \in I$, and $\gamma \notin \chi_I$,* (4.1) *has at most one solution u on* int (I) *satisfying*

$$(4.5) \qquad \lim_{t \to t_0} u(t) = u_0.$$

THEOREM 4.2. *Let $I = [s, T]$ and let f be a continuous function from I into $W_q(\Omega)$. For any given $u_0 \in V$, $t_0 \in I$ and $\gamma \notin \chi_I$,* (4.1) *has a unique solution u on I satisfying* (4.5).

THEOREM 4.3. *For any interval I and any fixed $\gamma \notin \chi_I$, let $u(t)$ be a solution of* (4.1) *on I. If $f(t)$ has continuous derivatives on I to order k as a function in $W_q(\Omega)$, then $u(t)$ has continuous derivatives on I to order* $\min(r, k) + 1$ *as a function in $W_{2m+q}(\Omega)$.*

The existence of a classical solution of the boundary value problem (4.2)–(4.4) can now be deduced from Theorems 4.1–4.3 and the Sobolev imbedding theorem as in § 2 provided f, Ω and the coefficients of $\mathscr{M}$, $\mathscr{L}$ and $\{B_j\}$ are sufficiently smooth. In particular, if these functions are $C^\infty(\bar{\Omega} \times (s, T))$, then every solution of (4.2)–(4.4) is $C^\infty(\bar{\Omega} \times (s, T))$ after correction on a set of measure zero.

Denote by $\mathscr{L}(W_q(\Omega), W_{2m+q}(\Omega))$ the Banach space of bounded linear operators from $W_q(\Omega)$ into $W_{2m+q}(\Omega)$ with the uniform operator topology. Theorems 4.1–4.3 are a consequence of the following lemma.

LEMMA 4.1. *For each $\gamma \notin \chi_I$, the mapping $t \to (1 - \gamma M(t))^{-1}$ of I into $\mathscr{L}(W_q(\Omega), W_{2m+q}(\Omega))$ is r times continuously differentiable.*

This result is due to H. Tanabe ([14]; cf. [5], [13]) and is based on the inequalities (2.2). In fact, if $f \in W_q(\Omega)$ and $u(t) = (1 - \gamma M(t))^{-1} f$, for each t the derivative $w_l(t) = u^{(l)}(t)$ is, formally, the solution of the boundary value problem

$$(4.6)\quad (1 - \gamma\mathscr{M}(x, t; D))w_l(x, t) = -\sum_{k=0}^{l-1} \binom{l}{k} \mathscr{M}^{(l-k)}(x, t; D)u^{(k)}(x, t), \qquad x \in \Omega,$$

$$(4.7)\quad B_j(x; D)w_l(x, t) = -\sum_{k=0}^{l-1} \binom{l}{k} B_j^{(l-k)}(x, t; D)u^{(k)}(x, t), \quad x \in \partial\Omega, \qquad 1 \leqq j \leqq m,$$

where $\mathscr{M}^{(l-k)}(x, t; D)$ (resp., $B_j^{(l-k)}$) is the differential operator obtained from $\mathscr{M}$ (resp., B_j) by differentiating the coefficients $l - k$ times with respect to t. Lemma 4.1 is proved by induction. The passage from $l - 1$ to l is made by applying the estimates (2.2) to the difference $w_l(t) - h^{-1}[u^{(l-1)}(t + h) - u^{(l-1)}(t)]$, where, for each t, $w_l(t)$ is the unique solution of (4.6), (4.7).

We now can prove Theorems 4.1–4.3. We first note that because of Lemma 4.1 and hypothesis (B_{qr}), the mapping $t \to A_\gamma(t) = (1 - \gamma M(t))^{-1} L(t)$ from I into $\mathscr{L}(V, W_{2m+q}(\Omega))$ is r times continuously differentiable. In particular, this mapping is continuous in $\mathscr{L}(V, V)$ so the evolution operators $\{G(t, \tau) : (t, \tau) \in I \times I\}$ exist for the equation $u'(t) = A_\gamma(t)u(t)$ (see, for example, [3, pp. 134–136]). Thus for any $u_0 \in V$ and any continuous $W_q(\Omega)$-valued function $f(t)$ on I, the problem (4.1), (4.5) has the unique solution

$$u(t) = G(t, t_0)u_0 + \int_{t_0}^{t} G(t, \tau)(1 - \gamma M(\tau))^{-1} f(\tau)\, d\tau, \qquad t \in I.$$

Theorems 4.1 and 4.2 are thereby proved. To prove Theorem 4.3 write

$$u'(t) = A_\gamma(t)u(t) + (1 - \gamma M(t))^{-1} f(t).$$

Because of the differentiability properties of $A_\gamma(t)$, $(1 - \gamma M(t))^{-1}$ and $f(t)$, an easy argument shows that $u'(t)$ has $\min(r, k)$ continuous derivatives as a function in $W_{2m+q}(\Omega)$. Arguing as in § 2, we also can conclude that $h^{-1}(u(t + h) - u(t))$ lies in $W_{2m+q}(\Omega)$ and converges to $u'(t)$ in that space.

REFERENCES

[1] S. AGMON, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math., 15 (1962), pp. 119–147.

[2] M. S. AGRANOVIC AND M. I. VISIK, *Elliptic problems with a parameter and parabolic problems of general type*, Uspekhi Mat. Nauk, 19 (1964), pp. 53–61 = Russian Math. Surveys, 19 (1964), pp. 53–157.

[3] R. W. CARROLL, *Abstract Methods in Partial Differential Equations*, Harper and Row, New York, 1969.

[4] I. C. GOHBERG AND M. G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Translations of Mathematical Monographs, vol. 18, American Mathematical Society, Providence, R.I., 1969.

[5] J. E. LAGNESE, *On equations of evolution and parabolic equations of higher order in t*, J. Math. Anal. Appl., 32 (1970), pp. 15–37.

[6] V. B. LIDSKII, *Summability of series in the principal vectors of nonselfadjoint operators*, Trudy Moskov. Mat. Obshch., 11 (1962), pp. 3–35; English transl., Amer. Math. Soc. Transl., 40 (1964), pp. 193–228.

[7] J. L. LIONS AND E. MAGENES, *Problèmes aux limites non homogènes et applications*, vol. 1, Dunod, Paris, 1968.

[8] A. S. MARKUS, *A basis of root vectors of a dissipative operator*, Soviet Math. Dokl., 1 (1960), pp. 599–602.

[9] R. E. SHOWALTER, *Well-posed problems for a partial differential equation of order 2m + 1*, this Journal, 1 (1970), pp. 214–231.

[10] ———, *The Sobolev equation I*, J. Appl. Anal., to appear.

[11] R. E. SHOWALTER AND T. W. TING, *Pseudoparabolic partial differential equations*, this Journal, 1 (1970), pp. 1–26.

[12] S. L. SOBOLEV, *Some new problems in mathematical physics*, Izv. Akad. Nauk SSSR Ser. Mat., 18 (1954), pp. 3–50.

[13] H. TANABE, *On differentiability and analyticity of solutions of weighted elliptic boundary value problems*, Osaka J. Math., 2 (1965), pp. 163–190.

[14] ———, *On regularity of solutions of abstract differential equations of parabolic type in Banach space*, J. Math. Soc. Japan, 19 (1967), pp. 521–542.

[15] V. N. VIZITEI AND A. S. MARKUS, *On convergence of multiple expansions in eigenvectors and associated vectors of an operator bundle*, Mat. Sb., 66 (1965), pp. 287–320; English transl., Amer. Math. Soc. Transl., 87 (1970), pp. 187–228.

[16] M. I. VISIK, *The Cauchy problem for equations with operator coefficients; mixed boundary value problems for systems of differential equations and approximation methods for their solutions*, Mat. Sb., 39 (81) (1956), pp. 51–148; English transl., Amer. Math. Soc. Transl., 24 (1963), pp. 173–278.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ON THE EXISTENCE OF SIMILAR SOLUTIONS OF SOME BOUNDARY LAYER PROBLEMS*

PHILIP HARTMAN†

Abstract. The first part deals with the continuity and monotonicity of functions of the parameters determining the range of existence of solutions of the Falkner–Skan equation considered by Iglisch and Kemnitz (and by Hartman). It also deals with the continuous and monotonic dependence on parameters of the maximal solution and its derivative. In the second part, we give existence theorems and physically significant properties for solutions of some problems associated with the names of Pohlhausen and Stewartson. Some of these problems have also been treated by Hastings. The method depends on Tikhonov's fixed-point theorem and on the consideration of a boundary value problem for a nonlinear second order equation on $[0, \infty)$. Existence for this singular boundary value problem is obtained by Nagumo's method of sub- and supersolutions, and uniqueness by properties of principal solutions of disconjugate linear second order equations. The third part is concerned with more general boundary value problems on $[0, \infty)$ associated with systems of equations, some of which are of order two and some of order three.

Consider the boundary value problem involving the Falkner–Skan [2] differential equation

$$u''' + uu'' + \lambda(1 - u'^2) = 0, \tag{I.1}$$

and boundary conditions

$$u(0) = \alpha, \qquad u'(0) = \beta, \tag{I.2}$$

$$u'(\infty) = 1; \tag{I.3}$$

also the side conditions

$$0 \leqq \beta < u' < 1 \quad \text{for } t > 0, \tag{I.4}$$

$$u'' > 0 \quad \text{for } t > 0, \tag{I.5}$$

$$0 < u' < 1 \quad \text{for } t > 0. \tag{I.6}$$

This problem occurs in boundary layer theory. Questions of existence and nonexistence, of uniqueness and nonuniqueness, of the asymptotic behavior of solutions have been settled (Weyl, Iglisch, Iglisch and Grohne, Iglisch and Kemnitz, Coppel [1], Hartman; see [6, pp. 519–537] for a systematic exposition and bibliography).

The situation as to existence is as follows: if $\lambda > 0$, $0 \leqq \beta < 1$ or $\lambda = 0$, $0 < \beta < 1$, then (I.1)–(I.5) has a solution for all α; if $\lambda = 0$, $\beta = 0$ or $\lambda < 0$, $0 \leqq \beta < 1$, then there exists a constant $A(\beta, \lambda)$ with the property that (I.1)–(I.5) has a solution if and only if $\alpha \geqq A(\beta, \lambda)$. It is known [3] that $A(0, 0) < 0$. Hastings [8] has pointed out that it is not known if $A(0, \lambda)$ is continuous at $\lambda = 0$, but his results imply $A(0, \lambda) < 0$ for small $-\lambda > 0$.

In Part I, we examine the function $A(\beta, \lambda)$ and solutions of (I.1)–(I.5) for $\lambda < 0$. It is observed that $A(\beta, \lambda)$ is continuous for $\lambda < 0$ and $0 \leqq \beta < 1$, monotone

* Received by the editors September 29, 1970.

† Department of Mathematics, The Johns Hopkins University, Baltimore, Maryland 21218. This work was supported by the Air Force Office of Scientific Research under Contract F44620-67-C-0098.

with respect to λ, and satisfies, as $\lambda \to 0-$, $A(\beta, \lambda) \to A(0,0)$ or $-\infty$ according as $\beta = 0$ or $0 < \beta < 1$. It is shown that the maximal solution of (I.1)–(I.5) and its first derivative are monotone in λ for $\lambda \leqq 0$. It will remain undecided if $A(\beta, \lambda)$ is monotone in β but it will be verified that if $0 \leqq \beta_0 \leqq \beta < 1$, then (I.1)–(I.3) has a solution satisfying (I.6), but possibly not (I.4), when $\alpha \geqq A(\beta_0, \lambda)$.

The results of Part I are used to obtain some existence theorems for the problems of Polhausen and Stewartson involving the system of differential equations

(II.1) $$u''' + uu'' + \lambda(h - u'^2) = 0,$$

(II.2) $$h'' + \sigma uh' = 0,$$

the boundary conditions

(II.3) $$u(0) = \alpha, \quad u'(0) = \beta, \quad h(0) = a,$$

(II.4) $$u'(\infty) = h'(\infty) = 1,$$

and one or more of the side conditions

(II.5) $$u' > 0 \qquad \text{for } t > 0,$$

(II.6) $$\beta < u' < 1 \qquad \text{for } t > 0,$$

(II.7) $$u'' > 0 \qquad \text{for } t > 0,$$

(II.8) $$h - u'^2 > 0 \qquad \text{for } t > 0,$$

(II.9) $$1 < u' < \beta \qquad \text{for } t > 0,$$

(II.10) $$u'' < 0 \qquad \text{for } t \geqq 0,$$

(II.11) $$h - u'^2 < 0 \qquad \text{for } t > 0.$$

Existence for (II.1)–(II.5) has been settled for $\sigma > 0$, $\lambda > 0$, $a > 0$, $\beta \geqq 0$ and $-\infty < \alpha < \infty$ by Ho and Wilson [9] in some cases, and by McLeod and Serrin [15] in general; see Lan [14] for systematic proofs. Hastings [8] has dealt with the case of $\sigma = 1$, $\alpha = \beta = 0$, $a \geqq 1$, and sufficiently small $-\lambda > 0$ (say, $\lambda_0(a) \leqq \lambda < 0$), and has raised the question of existence in corresponding cases for $0 < a < 1$. In Part II, we obtain sufficient conditions on the parameters a, α, β, $\lambda(< 0)$, σ for existence. The main result (Theorem 7.1) implies that if $\sigma = 1$, $a \geqq 0$, and either $\beta \neq 0$ and α is arbitrary, or $\beta = 0$ and $\alpha > A(0, 0)$, then (II.1)–(II.5) has a solution for sufficiently small $-\lambda > 0$. (After the completion of this paper, Mr. Lan called my attention to [8a] in which Hastings has announced a corresponding existence theorem for the case $\alpha = 0$, which has been incorporated into [8].) The condition $\sigma = 1$ can be relaxed to $\sigma > 0$ if $\alpha \geqq 0$. If $a - \beta^2 \geqq 0$ and $\sigma = 1$, then the solution obtained satisfies (II.8) and if, in addition, $\beta < 1$, then (II.6)–(II.7) hold. If $\beta \geqq (a+1)/2$ and $\sigma = 1$, then the solution obtained satisfies (II.11) and if, in addition, $\beta > 1$, then (II.9)–(II.10) hold. For references to Pohlhausen, Stewartson, Cohen and Reshotko, Libby and Lin, and other hydrodynamical literature, see [8] and [15].

Hastings [8] and Lan [14] obtain their existence theorems by the use of fixed-point theorems, not unlike the methods in Hartman [4, pp. 506–509]. The methods to be employed below are similar. The idea of using a solution of (I.1)–(I.5) as a subsolution of certain second order equations is suggested by Hastings [8], and is extensively exploited here in the proof of all existence theorems. In the proof of the continuity of $A(0, \lambda)$ at $\lambda = 0$ and of the proof of Remark 4 (following Theorem 7.1 which contains [8]), we use the same linear functional equations as Hastings [8]. In the other existence theorems, we use a nonlinear functional equation. The ingredients of the proofs are the choice of subsolutions (just mentioned), construction of a supersolution, a corollary of a theorem of Nagumo [16] on two-point boundary conditions for a nonlinear second order equation (cf. Proposition A3.1 in the Appendix), and a simple uniqueness theorem for linear second order equations (cf. Proposition A2.2). The latter should have other useful applications.

In Part III, it is pointed out that a similar procedure leads to existence theorems for a more general class of problems.

PART I. THE FUNCTIONS A, γ AND u_0

1. Statement of results. We shall need the following theorem of Iglisch and Kemnitz [12] on $\beta = 0$ (see [6, pp. 525–534] for $0 \leqq \beta < 1$).

PROPOSITION 1.1. *If $\lambda < 0$ and $0 \leqq \beta < 1$, then there exists a number $A = A(\beta, \lambda)$ and a continuous increasing function $\gamma(\alpha)$ defined for $\alpha \geqq A$ such that $\gamma(A) = 0$ and* (I.1)–(I.4) *has a solution if and only if $\alpha \geqq A$; in which case* (I.5) *holds. The solution of the initial value problem* (I.1), (I.2) *and*

$$u''(0) = \gamma \tag{1.1}$$

is a solution of (I.1)–(I.4) *if and only if $0 \leqq \gamma \leqq \gamma(\alpha)$; so that there is uniqueness if and only if $\alpha = A(\beta, \lambda)$.*

Below we shall prove the following result concerning $A(\beta, \lambda)$.

THEOREM 1.1. *The function $A(\beta, \lambda)$ is continuous for $\lambda < 0$ and $0 \leqq \beta < 1$, is a decreasing function of λ, and as $\lambda \to 0-$,*

$$A(\beta, \lambda) \to -\infty \quad \textit{uniformly on compacts of } 0 < \beta < 1, \tag{1.2}$$

$$A(0, \lambda) \to A(0, 0). \tag{1.3}$$

The monotonicity of $A(0, \lambda)$ has been proved by Iglisch and Kemnitz [12]; the proof for $A(\beta, \lambda)$ will be similar.

We shall write $\gamma(\alpha) = \gamma(\alpha, \beta, \lambda)$ and denote by $u = u_0(t) = u_0(t, \alpha, \beta, \lambda)$ the solution of (I.1)–(I.5) satisfying $u''(0) = \gamma(\alpha)$. If $\alpha > A(\beta, \lambda)$, so that $\gamma(\alpha) > 0$, then different solutions $u = u(t)$ of (I.1)–(I.5), i.e., those having $0 \leqq u''(0) < \gamma(\alpha)$, satisfy

$$\alpha < u < u_0, \quad \beta < u' < u_0' \quad \text{for } t > 0; \tag{1.4}$$

cf. the proof of (3.9) in § 3. The solution $u = u_0(t)$ will be called the *maximal solution* of (I.1)–(I.5). It can also be characterized by its asymptotic behavior at $t = \infty$; Hartman [5] (see [6, pp. 534–537]). For $\lambda = 0, u_0(t) = u_0(t, \alpha, \beta, 0)$ will be the unique solution of (I.1)–(I.5) when it exists; correspondingly, $\gamma(\alpha, \beta, 0) = u_0''(0)$. Analogously, let $A(\beta, \lambda) = -\infty$ for $\lambda = 0, 0 < \beta < 1$. Define the sets

$$(1.5)\quad \begin{aligned} \Omega^0 &= \{(\alpha, \beta, \lambda) : 0 \leqq \beta < 1, \lambda < 0, \alpha \geqq A(\beta, \lambda)\}, \\ \Omega_0 &= \{(\alpha, \beta, \lambda) : 0 \leqq \beta < 1, \lambda = 0, \alpha > A(\beta, 0)\} \quad \text{and} \quad \Omega = \Omega^0 \cup \Omega_0. \end{aligned}$$

THEOREM 1.2. *The functions* $\gamma(\alpha, \beta, \lambda)$, $u_0(t, \alpha, \beta, \lambda)$, *and* $u_0'(t, \alpha, \beta, \lambda)$ *are continuous on* $\{t \geqq 0\} \times \Omega$ *and are strictly increasing functions of* α *and of* λ.

The monotonicity of u_0, u_0' with respect to λ, when $\beta = 0$, is contained in [12]. One cannot expect continuity, for example, at $(\alpha, \beta, \lambda) = (A(0,0), 0, 0)$ for $\alpha = A(0, \lambda) \to A(0, 0)$ and $0 = \gamma(A(0, \lambda), 0, \lambda) \to 0$ as $\lambda \to 0-$, but no solution of (I.1)–(I.5) for $\lambda = 0$ can have $u''(0) = 0$.

In this part of the paper, we shall also prove the following theorem.

THEOREM 1.3. *Let* $\lambda < 0$, $0 \leqq \beta_0 < \beta < 1$, $\alpha \geqq A(\beta_0, \lambda)$. *Then* (I.1)–(I.3) *has a solution* $u = u(t)$ *satisfying* $\beta_0 < u' < 1$ (*but not necessarily* $u'' > 0$) *for* $t > 0$; *also*, $1 - u' < 1 - u_0'(t, \alpha, \beta_0, \lambda)$ *for* $t \geqq 0$.

The proofs of Theorems 1.1 and 1.2 will be given together; the monotonicity statements in § 3, the continuity statements for $\lambda < 0$ in § 4, and the assertions concerning $\lambda \to 0-$ in § 5. Theorem 1.3 will be proved in § 6.

2. Preliminaries. The proofs of Proposition 1.1 depend on some changes of variables, due to Iglisch [10] and [11], which will also be used below. Introduce the new independent and dependent variables,

$$(2.1_1)\qquad u = u(t) \quad \text{and} \quad z(u) = u'^2(t)$$

along solutions of (I.1) for which $u' > 0$ on some interval $0 < t < T(\leqq \infty)$, so that if $\dot{z} = dz/du$,

$$(2.1_2)\qquad u' = z^{1/2}, \quad u'' = \dot{z}/2, \quad u''' = z^{1/2}\ddot{z}/2.$$

Thus (I.1)–(I.5) and (1.1) become, respectively,

$$(2.2)\qquad z^{1/2}\ddot{z} + u\dot{z} + 2\lambda(1 - z) = 0,$$

where $\dot{z} = dz/du$,

$$(2.3)\qquad z(\alpha) = \beta^2,$$

$$(2.4)\qquad z(\infty) = 1,$$

$$(2.5)\qquad \beta^2 < z < 1 \quad \text{for } \alpha < u < \infty,$$

$$(2.6)\qquad \dot{z} > 0 \quad \text{for } \alpha < u < \infty,$$

$$(2.7)\qquad \dot{z}(\alpha) = 2\gamma.$$

Along a solution on which $\dot{z} > 0$, we can (following Iglisch [11]) also introduce z as a new independent variable. Let $u = U(z)$ be the function inverse to $z = z(u)$ and let $V(z) = \dot{z}(U(z))$. Then (2.2) becomes the system

$$(2.8)\qquad d(-U)/dz = -1/V, \qquad dV/dz = -U/z^{1/2} - 2\lambda(1-z)/z^{1/2}V.$$

When we wish to consider varying λ, add the following differential equation to this system:

$$d(-\lambda)/dz = 0.$$

The system of differential equations for $(-U, V, -\lambda)$ has the property that the right sides of these three equations are, respectively, nondecreasing functions of V and $-\lambda$, of $-U$ and $-\lambda$, of $-U$ and V for $0 < z < 1$. Hence, it follows from a theorem of Kamke [13] (cf. [6, Exercise 4.1, p. 28 and p. 558]) that if $(-U_i(z), V_i(z), -\lambda_i)$ for $i = 1, 2$ are solutions satisfying

$$(2.9) \qquad U_1 \leqq U_2, \quad V_1 \geqq V_2, \quad \lambda_1 \leqq \lambda_2$$

and $V_2 > 0$ at some $z = z_0$, $0 < z_0 < 1$, then the same inequalities persist on their common domain of existence in $z_0 \leqq z < 1$.

This leads easily to the following analogues of propositions of Iglisch and Kemnitz [12] (cf. [6, Steps (d) and (c), pp. 526–528]).

PROPOSITION 2.1. *Let* $z = z_1(u) = z_1(u, \lambda_1)$, $z = z_2(u) = z_2(u, \lambda_2)$ *be two solutions of* (2.2), *where* $\lambda = \lambda_1, \lambda_2$, *satisfying*

$$z_2(\alpha_2) = z_1(\alpha_1) = \beta^2, \quad 0 \leqq \dot z_2(\alpha_2) \leqq \dot z_1(\alpha_1), \quad \alpha_1 \leqq \alpha_2, \quad \lambda_1 \leqq \lambda_2 < 0,$$

$0 \leqq \beta < 1$, *and at least one of the following inequalities hold*:

$$\alpha_1 < \alpha_2, \quad \dot z_i(\alpha_2) < \dot z_1(\alpha_1), \quad \lambda_1 < \lambda_2.$$

Then, as long as both solutions satisfy $\beta^2 < z < 1$ *with increasing* u, $U_2(z) - U_1(z)$ *is positive and increasing, and* $V_2(z) - V_1(z) < 0$. *In particular, the arcs* $z = z_1(u)$, $z = z_2(u)$ *in the* (u, z)*-plane do not intersect in the half-strip* $u > \alpha_2$, $\beta^2 < z < 1$.

Remark. Proposition 2.1 remains valid for $\lambda_2 = 0$ provided that

$$(2.10) \qquad \text{either } \beta > 0 \quad \text{or} \quad \dot z_2(\alpha_2) > 0.$$

(If (2.10) fails to hold and $\lambda_2 = 0$, then $z_2(u)$ can be $z_2(u) \equiv \beta$ and the assertion becomes meaningless.) In the verification of Proposition 2.1 and this remark, one avoids the singularities of (2.8), if any, by exploiting the relationship between (I.1) and (2.2), and examining the Taylor expansion of corresponding solutions of (I.1) at $t = 0$, in order to first obtain the desired inequalities for small $z - \beta^2 > 0$.

PROPOSITION 2.2. *Let* $z = z(u)$ *be a solution of* (2.1) *for large* u *satisfying* $0 < z(u) < 1$ *and* $\dot z(u) > 0$. *Then* $z(\infty) = 1$.

3. Proofs. (a) *Monotonicity of* A, α *with* $\lambda(<0)$. We first show that if $0 \leqq \beta < 1$, then

$$(3.1) \qquad A(\beta, \lambda_2) \leqq A(\beta, \lambda_1) \quad \text{if } \lambda_1 < \lambda_2 < 0.$$

To this end, let

$$(3.2) \qquad A(\beta, \lambda_1) \leqq \alpha.$$

Let $z = z_1(u)$, $z_2(u)$ be the solutions of (2.2), with $\lambda = \lambda_1, \lambda_2$, determined by the initial conditions

$$(3.3) \qquad z_1 = z_2 = \beta^2, \quad \dot z_1 = \dot z_2 = 2\gamma(\alpha, \beta, \lambda_1) \geqq 0 \quad \text{at } u = \alpha.$$

Then $\beta^2 < z_1 < 1$, $\dot{z}_1 > 0$ for $u > \alpha$ and $z_1(\infty) = 1$, by Proposition 2.2. It follows from Proposition 2.1, since a solution of (2.2) cannot have a maximum value z on $0 < z < 1$, that $z_2(u)$ exists for $u \geqq \alpha$,

$$\beta^2 < z_2 < z_1 < 1, \quad \dot{z}_2 > 0 \quad \text{for } u > \alpha. \tag{3.4}$$

Hence $z_2(\infty) = 1$ by Proposition 2.2 and so $\alpha \geqq A(\beta, \lambda_2)$. This proves (3.1).

Also, since $2\gamma(\alpha, \beta, \lambda_2) \geqq \dot{z}_2(\alpha)$, equation (3.3) implies that

$$\gamma(\alpha, \beta, \lambda_2) \geqq \gamma(\alpha, \beta, \lambda_1) \quad \text{if } \lambda_1 < \lambda_2 < 0. \tag{3.5}$$

We now show that strict inequality holds in (3.5). Suppose, if possible, that

$$\gamma(\alpha, \beta, \lambda_2) = \gamma(\alpha, \beta, \lambda_1) \tag{3.6}$$

for some $\alpha \geqq A(\beta, \lambda_1)$, $0 \leqq \beta < 1$, $\lambda_1 < \lambda_2 < 0$. Then z_2, as well as z_1, corresponds to a maximal solution, so that $2\gamma(u, z_2^{1/2}(u), \lambda_2) \equiv \dot{z}_2(u)$. The application of Proposition 2.1 giving (3.4) also yields $V_1(z) > V_2(z)$ for $\beta^2 < z < 1$ for the corresponding functions V defined after (2.6). Hence, if $\beta^2 < z_0 < 1$ and $z_1(U_1) = z_2(U_2) = z_0$, then $U_1 < U_2$ and $\dot{z}_1(U_1) > \dot{z}_2(U_2)$. A solution $z = z^1(u)$ of (2.2), with $\lambda = \lambda_1$, determined by

$$z^1(U_2) = z_0, \quad \dot{z}_1(U_1) \geqq \dot{z}^1(U_2) > \dot{z}_2(U_2)$$

exists for $u \geqq U_2$ and satisfies $\dot{z}^1 > 0$ for $u \geqq U_2$ and $z^1(\infty) = 1$; Propositions 2.1 and 2.2. Hence

$$\gamma(U_2, z_0^{1/2}, \lambda_1) > \dot{z}_2(U_2)/2 = \gamma(U_2, z_0^{1/2}, \lambda_2).$$

This contradicts (3.5) and shows that $\gamma(\alpha, \beta, \lambda)$ is an increasing function of $\lambda(<0)$.

In particular, if $\alpha = A(\beta, \lambda_1)$, then $\gamma(\alpha, \beta, \lambda_2) > 0 = \gamma(\alpha, \beta, \lambda_1)$. Hence, the strict monotonicity of $A(\beta, \lambda)$ follows, that is, $A(\beta, \lambda_2) < A(\beta, \lambda_1)$ if $\lambda_1 < \lambda_2 < 0$.

(b) *Monotonicity of* u_0, u_0' *with* $\lambda(<0)$. Let $\lambda_1 < \lambda_2 < 0$, $\alpha \geqq A(\beta, \lambda_1)$, and z_{10}, z_{20} correspond to maximal solutions, that is, solutions of (2.2) for $\lambda = \lambda_1, \lambda_2$, respectively, satisfying

$$z_{i0}(\alpha) = \beta^2, \qquad \dot{z}_{i0}(\alpha) = 2\gamma(\alpha, \beta, \lambda_i). \tag{3.7}$$

Then the argument leading to the contradiction of (3.6) shows that

$$z_{20}(u) > z_{10}(u) \quad \text{for } u > \alpha. \tag{3.8}$$

Note that the fact that $\gamma(\alpha, \beta, \lambda)$ is an increasing function of λ implies that

$$u_0(t, \alpha, \beta, \lambda_1) < u_0(t, \alpha, \beta, \lambda_2) \quad \text{if } \lambda_1 < \lambda_2 < 0 \tag{3.9}$$

for small $t > 0$. Suppose, if possible, that there is a first t-value $t = t_0 > 0$ where (3.9) fails, so that $u_{10}(t_0) = u_{20}(t_0)$ and $u_{10}'(t_0) \geqq u_{20}'(t_0)$, where $u_{i0} = u_0(t, \alpha, \beta, \lambda_i)$. But, in view of (2.1_1), this contradicts (3.8). Hence (3.9) holds for $t > 0$.

The strict monotonicity of u_0' with respect to λ follows from (3.8), the monotonicity of z_{10}, and (3.9), that is, from

$$z_{20}(u_{20}(t)) > z_{10}(u_{20}(t)) > z_{10}(u_{10}(t)),$$

by virtue of (2.1_1).

(c) *Monotonicity of* γ, u_0, u_0' *with* α (*for* $\lambda < 0$). The strict monotonicity of $\gamma(\alpha, \beta, \lambda)$ with α follows at once from Proposition 2.1 and the "maximality" property of γ. The monotonicity of u_0, u_0' with α follows from Proposition 2.1 and the arguments just used to demonstrate their monotonicity with λ.

(d) *Monotonicity of* A, γ, u_0, u_0' *on* Ω. It merely has to be remarked that the above arguments extend to $\lambda \geqq 0$, $\alpha > A(\beta, \lambda)$, by virtue of the remark following Proposition 2.1. (Actually, the meaningfulness of this statement depends implicitly on the validity of (1.3) which will be proved in § 5.)

4. Proofs. Continuity for $\lambda < 0$.

(A) *Local estimates.* Let $\lambda < 0$ and let $z = z^*(u) = z^*(u, \lambda)$ be the maximal solution for $\beta = 0$ and $\alpha = A(0, \lambda)$, so that

$$z^* = 0, \quad \dot{z}^* = 0 \quad \text{at } u = A(0, \lambda). \tag{4.1}$$

Thus $0 < z^* < 1$, $\dot{z}^* > 0$ for $u > A(0, \lambda)$, and $z^*(\infty) = 1$. Let $u = U^*(z) = U^*(z, \lambda)$ be the function inverse to $z = z^*(u)$ and $V^*(z) = V^*(z, \lambda) = \dot{z}^*(U^*(z))$. Clearly,

$$A(\beta, \lambda) \leqq U^*(\beta^2, \lambda) \quad \text{for } 0 \leqq \beta < 1, \tag{4.2}$$

for $z = z^*(u)$ satisfies (2.2)–(2.6) with $\alpha = U^*(\beta^2)$. Also,

$$0 \leqq 2\gamma(\alpha, \beta, \lambda) \leqq V^*(\beta^2, \lambda) \quad \text{for } A(\beta, \lambda) \leqq \alpha \leqq U^*(\beta^2, \lambda) \tag{4.3}$$

by Proposition 2.1 and the "maximal" property of $\gamma(\alpha) = \gamma(\alpha, \beta, \lambda)$.

(B) *Upper semicontinuity of* $A(\beta, \lambda)$. The continuity of $U^*(\beta, \lambda)$ with respect to β and the monotonicity of $A(\beta, \lambda)$ with respect to λ imply that $A(\beta, \lambda)$ is locally bounded from above. Let $\lambda_0 < 0$, $0 \leqq \beta_0 < 1$, and

$$\sigma = \liminf A(\beta, \lambda) \quad \text{as } (\beta_0, \lambda_0) \neq (\beta, \lambda) \to (\beta_0, \lambda_0),$$

with $\lambda < 0, 0 \leqq \beta < 1$, so that $-\infty \leqq \sigma < \infty$. Let $\alpha > \sigma$; in particular, $\alpha > A(\beta, \lambda)$ for (β, λ) a member of a sequence $(\beta_1, \lambda_1), (\beta_2, \lambda_2), \cdots, (\beta_n, \lambda_n) \to (\beta_0, \lambda_0)$. Let $u = u^n(t)$ be the solution of (I.1), with $\lambda = \lambda_n$, satisfying $u(0) = \alpha$, $u'(0) = \beta_n$, $u''(0) = 0$; so that $u = u^n(t)$ satisfies the analogue of (I.1)–(I.5) for $n = 1, 2, \cdots$, and u^n and its derivatives tend uniformly to u^0 and its derivatives on compacts of $t \geqq 0$. It follows from $\lambda_0 < 0$ that $0 < u^{0'} < 1$, $u^{0''} > 0$ on $t > 0$, and that $u^{0'}(\infty) = 1$, by Proposition 2.2. Hence $\alpha \geqq A(\beta_0, \lambda_0)$. Since $\alpha (> \sigma)$ is arbitrary,

$$A(\beta_0, \lambda_0) \leqq \liminf A(\beta, \lambda) < \infty \quad \text{as } (\beta, \lambda) \to (\beta_0, \lambda_0).$$

(C) *Continuity of* $A(\beta, \lambda)$ *for* $\lambda < 0$. Let $\lambda_0 < 0$ and $0 \leqq \beta_0 < 1$. It suffices to show that if $0 > \lambda_n \to \lambda_0$, $0 \leqq \beta_n \to \beta_0$ and

$$\alpha = \lim_{n\to\infty} A(\beta_n, \lambda_n) \quad \text{exists (finite)}, \tag{4.4}$$

as $n \to \infty$, then

$$\alpha = A(\beta_0, \lambda_0). \tag{4.5}$$

To this end, let $z = z^*(u, \beta, \lambda)$ be the solution of (2.2)–(2.7), where $\alpha = A(\beta, \lambda)$ and $\gamma = 0$. The arguments of step (B) show that, since $\lambda_0 < 0$, $z^*(u, \beta_n, \lambda_n)$ and its derivatives tend to $z_0(u)$ and its derivatives, where $z = z_0(u)$ is the solution of

(2.2), with $\lambda = \lambda_0$, satisfying $z_0(\alpha) = \beta_0^2$, $\dot{z}_0(\alpha) = 0$. It has to be verified that

(4.6) $$z_0(u) = z^*(u, \beta_0, \lambda_0).$$

Let $v_{0\lambda}(u)$ be the solution of the Weber equation (A1.1) satisfying (A1.2) and let $S(\lambda)$ be defined by (A1.4) in the Appendix. Then $z^*(u, \beta, \lambda)$ satisfies

(4.7) $$-\dot{z}^*/(1 - z^*) \leqq \dot{v}_{0\lambda}(u)/v_{0\lambda}(u) \quad \text{for } u \geqq S(\lambda);$$

cf. [6, pp. 536–537]. Hence, by the continuity of $S(\lambda)$,

$$-\dot{z}_0/(1 - z_0) \leqq \dot{v}_{0\lambda}/v_{0\lambda} \quad \text{for } u \geqq S(\lambda_0), \qquad \lambda = \lambda_0.$$

Consequently, $1 - z_0(u) \leqq (\text{const.})v_{0\lambda}(u)$, $\lambda = \lambda_0$, for $u \geqq S(\lambda_0)$. This asymptotic behavior of $z_0(u)$ implies (4.6); see [6, pp. 536–537].

(D) *Continuity of* γ, u_0, u_0' *for* $\lambda < 0$. The proof is similar to that in step (B), but simpler because of (4.3), and will be omitted.

5. Proofs. Behavior as $\lambda \to 0-$.

(α) *On* (1.2). If $\lambda_0 = 0$ and $0 < \beta_0 < 1$, the arguments in step (C) of the last proof remain valid and show that if (4.4) holds for $\lambda_0 = 0$ and $0 < \beta_0 < 1$, then (I.1)–(I.5) has a solution with $u''(0) = 0$, $\lambda = 0$. But $\lambda = 0$, $u''(0) = 0$ imply that $u''(t) \equiv 0$, since (I.1) reduces to a first order linear equation for u'' when $\lambda = 0$. Hence, (4.4) cannot hold when $0 > \lambda_n \to 0, 0 < \beta_n \to \beta_0, 0 < \beta_0 < 1$. In particular, it follows, from the monotonicity of $A(\beta, \lambda)$ with respect to λ, that the limit relation in (1.2) holds for a fixed β, $0 < \beta < 1$. The relation (1.2) now follows from the continuity of $A(\beta, \lambda)$ for $\lambda < 0, 0 \leqq \beta < 1$.

(β) *On* $A(0, 0-) \leqq A(0, 0)$. Let $\varepsilon > 0$ and $\alpha = A(0, 0) + \varepsilon$. We shall show that there exists a number $\Lambda = \Lambda(\varepsilon) < 0$ such that (I.1)–(I.5), with $\beta = 0$, has a solution if $\Lambda \leqq \lambda < 0$. This, of course, implies that $A(0, \lambda) \leqq A(0, 0) + \varepsilon$ if $\Lambda \leqq \lambda < 0$, so that $A(0, 0-) \leqq A(0, 0)$.

Let $u = u_0(t) = u_0(t, A(0, 0), 0, 0)$ be the solution of (I.1)–(I.5) for $\lambda = 0$, $\alpha = A(0, 0)$ and $\beta = 0$. Put

(5.1) $$w(t) = u_0(t) + \varepsilon.$$

Then we have

(5.2) $$w(0) = A(0, 0) + \varepsilon, \qquad w'(0) = 0,$$

(5.3) $$0 \leqq w' = u_0' < 1, \qquad w'' = u_0'' > 0,$$

(5.4) $$w''' + ww'' + \lambda(1 - w'^2) = \varepsilon u_0'' + \lambda(1 - u_0'^2).$$

Choose $-\Lambda = -\Lambda(\varepsilon) > 0$ so small that

(5.5) $$\varepsilon u_0'' + \lambda(1 - u_0'^2) > 0 \quad \text{for } \Lambda \leqq \lambda, \qquad t \geqq 0.$$

The existence of Λ is clear from the fact that $u_0'' > 0$ and $2u_0'' \sim t(1 - u_0'^2)$ as $t \to \infty$; see Hartman [5] (cf. [6, p. 536]).

Let $\mathscr{C}^1$ be the space of functions $u \in C^1[0, \infty)$ with the topology of C^1 convergence on compacts of $t \geqq 0$. In terms of $M(T) = M(T; T, |\lambda|, 1)$ in (A3.9) in the Appendix, let $\mathscr{K}$ be the compact convex subset of $\mathscr{C}^1$ consisting of functions u satisfying

$$u(0) = w(0), \quad u'(0) = w'(0), \quad w'(t) \leqq u'(t) \leqq 1 \quad \text{for } t \geqq 1,$$
$$(5.6) \qquad |u'(t) - u'(s)| \leqq M(T)(t-s) \quad \text{for } 0 \leqq s \leqq t \leqq T,$$

where $w(0) = A(0,0) + \varepsilon$, $w'(0) = 0$. For $u \in \mathscr{K}$, consider the linear, inhomogeneous equation

$$(5.7) \qquad \mathscr{L}[y] \equiv y'' + u(t)y' + \lambda(1 - u'(t))(1 + y) = 0.$$

Note that this reduces to (I.1) if $y = u'$.

The function $y = 1$ is a supersolution,

$$(5.8) \qquad \mathscr{L}(1) = 2\lambda(1 - u') \leqq 0,$$

and the function $y = w'$ is a subsolution if $\Lambda \leqq \lambda < 0$,

$$(5.9) \qquad \mathscr{L}[w'] \geqq (u - w)w'' + \varepsilon u_0'' + \lambda(1 - u_0'^2) > 0.$$

It follows that (5.7) has a unique solution $y = y(t)$ satisfying

$$(5.10) \qquad y(0) = A(0,0) + \varepsilon, \qquad w' < y < 1 \quad \text{for } t > 0,$$

$$(5.11) \qquad |y'(t)| \leqq M(T) \quad \text{for } 0 \leqq t \leqq T, \quad T > 0.$$

In fact, the homogeneous part of (5.7),

$$(5.12) \qquad z'' + u(t)z' + \lambda(1 - u'(t))z = 0,$$

is disconjugate (i.e., no solution $z \not\equiv 0$ has two zeros) on $t \geqq 0$ since the coefficient of γ is

$$(5.13) \qquad \lambda(1 - u') \leqq 0.$$

Thus (5.7) has a unique solution $y = y_T(t)$ satisfying arbitrarily given two-point boundary conditions, say,

$$y(0) = A(0,0) + \varepsilon \quad \text{and} \quad y(T) = (w'(T) + 1)/2.$$

Such a solution satisfies (5.11), and (5.10) for $0 < t < T$ by a simple maximum principle. We obtain the solution $y = y(t)$ satisfying (5.10), (5.11) for $t > 0$ by letting $T \to \infty$ through a suitable sequence. (One recovers the strong inequalities in (5.10) by a maximum principle.) The solution $y = y(t)$ is unique, for the difference of two solutions is a solution $z = z(t)$ of (5.12) satisfying $z(0)$. But such a solution is either $z(t) \equiv 0$ or is monotone, by (5.13), and cannot satisfy $z(\infty) = 0$.

Define a map $\tau : \mathscr{K} \to \mathscr{K}$ by

$$(5.14) \qquad u \mapsto \alpha + \int_0^t y(s)\, ds,$$

where $y = y_u$ is the unique solution of (5.7), (5.10), (5.11). It is clear that τ is continuous. Hence, Tikhonov's fixed-point theorem implies the existence of a solution of (I.1)–(I.3) for $\alpha = A(0,0) + \varepsilon$, $\beta = 0$, $\lambda \geqq \Lambda(\varepsilon)$. Note, that by (5.6), a fixed point u satisfies $u''(0) \geqq u_0''(0) \geqq 0$, hence (I.4)–(I.5).

(γ) *On* (1.3). It has to be shown that $A(0, 0-) \geqq A(0, 0)$. Let $\alpha > A(0, 0-)$ be fixed. Then $\gamma(\alpha, 0, \lambda)$ increases to a limit, which is finite, as $\lambda \to 0-$. For (I.1) and $0 \leqq u' < 1$ imply that $|u'''| \leqq T|u''| + |\lambda|$ on $0 \leqq t \leqq T$, so that, for example,

$$\gamma(\alpha, 0, \lambda) = u_0''(0, \alpha, 0, \lambda) \leqq M(1; 1, |\lambda|, 1);$$

cf. (A3.9) in the Appendix. Since the limit $\gamma(\alpha, 0, 0-) > 0$, the arguments in step (C) of § 4 show that (I.1)–(I.5) has a solution if $\beta = 0, \lambda = 0$. Hence $\alpha \geqq A(0, 0)$, and so $A(0, 0-) \geqq A(0, 0)$.

(δ) *Completion of the proofs.* The continuity of u_0, u_0' and $\gamma(\alpha, \beta, \lambda)$ at points (α, β, λ) of Ω, where $\lambda = 0$, is an easy consequence of the uniqueness of solutions of (I.1)–(I.5) when $\lambda = 0$. This completes the proofs of Theorems 1.1 and 1.2.

6. Proof of Theorem 1.3. This theorem can be proved by the method of step (β) in the last section; cf. also the proof of Theorem 7.1, where $a = 1$ is permitted. A slightly simpler proof is as follows: Let $\lambda < 0$, $0 \leqq \beta_0 < \beta < 1$, $\alpha \geqq A(\beta, \lambda)$ and $z = z_0(u) = z_0(u, \alpha, \beta_0, \lambda)$ correspond to the maximal solution, i.e., $z_0 = \beta^2$, $\dot{z}_0 = 2\gamma(\alpha, \beta_0, \lambda)$ at $u = \alpha$. Note that $z \equiv 1$ is also a solution of (3.1) and, of course, $0 \leqq \beta_0^2 < z_0 < 1$ for $u > \alpha$. It follows from Proposition A3.1, and Remark 1 following it, in the Appendix, that (2.2), (2.3) has a solution $z = z(u)$ satisfying $z(\alpha) = \beta^2$, $(0<)z_0 < z < 1$ for $u > \alpha$. This implies the first part of Theorem 1.3. As to the last part, cf. the proof of the monotonicity of u_0, u_0' with respect to λ in § 3, step (b).

PART II. ON POHLHAUSEN'S AND STEWARTSON'S PROBLEMS

7. Existence theorems. In what follows, α, α_0, β, β_0, β_1, ε, σ, a are constants satisfying

$$-\infty < \alpha_0 \leqq \alpha < \infty, \quad 0 \leqq \beta_0 < 1, \quad \beta_0 \leqq \beta \leqq \beta_1,$$
$$0 < \varepsilon \leqq 1, \quad \sigma > 0, \quad a \geqq 0. \tag{7.1}$$

Note that the problem (II.1)–(II.4) is trivial if $\beta = 1$ for, in this case, $u = t + \alpha$, and h is determined by two quadratures. When $a = 1$, so that $h \equiv 1$, the problem (II.1)–(II.4) reduces to that discussed in Part I. When $c > 0$, the condition (II.4) is not less general than

$$u'^2(\infty) = h(\infty) = c. \tag{7.2}$$

For if $u(t)$, $h(t)$ are replaced by $c^{-1/4}u(c^{-1/4}t), c^{-1}h(c^{-1/4}t)$, then the problem (II.1)–(II.3), (7.2) is reduced to (II.1)–(II.4), with $\alpha, \beta, \lambda, a$ replaced by $\alpha/c^{1/4}$, $\beta/c^{1/2}$, $\lambda c^{1/2}$, a/c, respectively.

THEOREM 7.1. *Assume that either*

$$\{0 < \beta_0 < 1, \alpha_0 \textit{ arbitrary}\} \quad \textit{or} \quad \{\beta_0 = 0, \alpha_0 > A(0, 0)\} \tag{7.3}$$

and that either

$$\{\sigma = 1\} \quad \textit{or} \quad \{\sigma > 0, \alpha_0 \geqq 0\}. \tag{7.4}$$

Then there exists a number $\Lambda = \Lambda(\alpha_0, \beta_0, \beta_1, \varepsilon, \min(\sigma, 1), \max(a, 1))$ *with the property that if* (7.1) *holds and*

$$\Lambda \leqq \lambda < 0, \tag{7.5}$$

then (II.1)–(II.5) *has a solution satisfying* $0 \leqq \beta_0 < u' < \max(1 + \varepsilon, \beta_1)$ *for* $t > 0$.

It is not claimed that there is nonexistence if $\lambda < \Lambda$. The proof supplies estimates for $|u' - 1|$, hence implicitly for $|h - 1|$, for the solution (u, h) constructed. For example, there exists a positive constant $c = c(\alpha_0, \beta_0, \beta_1, \min(\sigma, 1), \max(a, 1))$ such that

$$|u' - 1| = O(\exp(-ct^2)) \quad \text{as } t \to \infty. \tag{7.6}$$

Remark 1. If $\beta \geqq (a + 1)/2$ and either

$$\{\sigma = 1\} \quad \text{or} \quad \{\sigma < 1, a < 1, \alpha \geqq 0\} \quad \text{or} \quad \{\sigma > 1, a > 1, \alpha \geqq 0\},$$

then the proof of Theorem 7.1 can be modified to yield a solution (u, h) satisfying, for $t > 0$,

$$u' > [(1 - \beta)h + (\beta - a)]/(1 - a) > h^{1/2}, \tag{7.7}$$

hence (II.11). If, in addition, $\beta = \beta_1 > 1$, then (II.9)–(II.10) hold; while if $(a + 1)/2 \leqq \beta < 1$, then u' either increases from β to 1 or first increases to a value >1 and then decreases to 1; cf. Remark 5.

If $\sigma = 1$ and $\beta \leqq a^{1/2}$, then the proof can be modified to yield solutions of (II.1)–(II.5), (II.8) (and (II.6)–(II.7) if $0 \leqq \beta < 1$) and, at the same time, simplify the definition of Λ. This is the content of the next three remarks.

Remark 2. Assume that $\sigma = 1$, $\beta_0 \leqq \beta \leqq a^{1/2} < 1$ and $\alpha \geqq A(\beta_0, \mu)$, where $\mu = 2\lambda/(1 + \beta_0) \leqq \lambda < 0$. Then (II.1)–(II.5) has a solution satisfying

$$w' < u' < [(1 - \beta)h + (\beta - a)]/(1 - a) < h^{1/2} < 1, \tag{7.8}$$

hence (II.8). In (7.8), $w(t) = u_0(t, \alpha, \beta_0, \mu)$ is the maximal solution of (I.1)–(I.5), with (α, β, λ) replaced by (α, β_0, μ). If $\beta = \beta_0$, then (II.6) and (II.7) hold. If $\alpha \geqq \max(A(\beta_0, \mu), 0)$, then condition $\sigma = 1$ can be relaxed to $\sigma \geqq 1$.

Remark 3. Assume that $\sigma = 1$, $0 < \beta_0 < 1$, $1 < \beta \leqq \beta_1 \leqq a^{1/2}$, and $\alpha \geqq A(\beta_0, \nu_0)$, where

$$\nu_0 = \lambda \max[(a - \beta_0^2)/(1 - \beta_0^2), 2\beta_1/(1 + \beta_0)] < \lambda < 0. \tag{7.9}$$

Then (II.1)–(II.5) has a solution satisfying

$$\begin{gathered} w' < u' < [(1 - \beta)h + (\beta - a)]/(1 - a) < h^{1/2}, \quad u' < \beta, \\ 1 < h < 1 + (a - 1)(1 - w')/(1 - \beta_0), \end{gathered} \tag{7.10}$$

hence (II.8), where $w = u_0(t, \alpha, \beta_0, \nu_0)$. If $\alpha \geqq \max(A(\beta_0, \nu_0), 0)$, then $\sigma = 1$ can be relaxed to $\sigma \geqq 1$. Furthermore, either u' decreases from β to 1 or first decreases to a value <1 and then increases to 1; cf. Remark 5.

Remark 4. Let $\beta_0 \leqq \beta < 1 < a$ and

$$\nu = \lambda(a - \beta_0^2)/(1 - \beta_0^2) < \lambda < 0. \tag{7.11}$$

(i) If $\sigma = 1$ and $\alpha \geqq A(\beta_0, \nu)$, then (II.1)–(II.5) has a solution satisfying

$$w' < u' < 1 < h < 1 + (a - 1)(1 - w')/(1 - \beta_0), \tag{7.12}$$

hence (II.8), where $w = u_0(t, \alpha, \beta_0, \nu)$. If $\beta = \beta_0$, then (II.6)–(II.7) hold. If $\alpha \geqq \max(A(\beta_0, \nu), 0)$, then $\sigma = 1$ can be relaxed to $\sigma \geqq 1$.

(ii) Also, if $0 < \sigma < 1$ and $\alpha \geqq \max(\sigma^{-1/2}A(\beta_0, \nu\sigma^{-1}), 0)$, then the conclusion of part (i) is valid with $w = \sigma^{-1/2}u_0(\sigma^{1/2}t, \sigma^{1/2}\alpha, \beta_0, \nu/\sigma)$ in (7.12).

In the verification of these remarks, we shall use the following remark.

Remark 5. If (u, h) is a solution of (II.1)–(II.4) satisfying $h^{1/2} - u' < 0\ (>0)$ for $t > 0$ and $u''(t_0) \leqq 0\ (\geqq 0)$ at some $t = t_0 \geqq 0$, then $u''(t) < 0\ (>0)$ for $t > t_0$. In fact, since $\lambda < 0$, $u''' + uu'' < 0\ (>0)$, so that $u'' \exp \int^t u\, ds$ is decreasing (increasing) and, hence, negative (positive) for $t > t_0$.

The proof of Theorem 7.1 is contained in §§ 8–9. Section 8 contains also some propositions needed for the verifications of Remarks 1–4. Remarks 1–4 are proved in §§ 10–13.

8. Subsolutions, supersolutions and uniqueness. Consider the system of differential equations

$$L[y] \equiv y'' + uy' + \lambda(h - y^2) = 0, \tag{8.1}$$

$$h'' + \sigma u h' = 0, \tag{8.2}$$

which reduces to (II.1), (II.2) if $y = u'$. If $u = u(t)$ is a given function of t such that $u \geqq$ const. > 0 for large t, then (8.2) has a unique solution $h(t) = h_u(t)$ satisfying

$$h(0) = a \quad \text{and} \quad h(\infty) = 1, \tag{8.3}$$

and $h' \neq 0$ if $a \neq 1$. It will always be assumed that $u = u(t)$ is a given function and $h = h_u(t)$, so that (8.1) reduces to a differential equation in one unknown function y. We shall assume, for convenience, that

$$a \neq 1. \tag{8.4}$$

Define the function $\theta = \theta(\sigma, \alpha_0)$ by

$$\theta = 1 = \sigma \quad \text{if } \alpha_0 < 0 \quad \text{and} \quad \theta = \min(1, \sigma) \quad \text{if } \alpha_0 \geqq 0. \tag{8.5}$$

Let $-\mu > 0$ be so small that

$$\theta^{1/2}\alpha_0 \geqq A(\beta_0, \mu/\theta). \tag{8.6}$$

Note that if $\alpha_0 < 0$, then $\theta = \sigma = 1$ and (7.3) implies the existence of μ; also, if $\alpha_0 \geqq 0$, then μ exists since $A(0, 0) < 0$; cf. Theorem 1.1. Put

$$w(t) = \theta^{1/2}u_0(\theta^{1/2}t, \theta^{1/2}\alpha_0, \beta_0, \mu/\theta), \tag{8.7}$$

where $u_0(t) = u_0(t, \theta^{1/2}\alpha_0, \beta_0, u/\theta)$ is the maximal solution of (I.1)–(I.5) when $(\alpha, \beta, \lambda) = (\theta^{1/2}\alpha_0, \beta_0, \mu/\theta)$. Then (8.7) satisfies

$$w''' + \theta w w'' + \mu(1 - w'^2) = 0, \tag{8.8}$$

$$w(0) = \alpha_0, \qquad w'(0) = \beta_0, \tag{8.9}$$

$$0 < 1 - w' = 1 - u_0'(\theta^{1/2}t) \sim c_1 t^{-2\mu/\theta} \exp(-\theta t^2/2 - c_2 t), \qquad t \to \infty, \tag{8.10}$$

where $c_1 > 0, c_2$ are constants; Hartman [5], cf. [6, pp. 536–537]. Also, we have

(8.11) $$X'' + \theta w X' - \mu(1 + w')X = 0,$$

where $X = 1 - w' > 0$, $X' < 0$ for $t > 0$; and the function

(8.12) $$X_0 = (1 - w')/(1 - \beta_0) > 0, \qquad X_0' \leqq 0,$$

is a solution of (8.11) and $X_0(0) = 1$, $X_0(\infty) = 0$.

Define $W = W(t)$ to be the solution of

(8.13) $$W'' + (\theta w - 1)W' = 0,$$

(8.14) $$W(0) = \max(1 + \varepsilon, \beta_1) > 1 \quad \text{and} \quad W(\infty) = 1,$$

where $0 < \varepsilon \leqq 1$, so that

(8.15) $$\begin{gathered} 1 < W \leqq \max(1 + \varepsilon, \beta_1) \quad \text{and} \quad W' < 0 \quad \text{for } t \geqq 0, \\ 0 < W - 1 = O(\exp(-\theta' t)), \quad t \to \infty, \quad \text{if } 0 < \theta' < \theta. \end{gathered}$$

Also, the function

(8.16) $$W_0 = (W - 1)/(W(0) - 1) > 0, \qquad W_0' < 0,$$

is the unique solution of (8.13) and $W(0) = 1$, $W(\infty) = 0$.

Note that if

(8.17) $$u \geqq w,$$

then, since $w \geqq 0$ when $\alpha_0 \geqq 0$,

(8.18) $$u \geqq \theta u \geqq \theta w \geqq \theta w - 1.$$

Hence, if the linear equation (8.2) is compared with (8.11) and with (8.13), we see that

(8.19) $$0 < (h - 1)/(a - 1) \equiv h_0 < X_0, W_0 \quad \text{for } t > 0,$$

where $h = h_0$ is the unique solution of (8.2) satisfying $h_0(0) = 1$, $h_0' < 0$; cf. (A2.10) and Proposition A2.1.

In the propositions to follow, we assume that L is defined in (8.1) and that the given function u satisfies

(8.20) $$u \geqq \text{const.} > 0 \quad \text{for large } t,$$

so that (8.2), (8.3) has a unique solution $h(t) = h_u(t)$.

PROPOSITION 8.1. *If $a < 1$ (so that $h < 1$ for $t \geqq 0$), then*

(8.21) $$L[1] = \lambda(h - 1) > 0 \qquad \text{for } t \geqq 0.$$

PROPOSITION 8.2. *If $a > 1$ (so that $h > 1$ for $t \geqq 0$), then*

(8.22) $$L[1] = \lambda(h - 1) < 0 \quad \text{for } t \geqq 0.$$

PROPOSITION 8.3. *Assume that*

(8.23) $$\beta \geqq (a + 1)/2,$$

that either

$$\{\sigma = 1\} \quad or \quad \{0 < \sigma \leqq 1, a < 1, u \geqq 0\} \quad or \quad \{\sigma \geqq 1, a > 1, u \geqq 0\}, \tag{8.24}$$

and that $m(t) = m_u(t)$ *is given by*

$$m(t) = [(1 - \beta)h + (\beta - a)]/(1 - a). \tag{8.25}$$

Then

$$m(0) = \beta, \quad m(\infty) = 1, \quad m > h^{1/2} \quad for\ t > 0, \tag{8.26}$$

$$m'' + \sigma u m = 0, \tag{8.27}$$

$$L[m] = (1 - \sigma)um' + \lambda(h - m^2) > 0 \quad for\ t > 0, \tag{8.28}$$

Proof. We have $(h + 1)/2 > h^{1/2}$ for $h > 0$, $h \neq 1$. The linear function of h on the right of (8.25) has the value β when $h = a$ and the value 1 when $h = 1$, so that $m \geqq (h + 1)/2 > h^{1/2}$ if h is between a and 1. This gives (8.26), while (8.27), (8.28) are clear.

PROPOSITION 8.4. *Assume that*

$$0 \leqq \beta \leqq a^{1/2}, \tag{8.29}$$

and that either

$$\{\sigma = 1\} \quad or \quad \{0 < \sigma \leqq 1, a > 1, u \geqq 0\} \quad or \quad \{\sigma \geqq 1, a < 1, u \geqq 0\}. \tag{8.30}$$

Then (8.25) *satisfies* (8.27),

$$m(0) = \beta, \quad m(\infty) = 1, \quad m < h^{1/2} \quad for\ t > 0, \tag{8.31}$$

$$L[m] = (1 - \sigma)um' + \lambda(h - m^2) < 0 \quad for\ t > 0, \tag{8.32}$$

Proof. The linear function of h on the right of (8.25) has the value $\beta, 1$ at $h = a, 1$, respectively, so that the concavity of the function $h^{1/2}$ implies that $m < h^{1/2}$ for h between a and 1.

PROPOSITION 8.5. *Let* $0 \leqq a < 1$ (*so that* $h < 1$ *for* $t \geqq 0$) *and let* $u \geqq w$. *Then*

$$\mu \leqq \lambda < 0 \tag{8.33}$$

implies that

$$L[w'] > 0 \quad for\ t > 0. \tag{8.34}$$

Proof. From (8.1) and (8.8),

$$L[w'] = (u - \theta w)w'' + \lambda(h - 1) + (\lambda - \mu)(1 - w'^2). \tag{8.35}$$

PROPOSITION 8.6. *Let* $a > 1$ (*so that* $h > 1$ *for* $t \geqq 0$) *and let* $u \geqq w$. *Then*

$$\mu(1 - \beta_0^2)/(a - \beta_0^2) \leqq \lambda < 0 \tag{8.36}$$

implies (8.34).

Proof. This follows from (8.35), (8.12) and (8.19), since $1 - w'^2 = (1 + w')(1 - w') \geqq (1 + \beta_0)(1 - w')$ and $(u - \theta w)w'' \geqq 0$.

PROPOSITION 8.7. *Let the function* W *in* (8.13)–(8.16) *and the constant* $\eta > 0$ *satisfy*

$$W'/(W-1) \leqq -\eta < 0 \tag{8.37}$$

and let $u \geqq w$. *Then*

$$-\eta\{\max(0, 1-a)/[\max(1+\varepsilon, \beta_1) - 1] + 3 + \beta_1\}^{-1} \leqq \lambda < 0 \tag{8.38}$$

implies that

$$L[W] < 0 \quad \text{for } t > 0. \tag{8.39}$$

Proof. Note that a constant η exists since, by l'Hôpital's rule,

$$W'/(W-1) \sim W''/W' = -\theta w + 1 \to -\infty, \qquad t \to \infty,$$

and $W' < 0$ for $t \geqq 0$. The assertion follows from

$$L[W] = (u - \theta w + 1)W' + \lambda(h-1) + \lambda(1 - W^2)$$
$$\leqq \{W'/(W-1) + \lambda(h-1)/(W-1) - \lambda(1+W)\}(W-1),$$

(8.16), (8.19) and $1 + W \leqq 3 + \beta_1$. Note that $\lambda(h-1) < 0$ if $a > 1$.

PROPOSITION 8.8. *Let* C *be a constant and* $u \geqq w$. *Then* (8.1) *has at most one solution* $y = y(t)$ *satisfying*

$$y(0) = \beta, \quad |y(t)| \leqq C, \quad |y(t) - 1| = O(t^{-N}) \quad \text{as } t \to \infty \tag{8.40}$$

for all N *if*

$$\mu(1 + \beta_0)/2C \leqq \lambda < 0. \tag{8.41}$$

Proof. If there are two solutions y_1, y_2, then their difference, $x = y_2 - y_1$, satisfies

$$\begin{aligned} &x'' + ux' - \lambda(y_1 + y_2)x = 0, \\ &x(0) = 0 \quad \text{and} \quad x(t) = O(t^{-N}) \quad \text{as } t \to \infty, \end{aligned} \tag{8.42}$$

for all N. If the differential equation in (8.42) is compared with (8.11), it is seen, by (A2.10) and Proposition A2.1, that (8.42) is disconjugate for $t \geqq 0$. Thus the assertion follows from Corollary A2.1.

9. Proof of Theorem 7.1. Let $\mathscr{C}^1$ be the space of functions $u \in C^1[0, \infty)$ with the topology of C^1 convergence on compacts on $t \geqq 0$. Put

$$\begin{aligned} &R(T) = \max W^2(t) \quad \text{for } 0 \leqq t \leqq T, \\ &C_1(T) = 1 + \max\left(|w(t)|, \left|\alpha + \int_0^t W(s)\,ds\right|\right), \\ &C_2(T) = \max(1, a, R(T)). \end{aligned} \tag{9.1}$$

In terms of the functions w, W and (9.1), define a compact convex subset $\mathscr{K} = \mathscr{K}[w, W]$ of $\mathscr{C}^1$ consisting of functions u satisfying

$$\begin{aligned} &u(0) = \alpha, \quad u'(0) = \beta, \quad w' \leqq u' \leqq W \quad \text{for } t \geqq 0, \\ &|u'(t) - u'(s)| \leqq M(T)|t-s| \quad \text{for } 0 \leqq s, t \leqq T, \quad T > 0, \end{aligned} \tag{9.2}$$

where $M(T) = M(T; C_1(T), C_2(T), R(T))$ in (A3.9) in the Appendix.

Note that if $u \in \mathcal{K}$, the solution $h = h_u(t)$ of (8.2), (8.3) satisfies

$$(9.3) \quad |h'(t)| \leqq M_1(T) = M(T; C_1(T), C_2(T), \max(a, 1)) \quad \text{for } 0 \leqq t \leqq T, \quad T > 0.$$

Also, if $y = y_u(t)$ is a solution of (8.1) and

$$(9.4) \qquad y(0) = \beta \quad \text{and} \quad w' < y < W \quad \text{for } t \geqq 0,$$

then

$$(9.5) \qquad |y'(t)| \leqq M(T) \quad \text{for } 0 \leqq t \leqq T, \quad T > 0.$$

Let Λ be the minimum of the quantities on the left of (8.33) if $a < 1$ or (8.36) if $a > 1$, of (8.38), and of (8.41) with $C = \max(1 + \varepsilon, \beta_1)$. Let $\Lambda \leqq \lambda < 0$. Then, by virtue of (8.34), (8.39) and $w' < 1 < W$, Proposition A3.1 implies that (8.1) has a solution $y = y_u(t)$ satisfying (9.4). This solution is unique by Proposition 8.8.

Define a map $\tau: \mathcal{K} \to \mathcal{K}$ by

$$u \mapsto \alpha + \int_0^t y_u(s)\, ds \in \mathcal{K}.$$

The definition of $\mathcal{K}$, (9.3), (9.5) and the uniqueness of $y = y_u(t)$ show that τ is continuous. Hence, by Tikhonov's theorem, τ has a fixed point u and (u, h_u) is the desired solution of (II.1)–(II.5).

10. On Remark 1. Let $u \in \mathcal{K}$. Then the function m in (8.25) satisfies

$$(10.1) \qquad w' < m < W \quad \text{for } t > 0.$$

The second inequality is clear if $\beta < 1$, so that $m < 1 < W$. If $\beta > 1$, (8.19) gives

$$(10.2) \qquad (h - 1)/(a - 1) = (m - 1)/(\beta - 1) < (W - 1)/(W(0) - 1),$$

so that the last inequality of (10.1) follows from $W(0) - 1 \geqq \beta_1 - 1 \geqq \beta - 1$. The first inequality is proved similarly. Thus, in § 9, we can use (8.28), instead of (8.34), and replace (9.4) by

$$(10.3) \qquad y(0) = \beta, \qquad (w' <) m < y < W \quad \text{for } t > 0.$$

11. On Remark 2. In the definition (9.1)–(9.2) of $\mathcal{K} = \mathcal{K}[w, W]$, choose W to be $W \equiv 1$. If $u \in \mathcal{K} = \mathcal{K}[w, 1]$, then the first part of (10.2) and (8.19) give

$$(11.1) \qquad w' < m < h^{1/2}.$$

If, in § 9, we use (8.32), instead of (8.39), we can replace (9.4) by

$$(11.2) \qquad y(0) = \beta, \qquad w' < y < 1 \quad \text{for } t > 0.$$

Also, $\beta = \beta_0$ implies that $y(0) = w'(0) = \beta$, hence $y'(0) \geqq w''(0) \geqq 0$. Thus (II.6), (II.7) hold in this case; cf. Remark 5.

Note that, in this argument, Proposition 8.7 and the function W of (8.13)–(8.16) are not involved. Thus, in the definition of Λ, (8.38) need not be considered and C can be taken to be 1 in (8.41).

12. On Remark 3. This remark can be proved in the same way except that we choose $\mathscr{K} = \mathscr{K}[w, \beta_1]$, i.e., $W \equiv \beta_1$ in (9.1)–(9.2). As to the last part of (7.10), cf. (8.19).

13. On Remark 4. Let $\mathscr{K} = \mathscr{K}[w, 1]$. Again we can use (8.32) instead of (8.39) in the arguments of § 9 to replace (9.4) by

$$y(0) = \beta, \qquad w' < y < 1 < h \quad \text{for } t > 0. \tag{13.1}$$

This would prove Remark 4 if condition (7.11) is strengthened to (7.9) with $\beta_1 = 1$. In order to obtain Remark 4, as stated, we replace (8.1), (8.2) by a linear system considered by Hastings [8],

$$L_0[y] \equiv y''' + uy' + \lambda(h^{1/2} - u')(h^{1/2} + y) = 0, \tag{13.2}$$

$$h'' + \sigma u h' = 0. \tag{13.3}$$

Let $\mathscr{K} = \mathscr{K}[w, 1]$. Then $w' \leqq u' \leqq 1 < h^{1/2}$ and the arguments in the proof of Proposition 8.6 show that

$$L_0[w'] > 0 \quad \text{for } t > 0 \tag{13.4}$$

when (8.36) holds. We also have

$$L_0[1] = \lambda(h^{1/2} - u')(h^{1/2} + 1) < 0. \tag{13.5}$$

The homogeneous part of (13.2),

$$x'' + ux' + \lambda(h^{1/2} - u')x = 0, \tag{13.6}$$

is disconjugate for $t \geqq 0$ since $\lambda(h^{1/2} - u') < 0$, so that, by Corollary A2.1, (13.2) has at most one solution y satisfying (8.40). The proof of Remark 4 now follows as in §9.

PART III. A MORE GENERAL PROBLEM

14. An existence theorem. Consider a system of differential equations for an $(n + 1)$-vector $(u, h) = (u, h^1, \cdots, h^n)$ of the form

$$u''' + G_{n+1}(t, u, u', u'', h, h')u'' + \lambda_{n+1}H_{n+1}(t, u, u', u'', h, h') = 0, \tag{14.1}$$

$$h^{k''} + G_k(t, u, u', u'', h, h')h^{k'} + \lambda_k H_k(t, u, u', u'', h, h') = 0 \tag{14.2$_k$}$$

for $k = 1, \cdots, n$, and the boundary conditions

$$u(0) = \alpha, \quad u'(0) = \beta, \quad h(0) = a, \tag{14.3}$$

$$u'(\infty) = 1 \quad \text{and} \quad h(\infty) = 0. \tag{14.4}$$

(Note that condition (II.4) in Theorem 7.1 reduces to (14.4) if the variable h there is replaced by $h + 1$.)

Partition the set of integers $\{1, \cdots, n\} = I \cup J$, where I and J are disjoint, possibly empty, sets. The index i will denote an element of I, j of J, while $k = 1, \cdots, n$ or $k = 1, \cdots, n + 1$. We make the following assumptions:

(H1) G_k, H_k are continuous for $t \geqq 0$, $u' \geqq 0$, arbitrary (u, u'', h, h') and $k = 1, \cdots, n + 1$.

(H2) There are constants $\sigma_k > 0$ and $c_0 \geqq 0$ such that

$$G_k \geqq \sigma_k(u - c_0) \quad \text{for } k = 1, \cdots, n + 1. \tag{14.5}$$

(H3) The functions H_j, H_{n+1} satisfy

$$H_j = 0 \quad \text{if } h^j = 0; \qquad H_{n+1} = 0 \quad \text{if } (u', h) = (1, 0). \tag{14.6}$$

(H4) H_j and $G_k, k = 1, \cdots, n + 1$, are subject to an estimate of the form

$$G_k, H_j = O(1) + o(|u''| + |h'|) \quad \text{as } |u''| + |h'| \to \infty, \tag{14.7}$$

uniformly on (t, u, u', h)-compacts.

(H5) H_i, H_{n+1} satisfy Lipschitz conditions of the form

$$|H_i^2 - H_i^1| \leqq B \sum_{k=1}^{i} |h_2^k - h_1^k|, \tag{14.8$_i$}$$

$$|H_{n+1}^2 - H_{n+1}^1| \leqq A|u_2' - u_1'| + B \sum_{k=1}^{n} |h_2^k - h_1^k|, \tag{14.8$_{n+1}$}$$

for $t \geqq 0; 0 \leqq u', |h_1|, |h_2| \leqq C; |u| \leqq C(1 + t)$; and arbitrary (u'', h'), where $C > 0$ is arbitrary, $A = A(C)$ and $B = B(C)$ are constants, $H_i^m = H_i(t, u, u', u'', h_m', \cdots, h_m^i, h^{i+1}, \cdots, h^n, h')$ and $H_{n+1}^m = H_{n+1}(t, u, u_m', u'', h_m, h')$ for $m = 1, 2$.

(H6) H_j is a nonincreasing function of h^j when h^j is on the interval $0 \leqq h^j \leqq a^j$ if $a^j \geqq 0$ or $a^j \leqq h^j \leqq 0$ if $a^j \leqq 0$.

THEOREM 14.1. *Assume* (H1)–(H6), $\lambda_j \geqq 0$ *fixed for* $j \in J$, *that* (α_0, β_0) *satisfy either*

$$\{0 < \beta_0 < 1, \alpha_0 \textit{ arbitrary}\} \quad \textit{or} \quad \{\beta_0 = 0, \alpha_0 > A(0,0)\}, \tag{14.9}$$

that either

$$\{\sigma_1 = \cdots = \sigma_{n+1}\} \quad \textit{or} \quad \{\sigma_k > 0 \textit{ arbitrary}, \alpha_0 \geqq 0\}, \tag{14.10}$$

and $\alpha, \alpha_0, \alpha_1, \beta, \beta_0, \beta_1, \varepsilon, a = (a^1, \cdots, a^n), a_1$ *satisfy*

$$\alpha_0 + c_0 \leqq \alpha \leqq \alpha_1, \quad \beta_0 \leqq \beta \leqq \beta_1, \quad |a^k| \leqq a_1, \quad 0 < \varepsilon \leqq 1. \tag{14.11}$$

Then there exists a positive number $\Lambda_1 = \Lambda_1(\alpha_0, \alpha_1, \beta_0, \beta_1, \varepsilon, \lambda_j$ *for* $j \in J$, $\min(\sigma_1, \cdots, \sigma_{n+1}), a_1)$ *with the property that if* $|\lambda_i|, |\lambda_{n+1}| < \Lambda_1$, *then* (14.1)–(14.4) *has a solution* (u, h) *satisfying*

$$0 \leqq \beta_0 < u' < \max(1 + \varepsilon, \beta_1) \quad \textit{for } t > 0, \tag{14.12}$$

$$h^j \textit{ is monotone}, \tag{14.13}$$

$$-\varepsilon < h^i < a^i \quad \textit{if} \quad a^i > 0 \quad \textit{or} \quad a^i < h^i < \varepsilon \quad \textit{if} \quad a^i < 0 \quad \textit{for } t > 0. \tag{14.14}$$

Also, if $H^{i_0} = 0$ *when* $h^{i_0} = 0$ *for some* $i_0 \in I$, *then*

$$0 < |h^{i_0}| < |a^{i_0}| \quad \textit{for } t > 0. \tag{14.15}$$

Except for the dependence of Λ_1 on α_1, Theorem 14.1 contains Theorem 7.1. The proof below also supplies (exponential) estimates for $|h|$ and $|u' - 1|$. It will be clear that, with slight modifications of the assumptions, with $h^{k'}$ playing the role of h^k, some or all of the equations (14.2_k) can be replaced by equations of the form

$$h^{k'''} + G_k h^{k''} + \lambda_k H_k = 0,$$

where the corresponding h_k'' occurs in the arguments of the G's and H's, and the boundary conditions for h^k are of the type $h^k(0) = \alpha^k$, $h^{k\prime}(0) = a^k$, $h^{k\prime}(\infty) = 0$.

15. Proof of Theorem 14.1. Without loss of generality, we can suppose that

$$a^k \geqq 0, \quad \text{where } a = (a^1, \cdots, a^n), \tag{15.1}$$

for otherwise the variable h^k is replaced by $-h^k$. Define the number $\theta = \theta(\sigma_1, \cdots, \sigma_{n+1}, \alpha_0)$ by

$$\theta = \sigma_1 = \cdots = \sigma_{n+1} \quad \text{if } \alpha_0 < 0, \qquad \theta = \min(\sigma_1, \cdots, \sigma_{n+1}) \quad \text{if } \alpha_0 \geqq 0. \tag{15.2}$$

Assume (8.6) and introduce the functions w, W in (8.7)–(8.16).

Let C be the constant

$$C = \max(1 + \varepsilon, |\alpha_0|, |\alpha_1|, \beta_1, a_1) > 1. \tag{15.3}$$

For $T \geqq 2C$, choose $C^0(T) = C^0(T, \lambda_j \text{ for } j \in J) \geqq 1$ so large that

$$\begin{gathered} |G_k|, |\lambda_j H_j| \leqq C^0(T) + (|u''| + |h'|)/32C, \\ |H_{n+1}|, |H_i| \leqq C^0(T), \end{gathered} \tag{15.4}$$

where $k = 0, \cdots, n$; $j \in J$; $i \in I$; $0 \leqq t \leqq T$; $0 \leqq u'$, $|h| \leqq C$; $|u| \leqq C(1 + t)$; and (u'', h') is arbitrary; cf. hypotheses (H4) and (H3), (H5). Finally, for $T \geqq 2C$, let

$$\begin{aligned} N(T) &= 8CC^0(T) > 1 \\ N_1(T) &= 2(C^0(T) + N(T)/16C)N(T) = 24C(C^0(T))^2. \end{aligned} \tag{15.5}$$

Let $\mathscr{C}^r$ be the space of functions of class $C^r[0, \infty)$ with the topology of C^r convergence on compacts of $t \geqq 0$. Let $\mathscr{K}$ be the compact convex subset of $\mathscr{C}^2 \times \mathscr{C}^1 \times \cdots \times \mathscr{C}^1 = \mathscr{C}^2 \times (\mathscr{C}^1)^n$ consisting of (u, h) satisfying

$$\begin{aligned} & u(0) = \alpha, \quad u'(0) = \beta, \quad w' \leqq u' \leqq W \quad \text{for } t \geqq 0, \\ & |u''(t)| \leqq N(T), \quad |u''(t) - u''(s)| \leqq N_1(T)|t - s| \quad \text{for } 0 \leqq s, t \leqq T, \\ & h(0) = a, \quad |h^k(t)| \leqq |a^k| \quad \text{for } t \geqq 0, \\ & |h^{k\prime}(t)| \leqq N(T), \quad |h^{k\prime}(t) - h^{k\prime}(s)| \leqq N_1(T)|t - s| \quad \text{for } 0 \leqq s, t \leqq T, \end{aligned} \tag{15.6}$$

and $T \geqq 2C$. In particular,

$$\begin{gathered} u \geqq w + \alpha - \alpha_0, \quad \text{so that } u - c_0 \geqq w, \\ \sigma_k(u - c_0) \geqq \theta w \quad \text{for } k = 1, \cdots, n + 1. \end{gathered} \tag{15.7}$$

For fixed $(u, h) \in \mathscr{K}$, consider the system of differential equations

$$L_k[y^k] \equiv y^{k''} + G_k^* y^{k'} + \lambda_k H_j^* = 0, \tag{15.8_k}$$

where $k = 1, \cdots, n + 1$, and the asterisk indicates that $u = u(t)$, $u' = u'(t)$,

$u'' = u''(t)$, $h = h(t)$, $h' = h'(t)$ except that, in G_k and H_k, $(h^1, \cdots, h^{k-1})$ and $(h^{1\prime}, \cdots, h^{k-1\prime})$ are replaced by $(y^1, \cdots, y^{k-1})$ and $(y^{1\prime}, \cdots, y^{k-1\prime})$; in addition, in H_k, h^k is replaced by y^k and, in H_{n+1}, u' is replaced by y^{n+1}. Thus a solution of (15.8) is obtained by solving successively for $y^1, \cdots, y^{n+1}$. In (15.8_k), the derivative $y^{k\prime}$ occurs only where it appears explicitly and y^k occurs only in H_k^*. Thus, if $y^1, \cdots, y^{k-1}$ are known, G_k^* becomes a known function of t alone and H_k^* is a function of (t, y^k) for $k = 1, \cdots, n+1$. The system (15.8) reduces to (14.1)–(14.2) if $(y^1, \cdots, y^{n+1}) = (h^1, \cdots, h^n, u')$.

The coefficients θw, $\theta w - 1$, G_k^* of (8.8), (8.13), (15.8_k) satisfy

$$G_k^* \geqq \sigma_k(u - c_0) \geqq \theta w \geqq \theta w - 1 \quad \text{for } k = 1, \cdots, n+1.$$

Thus, by (8.11) and (8.13),

$$X'' + G_k^* X' - \mu(1 + \beta_0)X \leqq 0 \quad \text{if } X = 1 - w', \tag{15.9}$$

$$W'' + (G_k^* + 1)W' \leqq 0. \tag{15.10}$$

PROPOSITION 15.1. *There exists a number* $\delta_n = \delta_n(\alpha_0, \alpha_1, \beta_0, \beta_1, \varepsilon, \lambda_j, \min(\sigma_k), a_1) > 0$ *with the property that if* $(u, h) \in \mathscr{K}$ *and*

$$|\lambda_i| < \delta_n \quad \text{for } i \in I, \tag{15.11}$$

then (15.8_k), $k = 1, \cdots, n$, *has a unique solution satisfying*

$$y^j(0) = a^j, \quad y^{j\prime} \leqq 0, \quad 0 \leqq y^j \leqq a^j X_0, \tag{15.12}$$

$$y^j(0) = a^j, \quad y^{j\prime} \leqq 0, \quad 0 \leqq y^j \leqq a^j W_0, \tag{15.13}$$

$$y^i(0) = a^i, \quad -\varepsilon X_0 < y^i < a^i X_0 \quad \text{for } t > 0, \tag{15.14}$$

$$y^i(0) = a^i, \quad -\varepsilon X_0 < y^i < a^i W_0 \quad \text{for } t > 0. \tag{15.15}$$

If, in addition, $H^{i_0} = 0$ *when* $h^{i_0} = 0$ *for some* $i_0 \in I$, *then* $\varepsilon = 0$ *is permitted in* (15.14), (15.15) *for* $i = i_0$. *Furthermore,*

$$|y^{k\prime}(t)| \leqq N(T) \quad \text{for } 0 \leqq t \leqq T, \quad T \geqq 2C. \tag{15.16$_k$}$$

The functions X_0, W_0 are defined in (8.12), (8.16).

Proof. Let $m < n$ and suppose that there exists a positive δ_m, depending on the specified parameters, with the property that if $|\lambda_i| \leqq \delta_m$ for $I \ni i \leqq m$, then $(y^1, \cdots, y^m)$ exist, are unique, and satisfy (15.12)–(15.16).

Suppose first that $m + 1 = j \in J$. Then, by the hypotheses (14.6) and (H6), the existence and uniqueness of a solution y^{m+1} of (15.8_{m+1}) satisfying (15.12) or (15.13) follows from Proposition A3.2, where the roles of $X, \ell[X] \leqq 0$ are played by either X_0, (15.9) or W_0, (15.10). Since uniqueness depends only on the conditions $y^{m+1}(0) = a^{m+1}$, $y^{m+1}(\infty) = 0$, the same solution satisfies both (15.12), (15.13).

Suppose that $m + 1 = i \in I$. By virtue of hypothesis (14.8_i) and the inequalities in (15.12)–(15.16_k) for $i, j, k \leqq m$, we can deduce, from Corollary A3.1(ii) or Corollary A3.2(ii), the existence of a $\delta_{m+1} > 0$ with the property that if $|\lambda_i| \leqq \delta_{m+1}$, $I \ni i \leqq m + 1$, then (15.8_{m+1}) has a solution y^{m+1} satisfying (15.14) and a solution satisfying (15.15). If $\delta_{m+1} > 0$ is sufficiently small, uniqueness follows from Corollary A3.1(iii).

In order to verify (15.16_k), for $k = m + 1$, note that, by (15.4) and (15.8_k),

$$|y^{k''}| \leqq (C^0(T) + N(T)/16C)(|y^{k'}| + 1) = (3C^0(T)/2)(|y^{k'}| + 1)$$

for $0 \leqq t \leqq T$, $T \geqq 2C$. Hence, on this t-range, $|y^{k'}(t)| \leqq M^0(T)$, where

$$M^0(T) \leqq M(T; 3C^0(T)/2, 3C^0(T)/2, C) \leqq 6CC^0(T) + 1 < N(T);$$

cf. (A3.9), (A3.10) in the Appendix.

PROPOSITION 15.2. *There exists a number* $\Lambda_1 = \Lambda_1(\alpha_0, \alpha_1, \beta_0, \beta_1, \varepsilon, \lambda_j, \min(\sigma_k), a_1)$, $0 < \Lambda_1 \leqq \delta_n$, *with the property that if* $(u, h) \in \mathscr{K}$; $|\lambda_i|, |\lambda_{n+1}| < \Lambda_1$; *and* $(y^1, \cdots, y^n)$ *is the solution of* (15.8_k), $k = 1, \cdots, n$, *provided by Proposition* 15.1, *then* (15.8_{n+1}) *has a unique solution* y^{n+1} *satisfying* (15.16_{n+1}) *and*

$$y^{n+1}(0) = \beta, \qquad w' < y^{n+1} < W \quad \text{for } t > 0. \tag{15.17}$$

Proof. By the arguments in the proofs of Corollaries A3.1 and A3.2, we can show the existence of a $\Lambda_1 > 0$ with the property that if $|\lambda_{n+1}| < \Lambda_1$, then

$$L_{n+1}[w'] > 0 \quad \text{and} \quad L_{n+1}[W] < 0.$$

In this argument, we use

$$L_{n+1}[w'] = (G^*_{n+1} - \theta w)w'' + \lambda_{n+1}[H^*_{n+1}]_{y^{n+1}=w'} - \mu(1 + w')(1 - w'),$$

$$L_{n+1}[W] = (G^*_{n+1} - \theta w + 1)W^1 + \lambda_{n+1}[H^*_{n+1}]_{y^{n+1}=W},$$

where, from (14.6) and (14.8_{n+1}),

$$|H^*_{n+1}| \leqq A|1 - y^{n+1}| + B \sum_{k=1}^{n} |y^k(t)|,$$

and $|y^k(t)| \leqq \max(\varepsilon, a^k)(1 - w')/(1 - \beta_0)$ and $|y^k(t)| \leqq \max(\varepsilon, a^k)(W - 1)/(W(0) - 1)$ by Proposition 15.1.

Thus, if $|\lambda_{n+1}| < \Lambda_1$, the existence of y^{n+1} follows from $w' < 1 < W$ and Proposition A3.1. In order to obtain uniqueness, note that the difference x of two solutions satisfies the conditions

$$x(0) = 0, \qquad x = O(W - w') = O((W - 1) + (1 - w')) = O(t^{-N}),$$

as $t \to \infty$, for all N, and a differential equation of the form

$$x'' + G^*_{n+1}x' + \lambda_{n+1}q_1(t)x = 0,$$

where $q_1(t)$ is a bounded measurable function of t, $|q_1(t)| \leqq A$ by (14.8_{n+1}). Hence, by the proof of Corollary A3.1(iii), uniqueness follows for small $|\lambda_{n+1}|$ from Corollary A2.1.

The proof of (15.16_k) for $k = n + 1$ is the same as for $k \leqq n$.

Completion of the proof. Let $|\lambda_i|, |\lambda_{n+1}| < \Lambda_1$ and for $(u, h) \in \mathscr{K}$, let $(y^1, \cdots, y^{n+1})$ be the solution of the system (15.8) supplied by Propositions 15.1 and 15.2. Define a map $\tau : \mathscr{K} \to \mathscr{K}$ by

$$(u, h) \mapsto \left(\alpha + \int_0^t y^{n+1}(s)\, ds, y^1, \cdots, y^n\right).$$

It is easily verified that τ is continuous and, therefore, has a fixed point (u, h). Clearly, such a point (u, h) is a desired solution of (14.1)–(14.4).

APPENDIX

A1. On Weber's equation. We record here, for easy reference, some facts about the Weber differential equation

$$v'' + tv' - 2\lambda v = 0. \tag{A1.1}$$

This equation has a pair of linearly independent solutions $v = v_{0\lambda}, v_{1\lambda}$ satisfying

$$\begin{gathered} v_{0\lambda} > 0, \quad v'_{0\lambda} < 0, \quad v''_{0\lambda} > 0 \quad \text{for large } t, \\ v'_{0\lambda}/v_{0\lambda} \sim -t, \quad v_{0\lambda} \sim t^{-1-2\lambda}\exp(-t^2/2) \quad \text{as } t \to \infty, \end{gathered} \tag{A1.2}$$

$v_{0\lambda}$ is unique up to positive constant factors; and

$$\begin{gathered} v_{1\lambda} > 0 \quad \text{for large } t, \\ v_{1\lambda} \sim t^{2\lambda} \quad \text{and} \quad v'_{1\lambda} \sim 2\lambda t^{2\lambda-1} \quad \text{as } t \to \infty, \end{gathered} \tag{A1.3}$$

$v_{1\lambda}$ is unique up to the addition of (const.) $v_{0\lambda}$; cf. [6, Exercise 17.6, p. 320].

Let $t = S(\lambda)$ be the largest zero of $v''_{0\lambda}(t)$, so that

$$v''_{0\lambda}(t) \gtreqqless 0 \quad \text{according as } t \gtreqqless S(\lambda). \tag{A1.4}$$

It is easy to see that $S(0) = 0$ and that $S(\lambda)$ is a continuous decreasing function of λ.

Finally, note that if $\sigma_0 > 0$ and $v = v(t)$ is a solution of (A1.1), then $v = v(\sigma_0^{1/2}t)$ is a solution of

$$v'' + \sigma_0 tv' - 2\lambda\sigma_0 v = 0. \tag{A1.5}$$

A2. Linear second order equations. Let the coefficients of

$$\ell[x] \equiv x'' + p(t)x' + q(t)x = 0 \tag{A2.1}$$

be real-valued and continuous on some t-interval I. Let $X = X(t)$ be a positive supersolution of class $C^2(I)$,

$$\ell[X] \leqq 0 \quad \text{and} \quad X > 0. \tag{A2.2}$$

Then the variation of constants

$$r = x/X(t) \tag{A2.3}$$

reduces (A2.1) to

$$r'' + (2X'/X + p(t))r' + (\ell[X]/X)r = 0. \tag{A2.4}$$

The implication of (A2.2) is the following proposition.

PROPOSITION A2.1. *Assume* (A2.2). *Then* (A2.1) *is disconjugate on* I (*i.e., no solution* $x(t) \not\equiv 0$ *has two zeros on* I). *If, in addition,* $I = [\alpha, \omega)$ *is a half-open interval, then* (A2.1) *has a solution* $x = x_0(t)$ *satisfying*

$$x_0(\alpha) = X(\alpha) \quad \text{and} \quad 0 < x_0 \leqq X, \quad \dot{x}_0/x_0 \leqq \dot{X}/X \quad \text{on } I; \tag{A2.5}$$

furthermore, any solution $x = x(t)$ *of* (A2.1) *satisfying*

$$x \geqq X, \qquad \dot{x}/x \geqq \dot{X}/X \tag{A2.6}$$

at $t = \alpha$, *satisfies these inequalities on* I. *The inequalities in* (A2.5), (A2.6) *are strict for* $\alpha < t < \omega$ *unless* $\ell[X] \equiv 0$ *for* t *near* ω.

See [5, pp. 357–358]. If X is a solution of

$$\ell_0[X] \equiv X'' + P(t)X' + Q(t)X = 0, \tag{A2.7}$$

then we can write $\ell[X]$ either as

$$\ell[X] = (p - P)X' + (q - Q)X \tag{A2.8}$$

or as

$$\ell[X] = (1 - p/P)X'' + (q - Qp/P)X, \tag{A2.9}$$

when the latter is meaningful. Thus, sufficient conditions for (A2.2) are any of the following:

$$X > 0, \quad X' \leqq 0 \quad \text{and} \quad p \geqq P, \quad q \leqq Q, \tag{A2.10}$$

$$X > 0, \quad X' \geqq 0 \quad \text{and} \quad p \leqq P, \quad q \leqq Q, \tag{A2.11}$$

$$X > 0, \quad X'' \geqq 0 \quad \text{and} \quad 1 - p/P \leqq 0, \quad q - Qp/P \leqq 0. \tag{A2.12}$$

(As to (A2.10), cf. [7, Lemma 2A], and as to (A2.11), cf. [6, Exercise 7.2, p. 363].)

PROPOSITION A2.2. *Let the coefficients of* (A2.1) *be continuous on* $\alpha \leqq t < \omega$ $(\leqq \infty)$ *and let* (A2.1) *be disconjugate on* $\alpha \leqq t < \omega$. *Let* $X(t)$ *be a function of class* C^2 *for* t *near* ω *satisfying* (A2.2). *Let* $x = x_0(t)$ *be a solution of* (A2.1) *satisfying* $x_0(t_0) = 0$ *for some* $t_0, \alpha \leqq t_0 < \omega$, *and*

$$x_0(t) = o(X(t)) \quad \textit{as } t \to \omega. \tag{A2.13}$$

Then $x_0(t) \equiv 0$.

Proof. If $x_0(t) \not\equiv 0$, then $x = x_0(t)$ is a principal solution of (A2.1); i.e., any solution $x = x(t)$ linearly independent of $x_0(t)$ satisfies $x_0(t)/x(t) \to 0$ as $t \to \omega$; cf. (A2.6) in Proposition A2.1. But, since (A2.1) is disconjugate on $[\alpha, \omega)$, a principal solution cannot vanish; cf. [6, pp. 350–361].

COROLLARY A2.1. *Let* (A2.1) *be disconjugate on* $[\alpha, \infty)$ *and let its coefficients satisfy*

$$p(t) \geqq \sigma_0 t \quad \textit{and} \quad q(t) \leqq c\sigma_0 \quad \textit{for large } t, \tag{A2.14}$$

where $\sigma_0 > 0$ *and* c *are constants. If* $x = x_0(t)$ *is a solution of* (A2.1) *satisfying*

$$x_0(t) = o(t^{-c}) \quad \textit{as } t \to \infty, \tag{A2.15}$$

then either $x_0(t) \equiv 0$ *or* $x_0(t) \neq 0$ *for* $t \geqq 0$.

Proof. If $v = v_{1\lambda}(t)$ is the solution of the Weber equation (A1.1), where $2\lambda = -c$, then one can choose $X(t) = v_{1\lambda}(\sigma_0^{1/2}t) \sim (\sigma_0^{1/2}t)^{-c}$ as $t \to \infty$; cf. (A1.5) and (A2.10).

A3. Existence theorems. Many of the existence theorems of this paper are obtained from Proposition A3.1 which is a consequence of a result of Nagumo [16].

PROPOSITION A3.1. *Let* $y = y_1(t), y_2(t) \in C^2[0, \infty)$ *and*

$$y_1(t) < y_2(t) \quad \textit{for } t > 0. \tag{A3.1}$$

Let $F(t, y, y')$ *be continuous for* $t \geqq 0$, $y_1(t) \leqq y \leqq y_2(t)$, y' *arbitrary, and such that solutions of*

$$L[y] \equiv y'' - F(t, y, y') = 0 \tag{A3.2}$$

are uniquely determined by initial conditions, that $y_1(t)$, $y_2(t)$ *are sub- and super-solutions,*

$$L[y_1] > 0, \qquad L[y_2] < 0 \quad \textit{for } t > 0, \tag{A3.3}$$

and that for every $T > 0$, *there exists a continuous positive function* $\phi_T(x)$ for $x \geqq 0$ *such that*

$$|F(t, y, y')| \leqq \phi_T(|y'|) \quad \textit{for } 0 \leqq t \leqq T, \qquad \int^{\infty} x\, dx/\phi_T(x) = \infty. \tag{A3.4}$$

If y_0 *is a number satisfying*

$$y_1(0) \leqq y_0 \leqq y_2(0), \tag{A3.5}$$

then there exists at least one solution $y = y(t)$ *of* (A3.2) *on* $0 \leqq t < \infty$ *satisfying*

$$y(0) = y_0 \quad \textit{and} \quad y_1 < y < y_2 \quad \textit{for } t > 0. \tag{A3.6}$$

Furthermore, if $|y_1(t)|, |y_2(t)| \leqq R = R(T)$ *for* $0 \leqq t \leqq T$ *and*

$$\int_{2R/T}^{M} x\, dx/\phi_T(x) = 2R, \tag{A3.7}$$

then any solution $y(t)$ *of* (A3.2) *on* $0 \leqq t \leqq T$ *satisfies* $|y'(t)| \leqq M$ *for* $0 \leqq t \leqq T$.

Actually, Nagumo's result concerns a finite interval and the existence of a solution $y = y(t)$ of a two-point boundary value problem where, in place of (A3.6), we require

$$y(0) = y_0, \quad y(T) = y_T \quad \text{and} \quad y_1(t) < y(t) < y_2(t) \quad \text{for } 0 < t < T,$$

and y_0, y_T are given satisfying (A3.5) and $y_1(T) \leqq y_T \leqq y_2(T)$. It is clear that Proposition A3.1 follows from a limit process, by letting $y_T = [y_1(T) + y_2(T)]/2$ and $T \to \infty$ through a suitable sequence.

In applications above, we shall choose

$$\phi_T(x) = C_1|x| + C_2, \tag{A3.8}$$

where $C_1 = C_1(T) > 0$, $C_2 = C_2(T) \geqq 0$ are constants. Obviously, such linear functions are admissible, and we write the corresponding M as

$$M(T) = M(T; C_1, C_2, R). \tag{A3.9}$$

Thus, if $C > 0$ and $T \geqq 2R$, (A3.7) shows that we can choose

$$M(T; C, C, R) = 4RC + 1. \tag{A3.10}$$

Remark. It is clear from Nagumo's proof that Proposition A3.1 remains valid, with a suitable modification of the inequalities in (A3.6), if either $y = y_1(t)$ or $y = y_2(t)$ is a solution of (A3.2).

COROLLARY A.31. *In* (A2.1), *let* $p(t)$, $q(t)$ *be continuous for* $t \geqq 0$; q_0 *a constant,*

(A3.11) $$q(t) \geqq q_0 > 0 \quad \textit{for } t \geqq 0;$$

$X(t) \in C^2[0, \infty)$ *satisfies*

(A3.12) $$X(0) = 1, \quad X > 0, \quad X' \leqq 0, \quad \ell[X] \leqq 0 \quad \textit{for } t \geqq 0.$$

(i) *Let* $H(t, y)$ *be continuous for* $t \geqq 0$, $y \geqq 0$ *and*

(A3.13) $$H(t, 0) = 0 \quad \textit{and} \quad |H(t, a_1 X(t))| \leqq AX(t) \quad \textit{for } t \geqq 0,$$

where a_1, $A > 0$ *are constants. Then, if*

(A3.14) $$|\lambda| < q_0 a_1 / A \quad \textit{and} \quad 0 < a \leqq a_1,$$

the equation

(A3.15) $$L[y] = y'' + p(t)y' + \lambda H(t, y) = 0$$

has a solution satisfying

(A3.16) $$y(0) = a, \quad 0 < y < a_1 X \quad \textit{for} \quad t > 0.$$

Of course, a_1 *can be chosen arbitrarily* (*with a suitable adjustment of* A) *if the second condition of* (A3.13) *is replaced by* $|H(t, y)| \leqq Ay$; *cf. also Proposition* A3.2.

(ii) *Let* A, B, a_1, ε *be positive constants. Let* $H(t, y)$ *be continuous for* $t \geqq 0$, y *arbitrary,*

(A3.17) $$|H(t, y)| \leqq A|y| + BX(t) \quad \textit{for } t \geqq 0.$$

Then, if $0 \leqq a \leqq a_1$ *and*

(A3.18) $$|\lambda| < q_0 \min(a_1(Aa_1 + B)^{-1}, \varepsilon(A\varepsilon + B)^{-1}),$$

the equation (A3.15) *has a solution satisfying*

(A3.19) $$y(0) = a, \quad -\varepsilon X < y < a_1 X \quad \textit{for } t > 0.$$

(iii) *Let* $p(t)$ *satisfy*

(A3.20) $$p(t) \geqq \sigma_0 t \quad \textit{for large } t \quad \textit{and some } \sigma_0 > 0,$$

and $X(t) = O(t^{-N})$, *as* $t \to \infty$, *for all* N. *Let* $H(t, y)$ *be continuous for* $t \geqq 0$, y *arbitrary, and*

(A3.21) $$|H(t, y_2) - H(t, y_1)| \leqq A|y_2 - y_1| \quad \textit{and} \quad |\lambda| < q_0/A.$$

Then (A3.15) *has at most one solution satisfying*

(A3.22) $$y(0) = a \quad \textit{and} \quad y = O(X) \quad \textit{as } t \to \infty.$$

Proof. On (i). We apply Proposition A3.1, using the solution $y = 0$ as a subsolution and $y = a_i X$ as a supersolution, for

$$L[a_1 X] = a_1(\ell[X] - qX) + \lambda H(t, a_1 X) \leqq X\{-a_1 q_0 + |\lambda| A\} < 0.$$

On (ii). We apply Proposition A3.1, using $y = a_1 X$ as a supersolution and $y = -\varepsilon X$ as a subsolution.

On (iii). The difference $x = y_2 - y_1$ of two solutions of (A3.15), (A3.22) satisfies a differential equation of the form

$$x'' + p(t)x' + \lambda q_1(t)x = 0, \tag{A3.23}$$

where $q_1(t)$ is a bounded measurable function, $q_1(t) = 0$ if $y_1(t) = y_2(t)$ and, otherwise,

$$q_1(t) = [H(t, y_2) - H(t, y_1)]/(y_2 - y_1),$$

so that $|q_1(t)| \leqq A$. Thus, condition (A3.21) on λ and Proposition A2.1 imply that (A3.23) is disconjugate on $t \geqq 0$, and (iii) follows from Corollary A2.1.

COROLLARY A3.2. *In* (A2.1), *let $p(t)$, $q(t)$ be continuous for $t \geqq 0$; p_1, p_2 positive constants;*

$$q(t) \geqq 0 \quad \textit{for } t \geqq 0; \tag{A3.24}$$

$X(t) \in C^2[0, \infty)$ satisfies (A3.12) *and*

$$X'/X \leqq -p_1 < 0 \quad \textit{for } t \geqq 0. \tag{A3.25}$$

Let $P(t)$, $H(t, y)$ be continuous for $t \geqq 0$, $y \geqq 0$;

$$P(t) - p(t) \geqq p_2 > 0 \quad \textit{for } t \geqq 0. \tag{A3.26}$$

(i) *If $H(t, y)$ satisfies* (A3.13), *and*

$$|\lambda| < a_1 p_1 p_2 / A \quad \textit{and} \quad 0 < a \leqq a_1, \tag{A3.27}$$

then the equation

$$L_0[y] = y'' + P(t)y' + \lambda H(t, y) = 0 \tag{A3.28}$$

has a solution satisfying (A3.16).

(ii) *If $A, B, a_1, \varepsilon, H$ are as in Corollary* A3.1(ii),

$$|\lambda| < p_2 p_1 \min\left(a_1(Aa_1 + B)^{-1}, \varepsilon(A\varepsilon + B)^{-1}\right) \quad \textit{and} \quad 0 < a \leqq a_1, \tag{A3.29}$$

then (A3.28) *has a solution satisfying* (A3.19).

Proof. On (i). This assertion follows from Proposition A3.1 if we verify that $y = a_1 X$ is a supersolution. Note that

$$\begin{aligned} L_0[a_1 X] &\leqq a_1(P - p)X' - a_1 q X + \lambda H(t, a, X) \\ &\leqq (a_1 p_2 X'/X + |\lambda|(A))X \leqq (-a_1 p_1 p_2 + |\lambda| A)X. \end{aligned}$$

Thus $L_0[a_1 X] < 0$ when λ satisfies (A3.27).

On (ii). This is proved by verifying that $y = a_1 X$ is a supersolution and $u = -\varepsilon X$ is a subsolution if (A3.29) holds.

PROPOSITION A3.2. *In* (A2.1), *let $p(t)$, $q(t)$ be continuous for $t \geqq 0$ and $X(t) \in C^2[0, \infty)$ satisfy* (A3.12). *Let $H(t, y)$ be continuous for $t \geqq 0$, $y \geqq 0$ and such that*

$$H(t, 0) = 0, \qquad q(t)y - H(t, X(t)y) \quad \textit{is nondecreasing in } y \tag{A3.30}$$

(e.g., let $q(t) \geqq 0$ and $H(t, y)$ be nonincreasing in y or let $q(t) \geqq q_0 > 0$ and $|H(t, y_2) - H(t, y_1)| \leqq q_0|y_2 - y_1|$). Then, for any $a > 0$, the equation (A3.15) *has a solution $y = y(t)$ satisfying*

$$y(0) = a, \quad 0 \leqq y \leqq aX \quad \text{and} \quad y' \leqq (X'/X)y \leqq aX' \not\equiv 0 \quad \text{for } t \geqq 0. \tag{A3.31}$$

If $H(t, y)$ is nonincreasing with respect to y and $X(t) \to 0$, as $t \to \infty$, then the solution y is unique.

Proof. Introduce the new dependent variable

$$x = y/X(t),$$

so that $y = xX(t)$, and (A3.15) becomes

$$x'' = -(p + 2X'/X)x' - x\ell[X]/X + \{q(t)x - H(t, Xx)\}/X. \tag{A3.32}$$

The right side vanishes for $x = x' = 0$ and is nondecreasing in x. Hence, (A3.32) has a solution satisfying

$$x(0) = a, \quad x \geqq 0 \quad \text{and} \quad x' \leqq 0 \quad \text{for } t \geqq 0;$$

see Hartman [4] or [6, Theorem 5.2 and Exercise 5.3, p. 434 and p. 575]. The corresponding solution y of (A3.15) satisfies (A3.31).

Since $y(0) = a$ and $y(\infty) = 0$, the uniqueness follows from the monotonicity of H; cf. [6, Exercise 4.6(c), p. 472 and pp. 574–575] for the analogous situation when $[0, \infty)$ is replaced by a compact t-interval.

REFERENCES

[1] W. A. Coppel, *On a differential equation of boundary layer theory*, Philos. Trans. Roy. Soc. London Ser. A, 253 (1960), pp. 101–136.

[2] V. M. Falkner and S. W. Skan, *Solutions of the boundary layer equations*, Philos. Mag., 12 (1931), pp. 865–896.

[3] D. Grohne and R. Iglisch, *Die laminare Grenzschicht an der längsangeströmten eben Platte mit schrägem Absaugen und Ausblasen*, Veröffentlichung Math. Inst. Hochschule, Braunschweig, 1945.

[4] P. Hartman, *On boundary value problems for systems of ordinary, nonlinear, second order differential equations*, Trans. Amer. Math. Soc., 96 (1960), pp. 493–509.

[5] ———, *On the asymptotic behavior of solutions of a differential equation in boundary layer theory*, Z. Angew. Math. Mech., 44 (1964), pp. 123–128.

[6] ———, *Ordinary Differential Equations*, John Wiley, New York, 1964.

[7] ———, *On third order, nonlinear, singular, boundary value problems*, Pacific J. Math., 36 (1971), pp. 165–180.

[8] S. P. Hastings, *An existence theorem for a class of nonlinear, boundary value problems including that of Falkner-Skan*, J. Differential Equations, 9 (1971), pp. 580–590.

[8a] ———, *A boundary value problem related to the Falkner-Skan equations*, Notices Amer. Math. Soc., 16 (1969), p. 1080; see [8].

[9] D. Ho and H. K. Wilson, *A boundary value problem arising in boundary layer theory*, Arch. Rational Mech. Anal., 27 (1967), pp. 165–174.

[10] R. Iglisch, *Elementarer Existenzbeweis für die Strömung in der laminaren Grenzschicht zur Potentialströmung $U = u_1x^m$ mit $m > 0$ bei Absaugen und Ausblasen*, Z. Angew. Math. Mech., 33 (1953), pp. 143–147.

[11] ———, *Elementarer Beweis für die Eindeutigkeit der Strömung in der laminaren Grenzschicht zur Potentialströmung $U = u_1x^m$ mit $m \geqq 0$ bei Absaugen und Ausblasen*, Ibid., 34 (1954), pp. 441–443.

[12] R. IGLISCH AND F. KEMNITZ, *Ueber die in der Grenzschictheorie auftretende Differentialgleichung* $f''' + ff'' + \beta(1 - f'^2) = 0$ *für* $\beta < 0$ *bei gewissen Absauge und Ausblasegezeten*, 50 Jahre Grenzschichtforschung, Braunschweig, 1955.
[13] E. KAMKE, *Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II*, Acta Math., 58 (1932), pp. 57–85.
[14] C. C. LAN, *On functional-differential equations and some laminar boundary layer problems*, Arch. Rational Mech. Anal., 42 (1971), pp. 24–39.
[15] J. B. MCLEOD AND J. B. SERRIN, *The existence of similar solutions for some boundary layer problems*, Ibid., 31 (1968), pp. 288–303.
[16] M. NAGUMO, *Ueber die Differentialgleichung* $y'' = f(x, y, y')$, Proc. Phys. Math. Soc. Japan (3), 19 (1937), pp. 861–866.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ON THE ASYMPTOTIC BEHAVIOR OF VOLTERRA INTEGRAL EQUATIONS*

JAMES L. KAPLAN†

Abstract. Suppose $y(t) = f(t) - \int_0^t a(t, s)y(s)\, ds$ is a system of Volterra integral equations, and let $r(t, s)$ be the resolvent kernel corresponding to this system. If $f(t)$ is continuous and ω-periodic, it is shown that under suitable restrictions on $r(t, s)$, the solution $y(t)$ is asymptotically ω-periodic. These conditions generalize a previous result of Miller, Nohel and Wong.

For the perturbed system $x(t) = f(t) - \int_0^t a(t, s)\{x(s) + g(s, x(s))\}\, ds$, if the resolvent kernel is "sufficiently close" to an L^1-function, then $|x(t) - y(t)| \to 0$ as $t \to \infty$ for a suitable class of perturbation terms $g(t, x)$. If the resolvent is of convolution type, this generalizes a theorem of A. Strauss. Finally, it is shown that if the resolvent kernel is of convolution type, and is in $L^1[0, \infty)$, then the Cesàro integral mean of $|x(t) - y(t)|$ converges to zero, for perturbations which are bounded and diminishing.

1. Introduction. Consider the systems of Volterra integral equations

$$y(t) = f(t) - \int_0^t a(t, s)y(s)\, ds, \tag{1.1}$$

$$x(t) = f(t) - \int_0^t a(t, s)\{x(s) + g(s, x(s))\}\, ds, \tag{1.2}$$

where x, y, f and g are vectors in R^n, $a(t, s)$ is an $n \times n$ matrix, and R^n is Euclidean n-dimensional space. Let $|\cdot|$ denote any vector norm in R^n.

The resolvent system corresponding to system (1.1) is

$$r(t, s) = a(t, s) - \int_s^t a(t, u)r(u, s)\, du \tag{1.3}$$

and its solution $r(t, s)$ is called the resolvent kernel. It is well known [3] that if $a(t, s)$ is locally L^1 in (t, s), and if $r(t, s)$ exists and is locally L^1 in (t, s), then systems (1.1) and (1.2) may be rewritten in the equivalent forms

$$y(t) = f(t) - \int_0^t r(t, s)f(s)\, ds, \tag{1.4}$$

$$x(t) = y(t) - \int_0^t r(t, s)g(s, x(s))\, ds. \tag{1.5}$$

Furthermore, we shall assume that $f(t)$, $a(t, s)$, $r(t, s)$ and $g(t, x)$ are sufficiently smooth to insure the local existence and uniqueness of solutions of (1.1) and (1.2) and the continuability of solutions so long as they remain bounded. Sufficient conditions for these hypotheses to be valid may be found in [3].

We are interested in providing sufficient conditions for the solution of (1.1) to be asymptotically periodic. Theorem 2.4 shows that if f is continuous and periodic, and if $r(t, s)$ is "sufficiently close" to being an L^1-function, then the solution $y(t)$ of (1.1) is asymptotically periodic.

* Received by the editors February 2, 1971.

† Department of Mathematics, Northwestern University, Evanston, Illinois 60201.

We also discuss a recent result of A. Strauss [5] comparing the solution of the unperturbed system (1.1) with the perturbed system (1.2) when $|g(t, x)| \leqq \varphi(t)(1 + |x|)$, where $\varphi(t)$ is small in some sense. Theorem 3.2 gives sufficient conditions so that $|x(t) - y(t)| \to 0$ as $t \to \infty$. Finally, Theorem 3.5 compares the solution of (1.1) with the solution of (1.2), provided that $|g(t, x)| \leqq \lambda(t)$, where $\lambda(t)$ is bounded and diminishing.

2. On the unperturbed system. We require the following lemmas.

LEMMA 2.1. *Let $f(t)$ be continuous and $g(t)$ be periodic. If $|f(t) - g(t)| \to 0$ as $t \to \infty$, then g is continuous.*

Proof. Let t_0 be given and let $\varepsilon > 0$. Then there exists $T > 0$ such that $|f(t) - g(t)| < \varepsilon/3$ for all $t \geqq T$. If ω is the period of g, let n be an integer chosen so that $n\omega \in [T, T + \omega]$. By the continuity of f at $t_0 + n\omega$, let $\delta = \delta(\varepsilon/3)$ be chosen. Now suppose that $|t - t_0| < \delta$. Then

$$\begin{aligned} |g(t) - g(t_0)| \leqq\ & |g(t) - g(t + n\omega)| + |g(t + n\omega) - f(t + n\omega)| \\ & + |f(t + n\omega) - f(t_0 + n\omega)| + |f(t_0 + n\omega) - g(t_0 + n\omega)| \\ & + |g(t_0 + n\omega) - g(t_0)|. \end{aligned}$$

The first and last terms are zero by the periodicity of g. The second and fourth terms are less than $\varepsilon/3$ since $|f(t) - g(t)| \to 0$ and $n\omega \geqq T$. The middle term is small by the continuity of f at $t_0 + n\omega$. Thus $|g(t) - g(t_0)| < \varepsilon$ and g is continuous at t_0.

DEFINITION 2.2. $f(t)$ is *asymptotically ω-periodic* if there exists a continuous ω-periodic function $p(t)$ such that $|f(t) - p(t)| \to 0$ as $t \to \infty$.

The following lemma gives a characterization of an asymptotically ω-periodic function in terms of its period. It is a corollary to a well-known similar result due to M. Fréchet [2] for almost periodic functions.

LEMMA 2.3. *The continuous function $f(t)$ is asymptotically ω-periodic if and only if given $\varepsilon > 0$, there exists $T = T(\varepsilon)$ such that*

$$|f(t + n\omega) - f(t)| < \varepsilon \quad \text{for all } t \geqq T, \qquad n = 1, 2, \cdots.$$

We are now prepared to state and prove a theorem about the asymptotic behavior of the solutions of (1.1). This result generalizes Lemma 4.1 of Miller, Nohel and Wong [4].

THEOREM 2.4. *Let $f(t)$ be continuous and ω-periodic. Suppose that the resolvent kernel $r(t, s)$ corresponding to* (1.1) *satisfies $r(t + \omega, s + \omega) = r(t, s)$,*

$$\lim_{t \to \infty} \int_{-\infty}^{0} |r(t, s)|\, ds = 0, \tag{2.1}$$

$$\lim_{h \to 0} \int_{t}^{t+h} |r(t + h, s)|\, ds + \int_{0}^{t} |r(t + h, s) - r(t, s)|\, ds = 0. \tag{2.2}$$

Then the solution $y(t)$ of (1.1) *is asymptotically ω-periodic.*

Proof. Let $\varepsilon > 0$ be given. Let $T = T(\varepsilon)$ be chosen by (2.1) so large that

$$\int_{-\infty}^{0} |r(t, s)|\, ds < \frac{\varepsilon}{\|f\|} \quad \text{whenever } t > T,$$

where $\|f\| = \sup|f(t)|$ for $t \in R$. Then using (1.4), we see that

$$|y(t + n\omega) - y(t)| \leqq \int_{-n\omega}^{0} |r(t, s)|\,|f(s)|\,ds \leqq \|f\| \int_{-\infty}^{0} |r(t, s)|\,ds < \varepsilon.$$

Moreover, by (2.2), $y(t)$ is continuous. By Lemma 2.3, $y(t)$ is asymptotically ω-periodic.

The proof of the corresponding results given in [4] required stronger hypotheses on $r(t, s)$ in order to be able to construct a contraction mapping. In addition to the hypotheses of Theorem 2.4, Miller, Nohel and Wong required that

$$\int_0^t |r(t, s)|\,ds \leqq B \quad \text{for all } t \geqq 0,$$

$$\lim_{t\to\infty} \int_0^T |r(t, s)|\,ds = 0 \quad \text{for each fixed } T > 0.$$

3. On a result of A. Strauss. We now turn our attention to a comparison of the solutions of (1.1) and (1.2). In order to do this, we first state a theorem due to A. Strauss [5].

THEOREM 3.1. *Let $r(t, s)$ be the resolvent kernel corresponding to* (1.1). *Let $y(t)$, $x(t)$ represent the solutions of* (1.1) *and* (1.2), *respectively. Suppose $r(t, s)$ satisfies*

$$\sup_{t\geqq 0} \int_0^t |r(t, s)|\,ds = B < \infty, \tag{3.1}$$

$$\lim_{t\to\infty} \int_0^T |r(t, s)|\,ds = 0 \quad \textit{for each fixed } T > 0. \tag{3.2}$$

Let $g(t, x)$ satisfy

$$|g(t, x)| \leqq \varphi(t)(1 + |x|) \quad \textit{for } t \geqq 0, \quad |x| < \infty, \tag{3.3}$$

where $\varphi(t) \to 0$ as $t \to \infty$. If $\|y\| < \infty$, then $|x(t) - y(t)| \to 0$ as $t \to \infty$.

It should be noted that if $r(t, s)$ is of convolution type, that is, $r(t, s) = r(t - s)$, and $r \in L^1[0, \infty)$, then r satisfies (3.1) and (3.2). In fact, for resolvents of convolution type, $r \in L^1$ if and only if (3.1) holds. We may think of conditions (3.1) and (3.2) as being, in some sense, the generalization of an L^1-function to functions of two variables. L^1-functions, however, have an additional property which is not shared by functions of two variables which satisfy (3.1) and (3.2). That property may be described as the property of having a "small" integral over sets of "small" measure. Using this property we are able to state the following theorem.

THEOREM 3.2. *Let x, y and r be as in Theorem* 3.1. *Suppose r satisfies* (3.1) *and* (3.2) *and has the following additional property*:

(i) *Given $\varepsilon > 0$ there exists $\delta > 0$ such that if A is any set contained in* $[0, t]$ *with $m(A) < \delta$, then*

$$\int_A |r(t, s)|\,ds < \varepsilon.$$

Suppose that $g(t, x)$ *satisfies* (3.3), *where* $\varphi(t)$ *is such that* $\|\varphi\| < \infty$ *and* φ *satisfies*:

(ii) *Given* $\alpha, \beta > 0$, *there exists* $T = T(\alpha, \beta)$ *such that*

$$m\{t : t \geqq T, |\varphi(t)| \geqq \alpha\} < \beta.$$

If $\|y\| < \infty$, *then* $|x(t) - y(t)| \to \infty$ *as* $t \to \infty$.

Remarks. Condition (ii) represents a relaxation of the condition $\varphi(t) \to 0$ as $t \to \infty$ to allow a broader class of perturbation terms. It holds, of course, when $\varphi(t) \to 0$ as $t \to \infty$, and will also hold if $\varphi(t) \in L^1[0, \infty) \cap L^\infty[0, \infty)$. In this way, if $r(t, s) = r(t - s)$, $r \in L^1$, then Theorem 3.2 generalizes Theorem 3.1.

Example 3.3. Let f be asymptotically ω-periodic. Let $r(t, s) = r(t - s) \in L^1[0, \infty)$. Let $|g(t, x)| \leqq \varphi(t)(1 + |x|)$, where φ is the function described as follows:

$$\varphi(t) = \begin{cases} 1/t & \text{for } 0 < t < \infty \text{ and } t \notin [n - 1/2^{n+1}, n + 1/2^{n+1}], \\ 2 & \text{for } t = n, \\ \text{linear on } [n - 1/2^{n+1}, n] \text{ and } [n, n + 1/2^{n+1}] \end{cases}$$

for $n = 1, 2, \cdots$. Then $\varphi(t)$ satisfies (ii) although $\varphi(t) \not\to 0$ and $\varphi(t) \notin L^1[0, \infty)$. By combining Theorem 2.6 and Theorem 3.2 we see that $x(t)$, the solution of (1.2), is asymptotically ω-periodic, in spite of the fact that the perturbation term is not periodic, nor is it small in either the L^1-norm or the L^∞-norm.

Proof of Theorem 3.2. The proof follows the proof of Theorem 2 in [5]. It proceeds in three stages.

PROPOSITION 1. *Every solution of* (1.2) *exists on* $[0, \infty)$.

The proof of Proposition 1 is identical to the proof of Proposition 1 in [5].

PROPOSITION 2. *Let* $\|y\| < \infty$. *Then* $\|x\| < \infty$.

Proof. Let δ be chosen by (i) so that $A \subset [0, t]$ and $m(A) < \delta$ implies

$$\int_A |r(t, s)|\, ds < \frac{1}{3\|\varphi\|}.$$

Now let T be chosen by (ii) so that if we let

$$A = \{t : t \geqq T, |\varphi(t)| \geqq 1/(3B)\},$$

then $m(A) < \delta$.

Since $x(t)$ exists on $0 \leqq t < \infty$ by Proposition 1, there exists $M > 1$ such that $|x(t)| \leqq M$ on $[0, T]$. Choose P so large that

$$\|y\| + B\|\varphi\|(1 + M) + 2/3 < P/3.$$

We claim that $|x(t)| < P$ for $t \in [0, \infty)$. If not, then there exists $t > T$ such that $|x(s)| < P$ for $0 \leqq s < t$ but $|x(t)| = P$. Then

$$\begin{aligned} |x(t)| \leqq \|y\| &+ \int_0^T |r(t, s)| g(s, x(s))|\, ds \\ &+ \int_{[T,t]-A} |r(t, s)|\, |g(s, x(s))|\, ds \\ &+ \int_A |r(t, s)|\, |g(s, x(s))|\, ds \end{aligned}$$

$$\leqq \|y\| + B\|\varphi\|(1 + M) + B(1 + P)\frac{1}{3B}$$

$$+ \|\varphi\|(1 + P)\frac{1}{3\|\varphi\|} < P.$$

This contradiction shows that $|x(t)| < P$ on $[0, \infty)$. Thus $\|x\| \leqq P < \infty$.

PROPOSITION 3. *Let* $\|x\| < \infty$. *Then* $|x(t) - y(t)| \to 0$ *as* $t \to \infty$.

Proof. Let $\|x\| = M$ and let $\varepsilon > 0$. By (i), let $\delta = \delta(\varepsilon/3\|\varphi\|(1 + M))$. By (ii), let $T = T(\varepsilon/3B(1 + M), \delta)$. Now let $\tau > T$ be chosen so large that

$$\int_0^T |r(t, s)|\, ds \leqq \frac{\varepsilon}{3\|\varphi\|(1 + M)} \qquad \text{for } t \geqq \tau.$$

Let

$$A = \left\{t : t \geqq T, |\varphi(t)| \geqq \frac{\varepsilon}{3B(1 + M)}\right\}.$$

Then $m(A) < \delta$. Now for all $t \geqq \tau$,

$$|x(t) - y(t)| \leqq \|\varphi\|(1 + M)\int_0^T |r(t, s)|\, ds$$

$$+ (1 + M)\int_{[T,t]-A} |r(t, s)|\,|\varphi(s)|\, ds$$

$$+ \|\varphi\|(1 + M)\int_A |r(t, s)|\, ds \leqq \varepsilon.$$

Thus $|x(t) - y(t)| \to 0$ as $t \to \infty$, and the proof is completed.

It should be noted that the above proof represents only a slight modification of the proof of Strauss [5].

This final example shows that Theorem 3.2 may not hold if we do not assume that $\|y\| < \infty$.

Example 3.4. Consider the following one-dimensional example. Let $\alpha(t) \in C^1(-\infty, \infty)$ satisfy

(i) $\alpha(t) > 0$ for all $t \in R$,

(ii) $\alpha(t) \to \infty$ as $t \to \infty$,

(iii) $\alpha'(t) \geqq 0$ for all $t \in R$.

Let $r(t, s) = -\alpha'(s)/\alpha(t)$, and suppose that $r(t, s)$ is the resolvent kernel of some function $a(t, s)$. Then $r(t, s)$ satisfies (3.1) and (3.2) since

$$\sup_{t \geqq 0} \int_0^t |r(t, s)|\, ds = \sup_{t \geqq 0} \int_0^t \frac{\alpha'(s)}{\alpha(t)}\, ds$$

$$= \sup_{t \geqq 0} \frac{\alpha(t) - \alpha(0)}{\alpha(t)} \leqq 1,$$

and

$$\lim_{t \to \infty} \int_0^T |r(t, s)|\, ds = \lim_{t \to \infty} \frac{\alpha(T) - \alpha(0)}{\alpha(t)} = 0.$$

Let $y(t)$ be the solution of (1.1). Then $y(t)$ may be represented using (1.4) as

$$y(t) = f(t) + \int_0^t \frac{\alpha'(s)}{\alpha(t)} f(s)\, ds.$$

It is obvious from this expression that $f(t)$ may be chosen so that $y(t)$ will satisfy $y(t) > 0$ for all $t \geqq 0$, and $y(t) \to \infty$ as $t \to \infty$. Define

$$g(t, x) = \frac{1}{|y(t)|}|x|.$$

Clearly,

$$g(t, x) = \frac{1}{|y(t)|}|x| \leqq \frac{1}{|y(t)|}(1 + |x|)$$

and $1/y(t) \to 0$ as $t \to \infty$. Hence $g(t, x)$ satisfies (3.3). Now $x(t)$, the solution of (1.2), may be written as

$$\begin{aligned} x(t) &= y(t) - \int_0^t r(t, s) g(s, x(s))\, ds \\ &= y(t) + \int_0^t \frac{\alpha'(s)}{\alpha(t)} \frac{|x(s)|}{y(s)}\, ds. \end{aligned}$$

Observe that

$$\frac{\alpha'(s)}{\alpha(t)} \frac{|x(s)|}{y(s)} \geqq 0 \quad \text{for } 0 \leqq s \leqq t.$$

This implies that $x(t) \geqq y(t)$. Therefore,

$$\begin{aligned} |x(t) - y(t)| = x(t) - y(t) &= \int_0^t \frac{\alpha'(s)}{\alpha(t)} \frac{|x(s)|}{y(s)}\, ds \\ &\geqq \int_0^t \frac{\alpha'(s)}{\alpha(t)} \frac{y(s)}{y(s)}\, ds = 1 - \frac{\alpha(0)}{\alpha(t)} \to 1 \quad \text{as } t \to \infty. \end{aligned}$$

Thus Theorem 3.2 fails.

We remark that we may take $\alpha(t) = e^t$ in the previous example and obtain the classic kernel $r(t, s) = -e^{-(t-s)}$.

Further remarks. We observe that the function $\varphi(t)$ of Example 3.3 is a diminishing function; that is, $\int_t^{t+1} \varphi(s)\, ds \to 0$ as $t \to \infty$. Further, the class of diminishing functions is closed under addition, and φ can be written as the sum of an L^1-function and a function which converges to zero, both of which are diminishing. This example, and others, lead to the belief that the class of diminishing functions is related to the class of perturbation terms for which Theorem 3.2 is valid. This belief is further enhanced by the fact that diminishing functions are known to play an important role in perturbation theory for asymptotically stable ordinary differential equations [6]. We remark that the class of bounded diminishing functions is the same as the class of bounded distributions converging to zero. Along these lines, we have the following theorem.

THEOREM 3.5. *Let $x(t)$ and $y(t)$ denote the solutions of* (1.1) *and* (1.2), *respectively. Suppose $r(t, s) = r(t - s)$, and $r \in L^1[0, \infty)$. Let $g(t, x)$ satisfy*

$$|g(t, x)| \leqq \lambda(t) \leqq K$$

for some constant K, where $\int_t^{t+1} \lambda(s)\, ds \to 0$ as $t \to \infty$. Then if $\|y\| < \infty$,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T |x(t) - y(t)|\, dt = 0.$$

Before we can give the proof of this theorem we require some notation. Denote by $L(f)$ the Laplace transform of the function f;

$$L(f)(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

Now let $\lambda(t) \geqq 0$ satisfy $\int_t^{t+1} \lambda(s)\, ds \to 0$ as $t \to \infty$ (that is, λ is diminishing), and define $\Lambda(t) = \int_t^{t+1} \lambda(s)\, ds$. We shall write $f(t) = O_L(g(t))$ for $t > 0$ if $f(t) \geqq -Mg(t)$ for t sufficiently large. We shall write $f(t) \sim Ag(t)$ as $t \to t_0$ if $f(t)/g(t) \to A$ as $t \to t_0$.

We now state several lemmas.

LEMMA 3.6. $L(\lambda - \Lambda(0) = \Lambda(0)$.

Proof. Consider $L(\Lambda)(s)$. Integrating by parts, and using the fact that $\Lambda'(t) = \lambda(t + 1) - \lambda(t)$, we have

$$\int_0^\infty e^{-st}\Lambda(t)\, dt = \frac{\Lambda(t)\, e^{-st}}{-s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}[\lambda(t + 1) - \lambda(t)]\, dt$$
$$= \frac{\Lambda(0)}{s} + \frac{1}{s}[L(\lambda(t + 1))(s) - L(\lambda(t))(s)].$$

Now $L(\lambda(t + 1))(s) = e^s[L(\lambda)(s) - \int_0^1 e^{-st}\lambda(t)\, dt]$. Therefore,

$$L(\Lambda)(s)\frac{s}{e^s - 1} = L(\lambda)(s) + \frac{\Lambda(0) - e^s\int_0^1 e^{-st}\lambda(t)\, dt}{e^s - 1}.$$

Now, using the fact that

$$\lim_{s \to 0} \int_0^1 e^{-st}\lambda(t)\, dt = \int_0^1 \lambda(t)\, dt = \Lambda(0)$$

and applying L'Hospital's rule, we obtain

$$L(\Lambda)(0) = L(\lambda)(0) - \Lambda(0).$$

The following lemma is a Tauberian theorem for Laplace transforms. Its proof may be found in [1].

LEMMA 3.7. *Let $L(h)(s)$ converge for $s > 0$. If $L(h)(s) \sim c/s^\gamma$ for $s \to 0$, with $\gamma > 0$, c real, and if $h(t) = O_L(t^{\gamma - 1})$ for $t > 0$, then*

$$\frac{1}{t^\gamma}\int_0^t h(\tau)\, d\tau \sim \frac{c}{\Gamma(\gamma + 1)} \quad \text{for } t \to \infty,$$

where Γ is the usual gamma function.

LEMMA 3.8. *If $f(t) = g(t)$ for all $t \geqq M > 0$, where $\|f\| < \infty$ and $\|g\| < \infty$, and if $\gamma \in L^1[0, \infty)$,*

$$\lim_{t \to \infty} \int_0^t |r(t - s)|(f(s) - g(s))\, ds = 0.$$

The proof of Lemma 3.8 is simple and is omitted. We may now proceed with the proof of Theorem 3.5.

Proof. Define

$$\bar{\lambda}(t) = \begin{cases} 0, & 0 \leqq t \leqq 2, \\ \lambda(t), & t > 2, \end{cases}$$

and let $\bar{\Lambda}(t) = \int_t^{t+1} \bar{\lambda}(s)\, ds$. Now since $r \in L^1[0, \infty)$, we know that $L(|r|)(0) < \infty$. Thus

$$L\left(\int_0^t |r(t - 2)|(\bar{\Lambda}(s) - \bar{\lambda}(s))\, ds\right)(0) = L(|r|)(0) \cdot L(\bar{\Lambda} - \bar{\lambda})(0)$$

$$= L(|r|)(0) \cdot \bar{\Lambda}(0) = 0$$

by Lemma 3.6, and because $\bar{\Lambda}(0) = 0$. Hence, by Lemma 3.7, with

$$h(t) = \int_0^t |r(t - 2)|(\bar{\Lambda}(s) - \bar{\lambda}(s))\, ds,$$

$\gamma = 1$, $c = 0$, we have

$$\frac{1}{T}\int_0^T \int_0^t |r(t - s)|(\bar{\Lambda}(s) - \bar{\lambda}(s))\, ds\, dt \sim 0.$$

By Lemma 3.8, since $\bar{\Lambda}(t) - \bar{\lambda}(t) = \Lambda(t) - \lambda(t)$ for $t > 2$, we have

$$\frac{1}{T}\int_0^T \int_0^t |r(t - s)|(\Lambda(s) - \lambda(s))\, ds\, dt \sim 0.$$

But $\int_0^t |r(t - s)|\Lambda(s)\, ds \to 0$ as $t \to \infty$, by Theorem 3.1, because $\Lambda(t) \to 0$ as $t \to \infty$. Therefore,

$$0 \leqq \frac{1}{T}\int_0^T |x(t) - y(t)|\, dt \leqq \frac{1}{T}\int_0^T \int_0^t |r(t - s)|\lambda(s)\, ds\, dt \to 0$$

as $T \to \infty$, and the theorem is proved.

Remarks. It is suspected that the stronger conclusion $\int_t^{t+1} |x(s) - y(s)|\, ds \to 0$ as $t \to \infty$ holds. It is also not known whether a result such as Theorem 3.5 is valid if we assume that $|g(t, x) \leqq \lambda(t)|x|$, where $\lambda(t)$ is bounded and diminishing.

REFERENCES

[1] G. DOETSCH, *Handbuch der Laplace-Transformation*, Verlag Birkhäuser, Basel, 1950.

[2] M. FRÉCHET, *Les fonctions asymptotiquement presque-periodiques continues*, C. R. Acad. Sci. Paris, 213 (1941), pp. 520–522.

[3] R. K. MILLER, *Nonlinear Volterra Integral Equations*, Lecture Note Series, W. A. Benjamin, New York, 1970.

[4] R. K. MILLER, J. A. NOHEL AND J. S. W. WONG, *Perturbations of Volterra integral equations*, J. Math. Anal. Appl., 25 (1969), pp. 676–691.

[5] A. STRAUSS, *On a perturbed Volterra integral equation*, Ibid., 30 (1970), pp. 564–575.

[6] A. STRAUSS AND J. A. YORKE, *Perturbing asymptotically stable differential equations*, Bull. Amer. Math. Soc., 74 (1968), pp. 992–996.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ON A NEW DISCRETE ANALOGUE OF THE LEGENDRE POLYNOMIALS*

M. WAYNE WILSON†

Abstract. A new system of orthogonal polynomials has been introduced recently by the author. These polynomials exhibit many of the "nice" properties of the Legendre polynomials. The evidence, theoretical and computational, implies that, as a discrete analogue to the Legendre polynomials, these polynomials are "superior" to the classical Hahn polynomials. In this paper, proofs of the announced results are presented and further development and generalizations are indicated.

1. Introduction. Recently we have introduced a new set of polynomials orthogonal on a finite point set (Wilson [19]). These polynomials, a striking analogue of the Legendre polynomials, have many of the behavioral characteristics of ultraspheric polynomials, yet they are not Fejér "generalized Legendre polynomials" (Szegö [14, § 6.5]). In this paper, we provide proofs of the results previously announced.

The usual discrete analogue of the Legendre polynomials, well studied in the literature (recently, Karlin and McGregor [10], Levit [11], Wilson [18]), is the special case of the Hahn polynomials defined by equal weights. (These are sometimes called Gram or Chebyshev polynomials of least squares.) Numerical evidence indicates that the polynomials discussed here are a superior analogue than the Hahn polynomials. In numerical applications, where the "closeness of the analogy" is of importance, the new polynomials perform better. (See Wilson [20], although the text has some minor errors.) The performance is expected, however, since the new polynomials converge like $1/N^2$ to the Legendre polynomials while the Hahn polynomials converge like $1/N$.

In contrast to the Hahn polynomials, the new polynomials do not have a difference "Rodrigues formula," nor do they have a known closed form recurrence formula. Rather, they are obtained by the method of Forsythe [7]. The Hahn polynomials are really a family of discrete analogues to the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$. The new polynomials are also a particular set belonging to a class of analogues of the ultraspherics (and perhaps, of Jacobi polynomials). Other known members of the set are $\{T_n\}$ and $\{U_n\}$, the Chebyshev polynomials of first and second kind. One may ask, what is the nature of other polynomials of this class? Are they "good" analogues? Section 7 discusses these questions.

Virtually all that is known about these new polynomials is the orthogonality relationship. A number of recent papers have been concerned with obtaining positive expansions of polynomial sets in terms of other polynomial sets (Askey [1], [2], [3], [4], [5], Wilson [16]). This paper illustrates the use of, and the power of, such results. By continually exploiting the positivity of expansion coefficients, we are able to obtain qualitative properties, where quantitative results have not (perhaps cannot) been obtained. It is somewhat surprising that so much can be obtained from the inner product.

* Received by the editors November 3, 1970, and in revised form May 24, 1971.

† IBM Scientific Center, Houston, Texas 77025.

2. The inner product. Let $\phi = \pi/(N+1)$ and $\psi = \phi/2$. (These two "angles" will appear frequently.) Let $t_i = \cos i\phi$ and $w_i = \sin i\phi$, $i = 1, 2, \cdots, N$. (Note that the t_i are the zeros of $U_N(x)$, and extrema of $T_N(x)$, a most propitious point set in approximation theory!) Let $[\cdot,\cdot]_N$ be the inner product defined by

$$[f, g]_N = 2\tan\psi \sum_{i=1}^{N} w_i f(t_i) g(t_i)$$

and $(\cdot,\cdot)$ be the usual integral inner product associated with the Legendre polynomials. For notational ease, we shall let

$$[f, g]_N^* = \sum_{i=1}^{N} w_i f(t_i) g(t_i).$$

Although $[\cdot,\cdot]_N$ and $[\cdot,\cdot]_N^*$ are, strictly speaking, only inner products on the space of polynomials of degree at most $N-1$, and bilinear forms on $C[-1, 1]$ and $L_2[-1, 1]$, we shall let context be the guide, and speak only of inner products.

In reality, we should use $\phi_N, \psi_N, t_i^{(N)}, w_i^{(N)}$, but, except for a few minor occurrences, N will always be understood. However, note that the usual economization of this point set is easily utilized. If $M = 2N+1$, then $\phi_M = \phi_N/2$, $t_{2i}^{(M)} = t_i^{(N)}$, $w_{2i}^{(M)} = w_i^{(N)}$, and $\tan\psi_M = (\cos\psi_N)(\tan\psi_N)/(1+\cos\psi_N)$.

Observe that the points are symmetric about the origin, and the weights at symmetric points are equal, so that $[1, f]_N = 0$, f an odd function, and $[f, g]_N = [g, f]_N = [1, fg]_N$, $f, g \in C[-1, 1]$.

Further, the trigonometric identity (Ryshik and Gradstein [13, 1.344])

$$\sum_{k=1}^{N} \sin kp\phi = \begin{cases} \cot p\psi, & p = 1, 3, 5, \cdots, \\ 0, & p = 0, 2, 4, \cdots, \end{cases} \tag{2.1}$$

shows that $[1, 1]_N = 2 = (1, 1)$ for all N. This innocent identity is the keystone to obtaining the results of this paper. Virtually every property of the polynomials defined herein ultimately rests upon it.

Although Riemann sum considerations (Rice [12, § 2–4]) show that $[t^n, t^n]_N \to (t^n, t^n)$ as $N \to \infty$, n fixed, we can show that this convergence is strictly increasing with N, and of order N^{-2}.

LEMMA 2.1. *For fixed n, $N > n > 1$,*

(i) $[t^n, t^n]_{n+1} \leqq [t^n, t^n]_N < [t^n, t^n]_{N+1} < (t^n, t^n) = 2/(2n+1)$,

(ii) $[t^n, t^n]_N = (t^n, t^n) + O(N^{-2})$ *as* $N \to \infty$,

(iii) $$[t^n, t^n]_N^* = \frac{2^{-2n}}{(2n+1)} \sum_{k=0}^{n} \binom{2n+1}{k} \frac{2n-2k+1}{\tan(2n-2k+1)\psi}.$$

Proof. We show part (iii) first. Note first that $[t^n, t^n]_N^* = [1, t^{2n}]_N^* = \sum_{i=0}^{N} \sin i\phi \cos^{2n} i\phi$. Applying the identity

$$\sin y \cos x = \tfrac{1}{2}[\sin(x+y) - \sin(x-y)] \tag{2.2}$$

to the identity

$$\cos^{2n} x = 2^{-2n}\left\{\sum_{k=0}^{n-1} 2\binom{2n}{k} \cos 2(n-k)x + \binom{2n}{n}\right\},$$

we obtain

$$\sin i\phi \cos^{2n} i\phi = 2^{-2n}\left\{\sin(2n+1)i\phi + \sum_{k=0}^{n-1}\left[\binom{2n}{k+1} - \binom{2n}{k}\right]\sin(2n-2k-1)i\phi\right\}.$$

Application of (2.1) after substitution of $\sin i\phi \cos^{2n} i\phi$ into $[t^n, t^n]_N^*$, and use of the identity

$$\binom{2n}{k+1} - \binom{2n}{k} = \frac{2n-2k-1}{2n+1}\binom{2n+1}{k+1},$$

yields (iii).

To show the monotonicity of (i), note that from (iii), $[t^n, t^n]_N$ is a positive combination of terms of the form $\tan\psi/\tan p\psi$, $p = 1, 3, 5, \cdots, 2n+1$. Elementary calculus shows that $g(x) = \tan x/\tan px$ is continuous and strictly decreasing on $[0, \pi/p)$, $p \geqq 2$. Thus, since $\psi = \pi/(2(N+1))$, for $N \geqq n$, $\tan\psi/\tan p\psi$, $p = 3, 5, \cdots, 2n+1$, is strictly increasing with N, with limit $1/p$. The positive combination implies that $[t^n, t^n]$ is strictly increasing with N, $N \geqq n$.

Taking the limit as $N \to \infty$ ($\psi \to 0$) in (iii) we obtain

$$\lim_{N\to\infty} [t^n, t^n]_N = \frac{2^{-2n+1}}{2n+1}\sum_{k=0}^{n}\binom{2n+1}{k}.$$

The summation is easily shown to be 2^{2n} so that the limit is $2/(2n+1)$, completing part (i).

To show the order of convergence, we note the series expansions

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots, \qquad x^2 < \frac{\pi^2}{4},$$

$$\cot x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \cdots, \qquad x^2 < \pi^2,$$

so that

$$\frac{\tan x}{\tan px} = \frac{1}{p} - \left(\frac{p^2-1}{3p}\right)x^2 - \frac{(p^2-1)(p^2+6)}{45p}x^4 - \cdots.$$

Since $[t^n, t^n]_N$ is a positive combination of terms $\tan\psi/\tan p\psi$, $p = 1, 3, 5, \cdots, 2n+1$, and since $\lim_{N\to\infty}[t^n, t^n]_N = (t^n, t^n)$ is obtained by letting $\psi \to 0$, the result follows, which completes the proof of the lemma.

Letting $Z_p = \tan\psi/\tan p\psi$, the first few inner products are explicitly

$$[1, 1]_N = 2,$$

$$[t, t]_N = [1 + Z_3]/2,$$

$$[t^2, t^2]_N = [2 + 3Z_3 + Z_5]/8,$$

$$[t^3, t^3]_N = [5 + 9Z_3 + 5Z_5 + Z_7]/32.$$

3. The polynomials. We let $q_k(t; N)$, $k = 0, 1, \cdots, N-1$, be the monic form of the orthogonal polynomials of $[\cdot, \cdot]_N$. Taking account of the symmetry of the inner product, the polynomials are given by the recurrence

$$q_0 \equiv 1, \quad q_1 \equiv t, \quad q_{n+1} = tq_n - \beta_n(N)q_{n-1}, \qquad n = 1, 2, \cdots, N-2,$$

where $\beta_n(N) = [q_n, q_n]_N/[q_{n-1}, q_{n-1}]_N$. The polynomials are even or odd with n.

Explicitly,

$$\beta_1(N) = \frac{1 + Z_3}{4},$$

$$\beta_2(N) = \frac{1 + Z_3 + Z_5 - Z_3^2}{4(1 + Z_3)},$$

$$\beta_3(N) = \frac{1 + Z_3 + Z_5 + Z_7 + Z_3(1 - Z_5 + Z_7) - Z_5^2}{4(1 + Z_3)(1 + Z_3 + Z_5 - Z_3^2)}.$$

Further, we have

$$[q_0, q_0]_N = 2,$$

$$[q_1, q_1]_N = \frac{1 + Z_3}{2},$$

$$[q_2, q_2]_N = \frac{1 + Z_3 + Z_5 - Z_3^2}{8}.$$

There does not appear to be a simple closed form for $\beta_n(N)$, so that, at the moment, the Forsythe [7] technique appears to be the most efficient method of calculating the polynomials.

Applying Wilson [17], which shows how the convergence of $[1, t^k]$ affects other quantities of a discrete system, we have $\beta_n(N) = n^2/(4n^2 - 1) + O(N^{-2})$. Computationally, it appears that the $\beta_n(N)$ are monotonic increasing with N for fixed n, and monotonic decreasing in n for fixed N.

Letting k_n be the leading coefficient of the Legendre polynomial $P_n(t)$, that is, $k_n = 2^n(1/2)_n/n!$, and defining $Q_n(t; N) \equiv k_n q_n(t; N)$, we have, by use of Wilson [17], again, the following theorem.

THEOREM 3.1. *Using the preceding notation,*

$$Q_n(t; N) = P_n(t) + O(N^{-2}). \tag{3.1}$$

4. Inner products of classical polynomials. Among the most remarkable features of this new system of polynomials $\{Q_n(t; N)\}$ are their expansion properties. Equally remarkable is that these properties are derivable from the behavior of the inner products of the Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$.

Observe that $[\cdot, \cdot]_N$ and the polynomials T_n and U_n are "made for one another," since

$$T_n(\cos y) = \cos ny, \qquad U_n(\cos y) = \frac{\sin (n + 1)y}{\sin y}.$$

LEMMA 4.1. *For $n + m$ even, $n \geqq m > 0$,*

(i) $[T_n, T_m]_N$

$$= -\tan\psi \sin\phi \left[\frac{1}{\cos\phi - \cos(n+m)\phi} + \frac{1}{\cos\phi - \cos(n-m)\phi}\right],$$

(ii) $[T_n, T_m]_N$ *is finite, strictly increasing with N, $2N + 1 > n + m$, with limit*

$$-\left[\frac{1}{(n+m)^2 - 1} + \frac{1}{(n-m)^2 - 1}\right].$$

Note that $[T_n, T_m]_N$ is zero if $n + m$ is odd, is negative if $n + m$ is even, $m \neq n$, and is positive for $n = m$ (which is necessary from the inner product definition).

Proof. Recalling that $T_n(x)T_m(x) = \frac{1}{2}[T_{n+m}(x) + T_{n-m}(x)]$ for $n \geqq m$, and that $[T_n, T_m]_N = [1, T_nT_m]_N$, it suffices to consider $[1, T_n]_N$, $2N \geqq n$. Now, for n even, $[1, T_n]_N^* = \sum_{i=1}^N \sin i\phi \cos in\phi$, which, by applying the identities (2.1) and (2.2), evaluates to $\frac{1}{2}[\cot(n+1)\psi - \cot(n-1)\psi]$. This can be rearranged to $-\sin\phi \cdot (\cos\phi - \cos n\phi)^{-1}$. Hence, for n even, $[1, T_n]_N = -2\tan\psi \sin\phi \cdot (\cos\phi - \cos n\phi)^{-1}$, thus showing (i).

To show the monotonic nature, we shall consider the reciprocal, observing that

$$\frac{-2}{[1, T_n]_N} = \frac{\cot\psi}{\sin\phi} \cdot (\cos\phi - \cos n\phi)$$

$$= -1 + \left(\frac{\sin n\psi}{\sin\psi}\right)^2, \qquad n = 0, 2, 4, \cdots,$$

using half-angle formulas ($\psi = \phi/2$). For $n = 2, 4, \cdots$, this is $-1 + [U_{n-1}(\cos\psi)]^2$. Now on $[\cos(\pi/n), 1]$, $U_{n-1}(x)$ is increasing from zero to n. ($\cos(\pi/n)$ is the zero on $[-1, 1]$ closest to 1.)

Now $\sin nx/\sin x \geqq 1$ if

$$\sin nx - \sin x = 2\sin\left(\frac{n-1}{2}\right)x \cos\left(\frac{n+1}{2}\right)x \geqq 0,$$

which is true for $0 \leqq ((n+1)/2)x \leqq \pi/2$, since $n = 2, 4, \cdots$. Thus, $\sin nx/\sin x > 1$ for $0 \leqq ((n+1)/2)x < \pi/2$, so $-2/[1, T_n]_N > 0$, strictly increasing with N if $\psi < \pi/(n+1)$, that is, if $n < 2N + 1$. Thus $[1, T_n]_N$ is negative, strictly increasing with N, with limit $-2/(n^2 - 1)$, thus showing (ii).

LEMMA 4.2. *For $n + m$ even,*

(i) $[T_m, U_n]_N = \tan\psi[\cot(n + m + 1)\psi + \cot(n - m + 1)\psi]$,

(ii) $[T_m, U_n]_N$ *is finite, positive, and strictly increasing with N, $N > n \geqq m > 0$, with limit $2(N+1)/((n+1)^2 - m^2)$.*

Again, $[T_m, U_n]_N$ is zero if $n + m$ is odd.

Proof. Since $(\sin i\phi)T_m(\cos i\phi)U_n(\cos i\phi) = \cos im\phi \cdot \sin i(n+1)\phi$, part (i) follows using identities (2.1) and (2.2) as in part (i) of Lemma 4.1. Part (ii) follows identically to part (i) of Lemma 2.1.

Although we already know $[P_n, P_m]_N = 0$, $n + m$ odd, and $[P_n, P_m]_N \to (P_n, P_m) = 2\delta_{m,n}/(2n+1)$, as $N \to \infty$, we need, for later use, the nature of this convergence. The proof of the following lemma is an immediate consequence of Lemma 4.1 and the two identities (Szegö [14, § 4.9 and exercise 84]).

$$P_n(t) = \sum_{k=0}^{[n/2]} g_k g_{n-k} T_{n-2k}(t), \tag{4.1}$$

where $g_0 = 1, g_k = (1/2)_k/k! > 0$, and

$$P_n(t)P_m(t) = \sum_{k=0}^{m} \alpha_k(n, m) P_{n+m-2k}(t), \qquad n \geqq m, \tag{4.2}$$

where $\alpha_k(n, m) > 0$.

LEMMA 4.3. *For $n + m$ even, $N \geqq n \geqq m \geqq 1$,*

(i) $[P_n, P_m]_N < 0$ *and strictly increases with N to limit* 0, *if $n > m$,*

(ii) $0 < [P_n, P_n]_N$ *and strictly increases with N to limit* $2/(2n+1)$.

5. Expansion properties. The ultraspherical polynomials $\{P_n^{(\alpha,\alpha)}(t)\}$ for $\alpha \in [-1/2, 1/2]$ have a very characteristic set of sign properties, when expanding one set in terms of another set. Writing

$$P_n^{(\alpha,\alpha)} = \sum_{k=0}^{n} D_k^n(\alpha, \beta) P_k^{(\beta,\beta)}, \tag{5.1}$$

it is trivially clear that $D_n^n(\alpha, \beta) = [D_n^n(\beta, \alpha)]^{-1} > 0$, and that $D_k^n = 0$, $n + k$ odd. For $1/2 \geqq \alpha > \beta \geqq -1/2$, and $n + k$ even, $D_k^n(\alpha, \beta) > 0$, and (except for $\alpha = 1/2$, $\beta = -1/2$) $D_k^n(\beta, \alpha) < 0$, $k > 0$. The excepted case $\alpha = 1/2$, $\beta = -1/2$ has the expansion

$$T_n = \tfrac{1}{2}[U_n - U_{n-2}]. \tag{5.2}$$

(Recall that for $\alpha = 1/2, 0, -1/2$, the ultraspherics are respectively the classical polynomials U_n, P_n, T_n.) Askey [1] gives an explicit formula for $D_k^n(\alpha, \beta)$.

For two given polynomial systems $\{p_n(t)\}$, $\{q_n(t)\}$ (normalized to have positive leading coefficients), if we expand polynomials of one system in terms of terms of the other,

$$p_n(t) = \sum_{k=0}^{n} D_k^n q_k(t), \qquad n = 1, 2, \cdots,$$

$$q_n(t) = \sum_{k=0}^{n} d_k^n p_k(t), \qquad n = 1, 2, \cdots,$$

then, if $D_k^n \geqq 0$, $d_k^n \leqq 0$, $k = 0, 1, \cdots, n-1$, $n = 1, 2, \cdots$, we shall say, by analogy with (5.1) and succeeding discussion, that the system $\{p_n(t)\}$ is *above* the system $\{q_n(t)\}$ and that the system $\{q_n(t)\}$ is *below* the system $\{p_n(t)\}$. (For precision, if the systems are finite, we assume they have the same number of elements.)

This section is primarily devoted to proving, via a sequence of lemmas, the following.

THEOREM 5.1. *The system* $\{Q_n\}$, *for fixed* N, *is below the system* $\{U_n\}$, *and above the systems* $\{P_n\}$, $\{T_n\}$.

To establish notation, we write, suppressing t and N dependence,

$$
\begin{aligned}
U_n &= A_n^n Q_n^n + A_{n-2}^n Q_{n-2} + \cdots,\\
Q_n &= a_n^n U_n + a_{n-2}^n U_{n-2} + \cdots,\\
Q_n &= P_n + B_{n-2}^n P_{n-2} + \cdots,\\
P_n &= Q_n + b_{n-2}^n Q_{n-2} + \cdots,\\
Q_n &= C_n^n T_n + C_{n-2}^n T_{n-2} + \cdots,\\
T_n &= c_n^n Q_n + c_{n-2}^n Q_{n-2} + \cdots.
\end{aligned} \tag{5.3}
$$

We show that A_j^n, B_j^n, C_j^n are positive, the a_n^n, c_n^n are positive, and all a_j^n, b_j^n, c_j^n are negative, $j = n-2, n-4, \cdots$. We shall also show a number of monotonicity properties which are useful later.

LEMMA 5.1. *For* $N > n$, *writing*

$$Q_n(t;N) = P_n(t) + B_{n-2}^n(N)P_{n-2}(t) + B_{n-4}^n(N)P_{n-4}(t) + \cdots,$$

we have $B_{n-2j}^n(N)$ *strictly decreasing with increasing* N *to a limit of zero with order* $O(N^{-2})$, $j = 1, 2, \cdots$.

Proof. We use a refinement of the argument in Wilson [16], which with Lemma 4.3 establishes the nonnegativity of the $B_{n-2j}^n(N)$. Note first that in expressing $Q_n(t;N)$ as above, we have utilized the fact that only polynomials P_j of the same parity of n appear in the expansion. For the sake of simplicity, we assume n even. The proof for n odd is the same except for notational changes. Let $n = 2m$.

Writing $Q_{2m}(t;N)$ as $\sum_{j=0}^m B_{2j}^n(N)P_{2j}$, where $B_n^n(N) \equiv 1$, and imposing the orthogonality conditions $[Q_{2m}, P_{2j}]_N = 0, j = 0, 1, \cdots, m-1$, and $[Q_{2m}, P_{2m}]_N = K > 0$, we observe that $B^n(N) \equiv (B_0^n(N), B_2^n(N), \cdots, B_{2m-2}^n, 1)^t$ is proportional to the solution $B^*(N) = (B_0^*(N), B_1^*(N), \cdots, B_m^*(N))^t$ of the system $A_m B^* = \delta$, where $\delta = (0, 0, \cdots, 0, 1)^t$, and $A_m = A_m(N)$ has elements $A_{ij}(N) \equiv [P_{2i}, P_{2j}]_N$.

Now A_m is a Stieltjes matrix, having positive diagonal elements and negative off-diagonal elements, and thus (see Varga [15, pp. 81–87]) it has a positive inverse. (It is positive definite since it is a Gram matrix; see Davis [6, p. 176].)

Partitioning

$$A_m = \begin{pmatrix} A_{m-1} & V_m \\ V_m^t & A_{m,m} \end{pmatrix},$$

where $V_m \equiv (A_{0,m}, A_{1,m}, \cdots, A_{m-1,m})^t < 0$, we have, by a formula of Frazer, Duncan and Collar [8, pp. 112–115],

$$A_m^{-1} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

where $M_{22} = |A_{m-1}|/|A_m|$, and $M_{12} = -M_{22}A_{m-1}^{-1}V_m$.

Therefore,

$$B^*(N) = \begin{pmatrix} M_{12} \\ M_{22} \end{pmatrix} \quad \text{and} \quad B^n(N) = \begin{pmatrix} -A_{m-1}^{-1}V_m \\ 1 \end{pmatrix}.$$

As stated earlier, $A_{m-1}^{-1} > 0$, and $-V_m > 0$ by Lemma 4.3, so $B^n(N) > 0$. Further, since $A_{m-1}(N+1) > A_{m-1}(N)$, and $-V_m(N+1) < -V_m(N)$, by Lemma 4.3, we have $A_{m-1}^{-1}(N+1) \leqq A_{m-1}^{-1}(N)$ (Varga [15, p. 87]) so that $B^n(N+1) < B^n(N)$. The limiting value and order is a consequence of (3.1).

LEMMA 5.2. *For $N > n$, writing*

$$Q_n(t;N) = C_n^n T_n(t) + C_{n-2}^n T_{n-2}(t) + \cdots,$$

we have that $C_n^n = 2g_0 g_n$ and $C_{n-2j}^n(N)$ strictly decrease with increasing N, of order $O(N^{-2})$, to the limit $g_j g_{n-j}$, given by the identity (4.1).

Proof. The proof is immediate from the identity (4.1) and Lemma 5.1, or by using a similar argument to Lemma 5.1.

LEMMA 5.3. *In the expansions* (5.3), *for $N > n$, we have $A_{n-2j}^n > 0$, $b_{n-2j}^n < 0$, $c_{n-2j}^n < 0$, $j = 1, 2, 3, \cdots$, while $A_n^n > 0$, $b_n^n > 0$, $c_n^n > 0$.*

Proof. $A_{n-2j}^n = [U_n, Q_{n-2j}]_N / [Q_{n-2j}, Q_{n-2j}]_N$ and Lemmas 5.2 and 4.2 show the numerator term positive, showing $A_{n-2j}^n > 0$. The b_{n-2j}^n result follows similarly from Lemmas 5.1 and 4.3, while the c_{n-2j}^n result follows from Lemmas 5.1 and 4.1.

LEMMA 5.4. *For $N > n$, writing*

$$Q_n(t;N) = a_n^n U_n(t) + a_{n-2}^n U_{n-2}(t) + \cdots,$$

we have $a_n^n > 0$, $a_{n-2j}^n < 0$ for $j = 1, 2, \cdots$.

Proof. Because of the length, we shall only sketch the proof. The $U_i(x)$, $i = 1, 2, \cdots, N-1$, are orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$, defined by

$$\langle f, g \rangle = \frac{2}{N+1} \sum_{i=1}^{N} \sin^2 i\phi \cdot f(\cos i\phi) g(\cos i\phi),$$

so that, as a Fourier coefficient, we have

$$a_{n-2p}^n(N) = \langle Q_n^N, U_{n-2p} \rangle, \qquad p \in \{0, 1, \cdots, [n/2]\}.$$

For $p = 0$, by examining the leading coefficients, it is clear that $a_n^n > 0$. Assume $p = 1, 2, \cdots, [n/2]$. Since $\sin i\phi = \sqrt{1 - \cos^2 i\phi}$, we have

$$a_{n-2p}^n = \frac{1}{(N+1)\tan\psi} [Q_n, \sqrt{1-x^2}\, U_{n-2p}].$$

Now, for $0 < x < \pi$,

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2 - 1},$$

so

$$\begin{aligned}(N+1)\tan\psi\, a_{n-2p}^n &= \frac{2}{\pi}[Q_n, U_{n-2p}] - \frac{4}{\pi}[Q_n, K \cdot U_{n-2p}] \\ &= -\frac{4}{\pi}[Q_n, K \cdot U_{n-2p}],\end{aligned}$$

where $n \geqq 2$, and

$$K(x) = \sum_{k=1}^{\infty} \frac{T_{2k}(x)}{4k^2 - 1}.$$

Since $T_{2q(N+1)\pm j}(\cos i\phi) = T_j(\cos i\phi)$, we can show

$$K(\cos i\phi) = \sum_{j=1}^{M} S_{N,j} T_{2j}(\cos i\phi) + S_N^0 T_{N+1}(\cos i\phi) + S_0^*,$$

where $N = 2M$ or $2M + 1$,

$$S_{N,j} = \frac{1}{4j^2 - 1} + \sum_{q=1}^{\infty} \left(\frac{1}{4[q(N+1) + j]^2 - 1} + \frac{1}{4[q(N+1) - j]^2 - 1} \right),$$

$$S_N^0 = \begin{cases} 0, & N = 2M, \\ \displaystyle\sum_{q=1}^{\infty} \frac{1}{(2q-1)^2(N+1)^2 - 1}, & N = 2M + 1, \end{cases}$$

$$S_0^* = \sum_{q=1}^{\infty} \frac{1}{4q^2(N+1)^2 - 1}.$$

By using the psi (digamma) function, the series may be summed, and it can be shown that

$$S_{N,j} > S_{N,j+1}, \qquad j = 1, 2, \cdots, M - 1,$$

and

$$S_{N,M} > 2S_N^0.$$

Now, substituting, and using orthogonality of Q_n, we have

$$a_{n-2p}^n = \frac{-4}{\pi(N+1)\tan\psi} \left[\sum_{j=p}^{M} [Q_n^N, U_{n-2p} T_{2j}]_N S_{N,j} + [Q_n, U_{n-2p} T_{N+1}] S_N^0 \right].$$

Using known identities on $U_n T_m$ to obtain expressions involving U-functions only, and using

$$U_{2(N+1)-(j+2)}(\cos i\phi) = -U_j(\cos i\phi), \quad \text{and} \quad [Q_n, U_N]_N \equiv 0,$$

one can show (for each of the four cases obtained from letting $N = 2M$, $N = 2M + 1$, $n = 2r$, $n = 2r + 1$) that the expression is negative (recall that $[Q_n, U_m]_N > 0, m = n, n + 1, \cdots, N - 1$, by Lemma 5.3), which completes the proof.

At this point, we have proved Theorem 5.1. Let us remark here that what we have shown is that the three matrices $(a_{n-2j}^{n-2i}), (b_{n-2j}^{n-2i}), (c_{n-2j}^{n-2i})$, with $i, j = 0, 1, \cdots, m$, where $n = 2m$ or $n = 2m + 1$, are M-matrices (Varga [15, § 3.5]), which have nonnegative inverses. The really difficult problem, illustrated by Lemma 5.4, involves determining when a nonnegative matrix has as its inverse an M-matrix. We conclude this section with the following lemma.

LEMMA 5.5. *For $N > n$, the coefficients satisfy, for n even, $C_n^n > C_{n-2}^n > \cdots > C_2^n > 2C_0^n$, and for n odd, $C_n^n > C_{n-2}^n > \cdots > C_1^n$.*

Proof. Substituting $T_k = \frac{1}{2}[U_k - U_{k-2}], T_0 = U_0$ and $T_1 = \frac{1}{2}U_1$ into the expansion for $Q_n = \sum C_{n-2j}^n T_{n-2j}$ to obtain $Q_n = \sum a_{n-2j}^n U_{n-2j}$, the inequalities above hold if and only if the a^n_{n-2j} satisfy $a^n_n > 0$, $a^n_{n-2j} < 0, j = 1, 2, \cdots$, so the inequalities follow directly from Lemma 5.4.

6. Properties of the Q-polynomials. We now prove a number of properties of the $\{Q_n(t; N)\}$ polynomials, and also state some properties which, from computational experiment, we suspect are valid.

THEOREM 6.1. *$Q_n(t; N)$ takes its maximum absolute value on $[-1, 1]$ at the endpoints. Further, $Q_n(1; N)$ is strictly decreasing, with order $O(N^{-2})$, to the limit value* 1, *with increasing N.*

Proof. The proof is an immediate consequence of Lemma 5.1.

By way of estimating $Q_n(1; N)$, we can obtain a rough estimate using the monotonicity stated in Lemma 5.5 and the explicit formula $C^n_n = 2 \cdot (2n)!/(4^n(n!)^2)$, so that $Q_n(1; N) < [(n + 1)/2] \cdot C^n_n$, where $[x]$ here means greatest integer containing x. Using Stirling's formula, we have, for $n > 1$, independently of N,

$$1 < Q_n(1; N) < (n + 2)/(\pi n)^{1/2}.$$

For $n = 29$, the right-hand value is 3.25, while $Q_{29}(1; 30) = 2.42$. We have had little success in attempting to find an explicit formula or a bound depending on both n and N for $Q_n(1; N)$.

THEOREM 6.2. *If the zeros of $Q_n(t; N)$ are designated* $\cos \theta^{(n)}_j, j = 1, 2, \cdots, n$, *then*

$$\frac{j - 1/2}{n + 1}\pi < \theta^{(n)}_j < \frac{j + 1/2}{n + 1}\pi, \qquad j = 1, 2, \cdots, n.$$

Proof. Writing (5.3) as

$$Q_n(\cos\theta; N) = C^n_n \cos n\theta + C^n_{n-2} \cos(n - 2)\theta + \cdots$$

and letting $n = 2m$, and $x = 2\theta$, we obtain

$$Q_n(\cos x/2) = C^n_n \cos mx + C^n_{n-2} \cos(m - 1)x + \cdots + C^n_0.$$

Lemmas 5.2 and 5.5 imply the result by direct application of Theorem 6.4 in Szegö [14]. The proof for $n = 2m + 1$ follows similarly.

There are two additional properties of ultraspherics (Legendre polynomials in particular) that the $Q_n(t; N)$ appear to satisfy from numerical evidence.

CONJECTURE 6.1. The sequence of successive relative maxima of $|Q_n(t; N)|$ on $[0, 1]$ is an increasing sequence.

CONJECTURE 6.2. For $N > n + m$, there is a positive product linearization formula. That is, $Q_n(t; N)Q_m(t; N)$ appears to be expandable as a nonnegative combination of $Q_j(t; N), j = 0, 1, \cdots, n + m$. With respect to the second conjecture, we have verified computationally that the $\beta_i(N)$ do not satisfy the appropriate inequalities that would allow the use of the results in Askey [3], [4].

Because these polynomials share, or appear to share, many of the properties of the ultraspheric polynomials, it is quite fair to ask, "Could these polynomials in essence be ultraspheric polynomials?" That is to say, could $Q_n(t; N) = h_n(N)P^{(\alpha,\alpha)}_n(t)$, where $\alpha = \alpha(N)$, such that $\alpha(N) \to 0$ as $N \to \infty$? Or perhaps they are Fejér "generalized Legendre polynomials" (Szegö [14, § 6.5])? We show the latter question is answered in the negative (which implies a negative answer for the first question also).

LEMMA 6.1. *For $N > 4$, N finite, the sequence Q_0, Q_1, Q_2, Q_3, Q_4 is not proportional to a sequence of Fejér "generalized Legendre polynomials."*

Proof. The Fejér "Legendre polynomials" associated with a sequence $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \cdots$ are defined by

$$F_n(\cos\theta) = 2\alpha_0\alpha_n \cos n\theta + 2\alpha_1\alpha_{n-1}\cos(n-2)\theta + \cdots$$
$$+ \begin{cases} 2\alpha_{(n-1)/2}\alpha_{(n+1)/2}, & n \text{ odd}, \\ \alpha_n^2, & n \text{ even}. \end{cases}$$

Without loss of generality, we take $\alpha_0 = 1$, and write explicitly the first 5 polynomials as

$$1, \quad 2\alpha_1 \cos\theta, \quad 2\alpha_2\cos 2\theta + \alpha_1^2, \quad 2\alpha_3\cos 3\theta + \alpha_1\alpha_2\cos\theta,$$
$$2\alpha_4\cos 4\theta + 2\alpha_1\alpha_3\cos 2\theta + \alpha_2^2.$$

For the monic forms $q_n(t; N)$ of § 3, writing

$$q_n(\cos\theta; N) = \sum_{j=0}^{[n/2]} \gamma_{nj}\cos(n-2j)\theta,$$

we have

$$\begin{aligned} &\gamma_{00} = 1, \\ &\gamma_{11} = 1, \\ &\gamma_{22} = 1/2, \qquad \gamma_{20} = (1-\beta_1)/2, \\ &\gamma_{33} = 1/4, \qquad \gamma_{31} = (3 - 4(\beta_1 + \beta_2))/4, \\ &\gamma_{44} = 1/8, \qquad \gamma_{42} = (1 - \beta_1 - \beta_2 - \beta_3)2, \\ &\gamma_{40} = (3 - 4(\beta_1 + \beta_2 + \beta_3 + 8\beta_1\beta_2))/8. \end{aligned}$$

If the two sets are proportional, that is, if there exists k_i such that $F_n(\cos\theta) = k_i q_i(\cos\theta; N)$, $i = 0, 1, \cdots, 4$, then, necessarily, $4\gamma_{20}\gamma_{40} = \gamma_{31}\gamma_{42}$. Writing everything in terms of $\cos\phi$, we were able to show, after some extremely painful algebra, that $4\gamma_{20}\gamma_{40} - \gamma_{31}\gamma_{42}$, as a rational function in $\cos\phi$, for $0 < \phi < \pi/5$, was one signed (never zero) so that the condition could not hold.

(The last step of the procedure involved the symbolic factoring of a 13th degree polynomial using the REDUCE system on the IBM 360/91 at the Watson Research Center.)

A few other comments are in order. First, because of the $O(N^{-2})$ convergence, these polynomials are not Hahn polynomials. The work of Wynn [21] and Hahn [9] then shows there is no difference analogue of "Rodrigues' formula" for these polynomials.

7. The general class of polynomials. Letting $\sum''$ represent summation with first and last terms weighted by 1/2, consider the general inner product, for $\alpha \geqq -1/2$,

$$[f, g]_N^\alpha = \sum_{i=0}^{N+1}{}'' (w_i)^{1+2\alpha} f(t_i) g(t_i),$$

where $w_i = \sin i\phi$, $t_i = \cos i\phi$, and letting $(0)^0 \equiv 1$. Let $q_n^{(\alpha)}(t; N)$ be the discrete orthogonal polynomials defined analogously to those in § 3. Riemann sum considerations (Rice [12, § 2–4]) show that these polynomials are discrete analogues of the ultraspherics $P_n^{(\alpha,\alpha)}(t)$. In fact, for $\alpha = +1/2$, $\alpha = -1/2$, the inner products are well known and generate U_n and T_n exactly (up to a constant factor). Since U_n and T_n (the latter requires the t_0 and t_{N+1} points) are (up to a constant factor) $P_n^{(\alpha,\alpha)}$ for $\alpha = 1/2, -1/2$ respectively, it is only natural to examine what happens when $\alpha = 0$ (the Legendre case). This motivated the current investigation.

This raises a whole host of questions. Do the $q_n^{\alpha}(t; N)$, suitably normalized, share similar properties to $P_n^{(\alpha,\alpha)}$? What is the convergence order of $[\cdot,\cdot]_N^{\alpha}$, for general α? Do the $q_n^{\alpha}(t; N)$ have similar interrelationships for different α as the ultraspherics have? For $\alpha = 1/2, 0, -1/2$ we have demonstrated some of these interrelationships.

Another very general question of interest concerns the characteristic sign configurations of expansions of § 5. We have shown in Theorem 5.1 that $\{Q_n\}$ is above $\{P_n\}$, and implicitly, $\{Q_n\}$ is above $\{P_n^{(\alpha,\alpha)}\}$ for $\alpha \in [-1/2, 0]$. Here we state, on very scanty numerical evidence, the following conjecture.

CONJECTURE 7.1. There exist numbers α_1, α_2, functions of N, satisfying $-1/2 \leqq \alpha_1 \leqq \alpha_2 \leqq 1/2$, and such that the system $\{Q_n\}$ is above the system $\{P_n^{(\alpha,\alpha)}\}$ for $\alpha \in [-1/2, \alpha_1]$ and below the system $\{P_n^{(\alpha,\alpha)}\}$ for $\alpha \in [\alpha_2, 1/2]$.

Clearly, if this is so, $\alpha_1 \geqq 0$ and tends to 0 with increasing N. On the other hand, $1/2 \geqq \alpha_2$, and we would suspect also that $\alpha_2 \to 0$ with increasing N. How large a gap is $\Delta_n(N) = \alpha_2(N) - \alpha_1(N)$?

REFERENCES

[1] R. ASKEY, *Orthogonal expansions with positive coefficients*, Proc. Amer. Math. Soc., 26 (1965), pp. 1191–1194.

[2] ———, *Jacobi polynomial expansions with positive coefficients*, Bull. Amer. Math. Soc., 74 (1968), pp. 301–304.

[3] ———, *Linearization of the product of orthogonal polynomials*, Problems in Analysis, R. Gunning, ed., Princeton Univ. Press, Princeton, N.J., 1970, pp. 131–138.

[4] ———, *Orthogonal polynomials and positivity*, Studies in Applied Mathematics 6, Special Functions and Wave Propagation, D. Ludwig and F. W. J. Olver, eds., SIAM, Philadelphia, Pa., 1970, pp. 64–85.

[5] ———, *Orthogonal expansions with positive coefficients. II*, this Journal, to appear.

[6] P. J. DAVIS, *Interpolation and Approximation*, Blaisdell, Waltham, Mass., 1963.

[7] G. E. FORSYTHE, *Generation and use of orthogonal polynomials for data-fitting with a digital computer*, J. Soc. Indust. Appl. Math., 5 (1957), pp. 74–88.

[8] R. A. FRASER, W. J. DUNCAN AND A. R. COLLAR, *Elementary Matrices, and Some Applications to Dynamics and Differential Equations*, Cambridge Univ. Press, Cambridge, 1957.

[9] W. C. HAHN, *Über Orthogalpolynome, die q-Differenzgleichungen genügen*, Math. Nachr., 2 (1949), pp. 4–34.

[10] S. KARLIN AND J. L. MCGREGOR, *The Hahn polynomials formulas, and an application*, Scripta Math., 26 (1961), pp. 33–46.

[11] R. J. LEVIT, *The zeros of the Hahn polynomials*, SIAM Rev., 9 (1967), pp. 191–203.

[12] J. R. RICE, *The Approximation of Functions*, vol. I., Addison-Wesley, Reading, Mass., 1964.

[13] I. M. RYSHIK AND I. S. GRADSTEIN, *Tables of Series, Products, and Integrals*, Veb Deutscher Verlag, Der Wissenschaften, Berlin, 1957.

[14] G. SZEGÖ, *Orthogonal Polynomials*, Colloquium Publications, vol. 23, 2nd rev. ed., American Mathematical Society, Providence, R.I., 1959.

[15] R. S. VARGA, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[16] M. W. WILSON, *Non-negative expansions of polynomials*, Proc. Amer. Math. Soc., 24 (1970), pp. 100–102.

[17] ———, *Convergence properties of discrete analogs of orthogonal polynomials*, Computing, 5 (1970), pp. 1–5.

[18] ———, *On the Hahn polynomials*, this Journal, 1 (1970), pp. 37–43.

[19] ———, *A new discrete analog of the Legendre polynomials*, Bull. Amer. Math. Soc., 76 (1970), pp. 345–348.

[20] ———, *Two discrete analogs of Legendre polynomials, a comparison*, RC 2318, IBM Watson Research Center, Yorktown Heights, N.Y., 1968.

[21] P. WYNN, *A general system of orthogonal polynomials*, Quart. J. Math. Oxford Ser., 18 (1967), pp. 81–96.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

A GENERALIZED CAUSALITY OF LINEAR CONTINUOUS OPERATORS DEFINED ON DISTRIBUTIONS*

VACLAV DOLEZAL†

Abstract. The paper deals with a qualitative property of linear continuous operators defined on distributions, that is, with a property appearing as a generalization of the traditional causality. The traditional causality of an operator A means that supp $Ax \subset S$ whenever supp $x \subset S$, where S is any interval $[T, \infty)$. Here, causality is defined in the same way, but S is any member from a certain family of subsets of R^m, called a scale.

A theorem is proved which gives necessary and sufficient conditions for an operator to be causal with respect to a given scale.

As an application there is considered a slightly generalized convolution-type operator; it is shown that this operator is causal with respect to a scale which consists of all translations of a fixed cone.

In this paper an extension of the traditional causality concept is discussed. First, the concept of a scale is introduced, that is, a scale $\mathscr{G}$ is defined as a family of subsets of R^m which satisfies certain requirements. It is shown that, in particular, we can take for a scale the collection of all sets which are translations of a fixed cone in R^m.

If A is a linear operator defined on distributions which is continuous (with respect to the convergence in $\mathscr{D}'$), then A is called causal with respect to a scale $\mathscr{G}$, if the support of Ax is contained in some $S \in \mathscr{G}$ whenever the support of x is contained in S. We use this concept and a certain representation of linear continuous operators given in [2] to prove a theorem which gives necessary and sufficient conditions for a linear continuous operator to be causal with respect to a given scale.

Based on this result, a theorem is then proved that describes a scale with respect to which an operator A is causal, provided A is a slight generalization of a convolution-type operator.

First, let us introduce several concepts and carry out some preliminary considerations. Let R^m stand for the Euclidean m-space; if $A \subset R^m$, let $\bar{A}$, A^c and Int A denote the closure, complement and the interior of A, respectively.

DEFINITION. Let S be a nonempty proper subset of R^m; S will be called a *normal* set if

(i) $\overline{\text{Int } S} = S$,

(ii) there exists an $x_0 \in R^m$ such that $\overline{S^c} - \lambda x_0 \subset S^c$ for every $\lambda > 0$.

Furthermore, let $\mathscr{D}$ stand for the space of all smooth functions on R^m with compact support, and let the convergence in $\mathscr{D}$ be defined in the usual way; let $\mathscr{D}'$ be the dual of $\mathscr{D}$, that is, the space of all linear functionals on $\mathscr{D}$ which are continuous with respect to the convergence in $\mathscr{D}$ (see [1]). Then we have the following lemma.

LEMMA 1. *Let S be a normal set, and let $f \in \mathscr{D}'$; then $f = 0$ on S^c if and only if $\langle f, \varphi \rangle = 0$ for every $\varphi \in \mathscr{D}$ with* supp $\varphi \subset \overline{S^c}$.

* Received by the editors December 17, 1970.

† Department of Applied Mathematics and Statistics, State University of New York at Stony Brook, Stony Brook, Long Island, New York 11790.

Proof. Owing to (i), S^c is an open set; hence, sufficiency is trivial. Conversely, let $f = 0$ on S^c and let $\varphi \in \mathscr{D}$ be such that $\operatorname{supp} \varphi \subset \overline{S^c}$. If x_0 is the point with property (ii), define functions $\varphi_n(t)$, $n = 1, 2, \cdots$, by $\varphi_n(t) = \varphi(t + x_0/n)$; then clearly $\varphi_n \in \mathscr{D}$ and $\operatorname{supp} \varphi_n = \operatorname{supp} \varphi - x_0/n$, i.e., $\operatorname{supp} \varphi_n \subset \overline{S^c} - x_0/n$. Hence, by (ii), $\operatorname{supp} \varphi_n \subset S^c$, and consequently, $\langle f, \varphi_n \rangle = 0$.

On the other hand, it can be easily verified that $\varphi_n \to \varphi$ in $\mathscr{D}$; thus, by continuity of f, $\langle f, \varphi \rangle = 0$ as required.

LEMMA 2. *Let S be a normal set and let $f \in \mathscr{D}'$; then $f = 0$ on S^c if and only if $\langle f, \varphi \rangle = 0$ for every $\varphi \in \mathscr{D}$ with $\varphi = 0$ on S.*

Proof. In view of Lemma 1 it suffices to show that $\operatorname{supp} \varphi \subset \overline{S^c}$ if and only if $\varphi = 0$ on S.

1. Let $\varphi = 0$ on S; then $A = \{t : \varphi(t) \neq 0\} \subset S^c$, and consequently, $\operatorname{supp} \varphi = \overline{A} \subset \overline{S^c}$.

2. Conversely, let $\operatorname{supp} \varphi \subset \overline{S^c}$; then $A \subset \operatorname{supp} \varphi \subset \overline{S^c}$, so that $A^c = \{t : \varphi(t) = 0\} \supset (\overline{S^c})^c = \operatorname{Int} S$. Hence, $\varphi = 0$ on $\operatorname{Int} S$, and since φ is continuous, we have $\varphi = 0$ on $\overline{\operatorname{Int} S} = S$ by (i), which finishes the proof.

DEFINITION. The family $\mathscr{G} = \{S_a : S_a \subset R^m, a \in R^m\}$ will be called *a scale*, if

(i)* each S_a is a normal set,

(ii)* $a \in S_a$ for every $a \in R^m$,

(iii)* $b \in S_a$ implies that $S_b \subset S_a$.

As an example of a scale, let us consider the following situation: If $t \in R^m$ and $t = (t_1, t_2, \cdots, t_m)$, let $\|t\| = (\sum_{i=1}^m t_i^2)^{1/2}$. Let $\mu > 0$ be fixed, and for every $a \in R^m$ define

$$S_a = \{t : \|t\|^2 - t_m^2 \leqq \mu(t_m - a_m) + \|a\|^2 - a_m^2\}. \tag{1}$$

Clearly, each S_a is a paraboloid opening to the right of the t_m-axis, and satisfies the condition (i); moreover, taking for x_0 any point of the positive part of the t_m-axis, inclusion (ii) is satisfied. Hence, every S_a is a normal set.

On the other hand, we have $a \in S_a$, and it can be easily verified that implication (iii)* holds too; consequently, the family $\mathscr{G} = \{S_a : a \in R^m\}$ is a scale.

Let us now discuss a more sophisticated example of a scale.

DEFINITION. Let S be a proper subset of R^m; the set S will be called *a cone*, if S is closed, convex with nonempty interior and such that $\lambda x \in S$ whenever $x \in S$ and $\lambda \geqq 0$.

Then we have the following theorem.

THEOREM 1. *Let S be a cone; then the family*

$$\mathscr{G} = \{S + a : a \in R^m\} \tag{2}$$

is a scale.

Proof. First of all, we shall show that each $S + a$ is a normal set. It is clear that for proving this it suffices to show that S is normal.

Thus, choose $x_0 \in \operatorname{Int} S$ and let us show that $\overline{S^c} - x_0 \subset S^c$. Actually, let $x \in S$; then, due to convexity of S, $x + x_0 \in S$, and we even have $x + x_0 \in \operatorname{Int} S$. Indeed, by assumption there exists an $\varepsilon > 0$ such that $z \in R^m$, $\|z - x_0\| < \varepsilon$ implies $z \in S$. Consequently, if $\xi \in R^m$ and $\|\xi - (x + x_0)\| < \varepsilon$, we have $\xi - x \in S_0$.

Thus, $x + (\xi - x) = \xi \in S$, i.e., $x + x_0 \in \operatorname{Int} S$; hence, $\overline{S} \subset \operatorname{Int} S - x_0$, and consequently, $(\operatorname{Int} S)^c - x_0 \subset S^c$. The relation $(\operatorname{Int} A)^c = \overline{A^c}$ concludes the proof of the above inclusion.

Next, observe that if $x_0 \in \operatorname{Int} S$, then also $\lambda x_0 \in \operatorname{Int} S$ for any $\lambda > 0$; consequently, we have $\overline{S^c} - \lambda x_0 \subset S^c$, and condition (ii) holds.

As for condition (i), we can reason as follows: we have $\operatorname{Int} S \subset S$, and consequently, $\overline{\operatorname{Int} S} \subset \overline{S} = \bar{S}$. Conversely, let $x \in S$ and show that $x \in \overline{\operatorname{Int} S}$. Actually, choose an $x_0 \in \operatorname{Int} S$ and show that every point $z_\lambda = \lambda x_0 + (1 - \lambda)x$ with $0 < \lambda \leqq 1$ belongs to $\operatorname{Int} S$. By assumption, there exists $\varepsilon > 0$ such that $\|\xi - x_0\| < \varepsilon$ implies $\xi \in S$; choosing a fixed λ with $0 < \lambda \leqq 1$, let η be a point such that $\|\eta - z_\lambda\| < \lambda\varepsilon$, so that $\|\eta - \lambda x_0 - (1 - \lambda)x\| < \lambda\varepsilon$. Then we have $\|\lambda^{-1}\eta - \lambda^{-1} \cdot (1 - \lambda)x - x_0\| < \varepsilon$, and consequently, $\lambda^{-1}\eta - \lambda^{-1}(1 - \lambda)x \in S$. Since $\lambda^{-1}(1 - \lambda)x \in S$, we have $\lambda^{-1}\eta \in S$ and thus $\eta \in S$. Hence, $z_\lambda \in \operatorname{Int} S$ for $0 < \lambda \leqq 1$.

On the other hand, $\|x - z_\lambda\| = \lambda\|x - x_0\|$, so that $x \in \overline{\operatorname{Int} S}$; hence, $S \subset \overline{\operatorname{Int} S}$, and requirement (i) is satisfied. Thus, S is normal.

Next, since $\theta \in S$, we have $a \in S + a$ for any $a \in R^m$; consequently, (ii)* holds.

Finally, if $b \in S + a$, then $b - a \in S$; if $x \in S + b$, we have $x = b + y$ with $y \in S$. Consequently, writing $x = a + (b - a) + y$, it follows that $(b - a) + y \in S$, so that $x \in S + a$. Hence, (iii)* is satisfied, and the theorem is proved.

Let us now turn to causality of operators. Let $\mathscr{D}'_*$ stand for the space of all distributions in $\mathscr{D}'$ which have a finite order, that is, $f \in \mathscr{D}'_*$ exactly if there exists a continuous function $F(t)$ on R^m and a multi-index k such that $\langle f, \varphi \rangle = (-1)^{\bar{k}} \cdot \int_{R^m} F(t) D^k \varphi(t)\, dt$ for every $\varphi \in \mathscr{D}$ (see [1]).

Next, let $\mathscr{P}$ be a linear subspace of $\mathscr{D}'$ such that $\mathscr{D}'_* \subset \mathscr{P} \subset \mathscr{D}'$; a linear operator $A : \mathscr{P} \to \mathscr{D}'$ is called continuous if $x, x_n \in \mathscr{P}$, $n = 1, 2, \cdots$, and $x_n \to x$ in $\mathscr{D}'$ implies that $Ax_n \to Ax$ in $\mathscr{D}'$. In [2] we have proved a representation theorem for a linear continuous operator $A : \mathscr{P} \to \mathscr{D}'$; for the present purposes we shall need only the following assertion.

Let $A : \mathscr{P} \to \mathscr{D}'$ *be a linear continuous operator, and let* $f_a = A\delta_a$ *for every* $a \in R^m$; *then, for any fixed* $\varphi \in \mathscr{D}$, $\psi_\varphi(a) = \langle f_a, \varphi \rangle \in \mathscr{D}$ *and*

$$\langle Ax, \varphi \rangle = \langle x, \langle f_a, \varphi \rangle \rangle \tag{3}$$

for every $x \in \mathscr{P}$ *and* $\varphi \in \mathscr{D}$.

Note. In order to avoid misunderstanding, equation (3) should be interpreted as

$$\langle (Ax)(t), \varphi(t) \rangle = \langle x(a), \langle f_a(t), \varphi(t) \rangle \rangle,$$

where the "arguments" t and a indicate the involved relations; having this in mind, we shall use the shorter notation of (3) in the sequel.

Definition. Let $\mathscr{G} = \{S_a : S_a \subset R^m, a \in R^m\}$ be a scale, and let $A : \mathscr{P} \to \mathscr{D}'$ be a linear continuous operator; the operator A will be called *causal with respect to* $\mathscr{G}$ if $\operatorname{supp} Ax \subset S_a$ whenever $x \in \mathscr{P}$ and $\operatorname{supp} x \subset S_a$.

Observe that the traditional causality ($m = 1$) is obtained by setting $S_a = [a, \infty)$.

Now, we are ready for stating the main result.

Theorem 2. *Let* $\mathscr{G} = \{S_a : S_a \subset R^m, a \in R^m\}$ *be a scale, and let* $A : \mathscr{P} \to \mathscr{D}'$ *be a linear continuous operator; then* A *is causal with respect to* $\mathscr{G}$ *if and only if* $\operatorname{supp} A\delta_a \subset S_a$ *for every* $a \in R^m$.

Proof. 1. Assume first that $\operatorname{supp} A\delta_a \subset S_a$ for every $a \in R^m$, i.e., $f_a = A\delta_a = 0$ on S_a^c, and let $x \in \mathscr{P}$ be such that $\operatorname{supp} x \subseteq S_b$, i.e., $x = 0$ on S_b^c. Referring to Lemma 1, let $\varphi \in \mathscr{D}$ be such that $\operatorname{supp} \varphi \subseteq \overline{S_b^c}$. Now, if $a \in S_b$, then $S_a \subset S_b$ by property (iii)* of a scale, and consequently, $\overline{S_b^c} \subset \overline{S_a^c}$. Thus, $\operatorname{supp} \varphi \subset \overline{S_a^c}$, and since $f_a = 0$ on S_a^c, we have by Lemma 1, $\psi_\varphi(a) = \langle f_a, \varphi \rangle = 0$. Hence, $\psi_\varphi = 0$ on S_b, and consequently, by Lemma 2, $\langle x, \psi_\varphi \rangle = 0$. However, by the above proposition, $\langle x, \psi_\varphi \rangle = \langle Ax, \varphi \rangle$. Thus, summarizing, $\langle Ax, \varphi \rangle = 0$ for any $\varphi \in \mathscr{D}$ with $\operatorname{supp} \varphi \subset \overline{S_b^c}$, so that, in view of Lemma 1, $Ax = 0$ on S_b^c. Consequently, $\operatorname{supp} Ax \subset S_b$, so that A is causal with respect to $\mathscr{G}$.

2. Conversely, let A be causal with respect to $\mathscr{G}$. Since $\delta_a \in \mathscr{P}$ for any $a \in R^m$ and $\operatorname{supp} \delta_a = \{a\}$, we have $\operatorname{supp} \delta_a \subset S_a$ by requirement (ii)* in the definition of a scale; hence, by causality of A, $\operatorname{supp} A\delta_a \subset S_a$ and the theorem is proved.

In the majority of physical applications the operators involved are of convolutional type. It turns out that a scale can easily be constructed with respect to which such an operator is causal; actually, this can be done for operators that are slightly more general than convolutional ones, as is apparent from the next theorem.

If A is a subset of R^m, let $\mathscr{K}(A)$ signify the convex hull of A; then we have the following theorem.

THEOREM 3. *Let $k_i \in \mathscr{D}'$, $i = 1, 2, \cdots, n$, have a bounded support and let $\lambda_i(t)$, $\mu_i(t)$, $i = 1, 2, \cdots, n$, be infinitely smooth functions on R^m; furthermore, let the operator $A : \mathscr{D}' \to \mathscr{D}'$ be defined by*

$$Ax = \sum_{i=1}^{n} \lambda_i(k_i * (\mu_i x)). \tag{4}$$

If the set $K = \mathscr{K}(\bigcup_{i=1}^{n} \operatorname{supp} k_i)$ has nonempty interior and if

$$S = \overline{\{\alpha z : z \in K, \alpha \geqq 0\}} \neq R^m, \tag{5}$$

then the family $\mathscr{G} = \{S + a : a \in R^m\}$ is a scale and the operator A is causal with respect to $\mathscr{G}$.

Proof. First of all, since each k_i has a bounded support, $k_i * x$ is defined for every $x \in \mathscr{D}'$ and the operator $A_i = k_i *$ is linear and continuous (see [1]); hence, A defined by (4) is clearly linear and continuous. Moreover, for any $a \in R^m$ we have

$$A\delta_a = \sum_{i=1}^{n} \lambda_i(k_i * \mu_i(a)\delta_a) = \sum_{i=1}^{n} \mu_i(a)\lambda_i(P_a k_i), \tag{6}$$

where P_a signifies the operator of shifting by vector a; however, properties of the support of a distribution and (6) imply immediately that

$$\operatorname{supp} A\delta_a \subset a + \bigcup_{i=1}^{n} \operatorname{supp} k_i. \tag{7}$$

Consequently, by (5),

$$\operatorname{supp} A\delta_a \subset S + a. \tag{8}$$

On the other hand, the set S is clearly closed with nonempty interior and has the property that $y \in S$, $\varkappa \geqq 0$ implies $\varkappa y \in S$. Moreover, the set $S' = \{\alpha z : z \in K,$

$\alpha \geqq 0\}$ is convex; actually, let $y_1, y_2 \in S'$, $\beta \in [0, 1]$ and $y = \beta y_1 + (1 - \beta)y_2$. If either y_1 or y_2 is zero, then clearly $y \in S'$. Thus, assuming that both y_1 and y_2 are nonzero, we have $y_1 = \alpha_1 z_1$, $y_2 = \alpha_2 z_2$, $\alpha_1, \alpha_2 > 0$ and $z_1, z_2 \in K$. Setting $\gamma = [\beta\alpha_1 + (1 - \beta)\alpha_2]^{-1}\beta\alpha_1$, we have $\gamma \in [0, 1]$, and consequently, $z = \gamma z_1 + (1 - \gamma)z_2 \in K$; hence, $\mu = (\beta\alpha_1 + (1 - \beta)\alpha_2)z \in S'$. An easy calculation shows that $\mu = y$; thus, S' is convex, and since $S = \overline{S'}$, the set S is convex, too.

Summarizing our results, it follows that S is a cone; hence, by Theorem 1, our family $\mathscr{G}$ is a scale. Finally, Theorem 2 and inclusion (8) conclude the proof of our theorem.

The above results can be easily modified for the case that the restriction on boundedness of supports of k_i is removed; then, of course, the domain of operator A has to be restricted in order that $k_i * x$ remain meaningful. Let us consider a simple situation of this kind.

Let $R^m_+ = [0, \infty)^{\times m}$, and let $\mathscr{D}'_+ = \{x : x \in \mathscr{D}', \operatorname{supp} x \subset R^m_+\}$; assuming that $k_i \in \mathscr{D}'_+$, $i = 1, 2, \cdots, n$, define A again by (4). Then $A : \mathscr{D}'_+ \to \mathscr{D}'_+$, and A is linear and continuous, that is, x_p, $x \in \mathscr{D}'_+$, $x_p \to x$ in $\mathscr{D}'$ implies $Ax_p \to Ax$ in $\mathscr{D}'$ (see [1]). We cannot apply Theorem 2 directly, since $\mathscr{P} = \mathscr{D}'_+ \not\supset \mathscr{D}'_*$; however, we can overcome this difficulty as follows: Let $\varkappa(\xi)$ be an infinitely smooth function on R^1 such that $\varkappa(\xi) = 1$ for $t \geqq -1$, $\varkappa(\xi) = 0$ for $t \leqq -2$ and $0 \leqq \varkappa(\xi) \leqq 1$ everywhere, and set, for every $t \in R^m$, $v(t) = \prod_{i=1}^m \varkappa(t_i)$.

Next, define an operator $\tilde{A} : \mathscr{D}' \to \mathscr{D}'$ by $\tilde{A}x = A(vx)$; clearly, this definition is meaningful, and $\tilde{A}$ is linear and continuous. Moreover, $\tilde{A} = A$ on $\mathscr{D}'_+$ and we have as before, $\operatorname{supp} \tilde{A}\delta_a \subset S + a$ for every $a \in R^m$. Observe also that here $S \subset R^m_+$. Thus, if Int K is not empty, we conclude as above that $\operatorname{supp} Ax \subset S + a$ whenever $x \in \mathscr{D}'_+$, $\operatorname{supp} x \subset S + a$ and $a \in R^m_+$.

In conclusion, let us make a few comments on a possible extension of the above results.

The members of a scale, as defined above, are clearly unbounded sets with nonempty interior. On the other hand, we would get a meaningful definition of causality of an operator if we replaced the scale by any nonempty family of subsets of R^m. This fact suggests the following generalization.

A family $\mathfrak{S} = \{S_a : S_a \subset R^m, a \in R^m\}$ will be called a *generalized scale* if there exists a collection $\{\mathfrak{S}_\alpha\}$ of scales $\mathfrak{S}_\alpha = \{S^\alpha_a : S^\alpha_a \subset R^m, a \in R^m\}$ such that $S_a = \bigcap_\alpha S^\alpha_a$ for every $a \in R^m$.

Clearly, every scale is a generalized scale, but not conversely, since requirement (i)* need not be satisfied.

Then we can easily verify that Theorem 2 remains true if the term "scale" is replaced by "generalized scale".

However, a more direct characterization of a generalized scale is still to be resolved.

REFERENCES

[1] A. H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw-Hill, New York, 1965.

[2] V. Dolezal, *A representation of linear continuous operators on testing functions and distributions*, this Journal, 1 (1970), pp. 491–506.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

ERRATUM: ASYMPTOTIC ANALYSIS OF A DIFFERENTIAL EQUATION OF TURRITTIN*

B. L. J. BRAAKSMA†

In his paper [1] H. L. Turrittin remarks that some corrections have to be made in the paper mentioned in the title. These are as follows.

In the paper I distinguished two cases: the general case $m \neq 0, \pm 1, \cdots, \pm(n-1)$ and the exceptional case $m = \pm 1, \cdots, \pm(n-1)$ (cf. p. 3, lines 8 and 12, p. 5, lines 3 and 7; p. 10, lines 13 and 14). This distinction should be as follows.

The general case. $m \neq 0$ and m is not a fraction of the form

$$(1) \quad m = \pm P/Q, \quad \text{where } P = 1, 2, \cdots, n-1, \quad Q = 1, 2, \cdots, \quad (P, Q) = 1.$$

The exceptional case. m is of the form (1).

Then (1.10) should read

$$0 \leqq hP + j \leqq n-1, \quad 0 \leqq h \leqq n-1, \quad 0 \leqq j \leqq P-1.$$

In lines 12 and 2 from below on p. 3 the values $v = -hQ, -hQ+1, \cdots, -1$ should be included if $m < 0$. In (1.16) the summation over v should extend from $-hQ$ to ∞ if $m < 0$.

In (2.7) the factor $(-1)^{(h+1)(h/2+1)}$ should be replaced by $(-1)^{(h+1)(Qh/2+1)}$, and in the two lines after (2.7) the symbol $|m|$ should be replaced by P.

REFERENCE

[1] H. L. TURRITTIN, *Stokes multipliers for the equation* $d^3y/dx^3 - y/x^2 = 0$, Analytic Theory of Differential Equations, Lecture Notes in Mathematics, vol. 183, Springer-Verlag, Berlin, 1971, pp. 145–157.

* This Journal, 2 (1971), pp. 1–16. Received by the editors June 25, 1971.

† Technische Hogeschool Delft, Delft, Julianalaan 132, the Netherlands.

SIAM J. MATH. ANAL.
Vol. 3, No. 1, February 1972

PERIODIC SOLUTIONS OF HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS IN THE LARGE*

A. K. AZIZ† AND M. G. HORAK‡

Abstract. Sufficient conditions are given for the existence and uniqueness of periodic solutions in the large for the nonlinear hyperbolic partial differential equation $u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y, u, u_x, u_y)$. The method of proof consists of reducing the given problem to the finding of a unique fixed point of a certain integral operator in a suitable function space.

Introduction. In this paper we shall investigate the questions of existence and uniqueness of solutions $u(x, y)$ of the problem

$$Lu = u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y,u,u_x,u_y), \tag{1.1}$$

$$u(x + T, y) = u(x, y) = u(x, y + T), \tag{1.2}$$

provided the functions a, b, c and f are periodic in x and in y with the same period T.

In [1] we dealt with the questions of existence, uniqueness and stability of periodic solutions of (1.1) in a strip, that is, a solution of (1.1) such that

$$u(x,0) = \theta(x), \qquad u(x + T, y) = u(x, y) \tag{1.3}$$

in the strip

$$S = \{(x,y);\ -\infty < x < \infty,\ |y| \leqq \alpha\},$$

where $\theta(x)$ is a prescribed continuously differentiable periodic function.

Problem (1.1), (1.2) has been discussed by many authors, and the reader may consult the recent survey article of Cesari [5] and the paper of Hale [6] and the references contained therein for an extensive bibliography.

In [4] two criteria are given for the existence in the large of periodic solutions for the equation

$$u_{xy} = f(x, y, u, u_x, u_y). \tag{1.4}$$

In the first criterion [4, p. 183], among other regularity and smoothness assumptions it is required that $f(x, y, u, u_x, u_y)$ be of the form

$$f = \varepsilon[\phi(x,y) + cu + \psi_1(y)u_x + \psi_2(x)u_y] + \varepsilon^2 g(x,y,u,u_x,u_y). \tag{1.5}$$

Equation (1.5) is also discussed in [6] by Hale as an example with the conditions

$$\int_0^T \psi_1(y)\,dy \neq 0, \qquad \int_0^T \psi_2(x)\,dx \neq 0.$$

The methods used in both [4] and [6], which are very similar, require first the solution of a modified problem; then using this result, one proceeds to the solution of the original problem.

* Received by the editors March 9, 1971.

† Division of Mathematics, University of Maryland Baltimore County, Baltimore, Maryland 21228 and Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland. The work of this author was supported in part by the Atomic Energy Commission under Contract AEC-AT-(40-1)3443.

‡ Department of Mathematics, Towson State College, Baltimore, Maryland 21204.

The aim of the present paper is to obtain in a direct and simple manner a criterion for the existence of periodic solutions in the large for (1.1). This is accomplished by an extension of the method used by us in [1]. In the process of this extension we also weaken certain hypotheses assumed in [1] (e.g., instead of requiring $b(x, y) > 0$, we merely assume that $\int_0^T b(t, y)\,dt \neq 0$).

The result of the present paper is not contained in the results of [4] and [6], and on the other hand our result does not include the main results of [4] and [6] since our hypotheses exclude the cases where $\int_0^T a(x, t)\,dt = 0$, $\int_0^T b(s, y)\,ds = 0$ (see the statement of Theorem 2.1 below).

2. Existence and uniqueness. In this section we give a criterion for the existence and uniqueness of the solutions $u(x, y)$ for the problem

$$Lu = u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y, u, u_x, u_y), \tag{2.1}$$

$$u(x + T, y) = u(x, y) = u(x, y + T). \tag{2.2}$$

It is known [1, p. 168] that

$$Lu = f(x, y, u, u_x, u_y)$$

if and only if

$$u(x, y) = \frac{1}{\beta(x, y)}\left\{\psi(x) + \int_0^y \frac{1}{\alpha(x, \eta)}\left[\phi(\eta) + \int_0^x A(\xi, \eta, u, u_x, u_y)\,d\xi\right]d\eta\right\}, \tag{2.3}$$

where

(i) $\beta_y/\beta = a$,

(ii) $(\alpha\beta)_x/\alpha\beta = b$,

(iii) $\gamma(x, y, u, u_x, u_y) = f(x, y, u, u_x, u_y) + [a_x + ab - c]u$,

(iv) $A(x, y, u, u_x, u_y) = (\alpha\beta)(x, y)\gamma(x, y, u, u_x, u_y)$,

and $\psi(x)$ and $\phi(y)$ are arbitrary.

We seek first to find a function $u(x, y)$ defined on $R = I \times I$, $I = [0, T]$ such that

$$\begin{aligned} &Lu = f(x, y, u, u_x, u_y), \\ &u(x, 0) = u(x, T), \quad u(0, y) = u(T, y), \\ &u_x(0, y) = u_x(T, y), \quad u_y(x, 0) = u_y(x, T). \end{aligned} \tag{2.4}$$

Suppose $u(x, y)$ satisfies (2.4). Then $u(x, y)$ satisfies (2.3) along with the conditions

$$u(x, 0) = u(x, T), \quad u(0, y) = u(T, y), \quad u_y(0, y) = u_y(T, y). \tag{2.5}$$

Since

$$u_y(0, y) + a(0, y)u(0, y) = u_y(T, y) + a(T, y)u(T, y),$$

we obtain from (2.3),

$$\phi(y) = \frac{(\alpha\beta)(0, y)}{(\alpha\beta)(T, y) - (\alpha\beta)(0, y)}\int_0^T A(\xi, y, u, u_x, u_y)\,d\xi, \tag{2.6}$$

which is well-defined provided

$$\int_0^T b(s, y)\, ds \neq 0 \quad \text{for } y \in I. \tag{2.7}$$

From $u(x, 0) = u(x, T)$ and (2.6) for ϕ above, we see that

$$\psi(x) = \frac{\beta(x, 0)}{\beta(x, T) - \beta(x, 0)}\Bigg[\int_0^T \int_0^T \frac{(\alpha\beta)(0, \eta)}{\alpha(x, \eta)[(\alpha\beta)(T, \eta) - (\alpha\beta)(0, \eta)]} \cdot A(\xi, \eta, u, u_x, u_y)\, d\xi\, d\eta + \int_0^T \int_0^x \frac{1}{\alpha(x, \eta)} A(\xi, \eta, u, u_x, u_y)\, d\xi\, d\eta\Bigg] \tag{2.8}$$

which is well-defined provided

$$\int_0^T a(x, t)\, dt \neq 0 \quad \text{for } x \in I. \tag{2.9}$$

The substitution of ϕ and ψ as given by (2.6) and (2.8) in (2.3) yields

$$\begin{aligned} u(x, y) = {} & \frac{1}{p(x)} \int_0^T \int_0^T \gamma(\xi, \eta, u, u_x, u_y) \frac{s(\xi, \eta; x, y)}{q(\eta)}\, d\xi\, d\eta \\ & + \frac{1}{p(x)} \int_0^T \int_0^x \gamma(\xi, \eta, u, u_x, u_y) s(\xi, \eta; x, y)\, d\xi\, d\eta \\ & + \int_0^y \int_0^T \gamma(\xi, \eta, u, u_x, u_y) \frac{s(\xi, \eta; x, y)}{q(\eta)}\, d\xi\, d\eta \\ & + \int_0^y \int_0^x \gamma(\xi, \eta, u, u_x, u_y) s(\xi, \eta; x, y)\, d\xi\, d\eta, \end{aligned} \tag{2.10}$$

where

$$s(\xi, \eta; x, y) = \exp\left\{-\int_\eta^y a(x, t)\, dt - \int_\xi^x b(s, \eta)\, ds\right\},$$

$$p(x) = \exp\left\{\int_0^T a(x, t)\, dt\right\} - 1, \qquad q(y) = \exp\left\{\int_0^T b(s, y)\, ds\right\} - 1.$$

Hence if u satisfies (2.1) and (2.5) on R and if (2.7) and (2.9) hold, then u satisfies (2.10) on R. Conversely, if u satisfies (2.10) on R, then it also satisfies (2.1) and (2.5).

Before stating our main result, we first introduce certain definitions and hypotheses which we shall need in the sequel.

Let E denote the set of real numbers and assume that the following hypotheses hold:

H_1 : $f(x, y, u, u_x, u_y)$ is continuous on E^5 and it is periodic in x and in y with period T; f satisfies the Lipschitz condition

$$|f(x, y, u, p, q) - f(x, y, \bar{u}, \bar{p}, \bar{q})| \leqq L[|u - \bar{u}| + |p - \bar{p}| + |q - \bar{q}|].$$

H_2 : $a(x, y)$, $a_x(x, y)$, $b(x, y)$ and $c(x, y)$ are continuous on E^2 and a, b and c are periodic in both variables with period T; moreover,

$$\int_0^T a(x, t)\, dt \neq 0, \qquad \int_0^T b(s, y)\, ds \neq 0 \quad \text{on } I = [0, T].$$

Let

$$\mu_1 = \sup_R \left[\left\{ \frac{1}{|p(x)|} \int_0^T \int_0^T + \int_0^y \int_0^T \right\} \frac{s(\xi, \eta; x, y)}{|q(\eta)|} d\xi\, d\eta + \left\{ \frac{1}{|p(x)|} \int_0^T \int_0^x + \int_0^y \int_0^x \right\} s(\xi, \eta; x, y)\, d\xi\, d\eta \right],$$

$$\mu_2 = \sup_R \left[\left\{ \frac{1}{|p(x)|} \int_0^T + \int_0^y \right\} \exp\left[-\int_\eta^y a(x, t)\, dt \right] d\eta \right],$$

$$\mu_3 = \sup_R \left[\left\{ \frac{1}{|q(y)|} \int_0^T + \int_0^x \right\} \exp\left[-\int_\xi^x b(s, y)\, ds \right] d\xi \right],$$

$$\mu_4 = \sup_{0 \leq \eta \leq T, (x,y) \in R} \left| \int_\eta^y a_x(x, t)\, dt + b(x, \eta) \right|,$$

$$\mu_5 = \sup_R \left[\int_0^T \int_0^T \frac{s(\xi, \eta; x, y)}{|q(\eta)|} d\xi\, d\eta + \int_0^T \int_0^x s(\xi, \eta; x, y)\, d\xi\, d\eta \right],$$

$$A = \sup_R |a(x, y)|, \qquad B = \sup_I \frac{p(x) + 1}{p^2(x)} \left| \int_0^T a_x(x, t)\, dt \right|,$$

$$C = \sup_R |(a_x + ab - c)(x, y)|,$$

$$\mu = (1 + A + \mu_4)\mu_1 + \mu_2 + \mu_3 + B\mu_5.$$

Let C_p denote the space of all continuous functions u on R, such that u_x, u_y are continuous on R and $u(0, y) = u(T, y)$, $u(x, 0) = u(x, T)$, $u_x(0, y) = u_x(T, y)$, $u_y(x, 0) = u_y(x, T)$. The norm in C_p is defined by

$$\|u\| = \sup_{(x,y) \in R} |u| + \sup_{(x,y) \in R} |u_x| + \sup_{(x,y) \in R} |u_y|.$$

The equation (2.10) leads us to consider the integral operator K on C_p defined by

$$\begin{aligned}(Ku)(x, y) = {} & \frac{1}{p(x)} \int_0^T \int_0^T \gamma(\xi, \eta, u, u_x, u_y) \frac{s(\xi, \eta; x, y)}{q(\eta)} d\xi\, d\eta \\ & + \frac{1}{p(x)} \int_0^T \int_0^x \gamma(\xi, \eta, u, u_x, u_y) s(\xi, \eta; x, y)\, d\xi\, d\eta \\ & + \int_0^y \int_0^T \gamma(\xi, \eta, u, u_x, u_y) \frac{s(\xi, \eta; x, y)}{q(\eta)} d\xi\, d\eta \\ & + \int_0^y \int_0^x \gamma(\xi, \eta, u, u_x, u_y) s(\xi, \eta; x, y)\, d\xi\, d\eta,\end{aligned} \tag{2.11}$$

with $(Ku)_x$ and $(Ku)_y$ given by

$$
\begin{aligned}
(Ku)_x(x, y) = {} & \frac{1}{p(x)} \int_0^T \int_0^T \gamma(\xi, \eta, u, u_x, u_y) \frac{s_x(\xi, \eta; x, y)}{q(\eta)} \, d\xi \, d\eta \\
& + \frac{1}{p(x)} \int_0^T \int_0^x \gamma(\xi, \eta, u, u_x, u_y) s_x(\xi, \eta; x, y) \, d\xi \, d\eta \\
& + \int_0^y \int_0^T \gamma(\xi, \eta, u, u_x, u_y) \frac{s_x(\xi, \eta; x, y)}{q(\eta)} \, d\xi \, d\eta \\
& + \int_0^y \int_0^x \gamma(\xi, \eta, u, u_x, u_y) s_x(\xi, \eta; x, y) \, d\xi \, d\eta \\
& + \frac{1}{p(x)} \int_0^T \gamma(x, \eta, u, u_x, u_y) \exp\left\{ -\int_\eta^y a(x, t) \, dt \right\} d\eta \\
& + \int_0^y \gamma(x, \eta, u, u_x, u_y) \exp\left\{ -\int_\eta^y a(x,t) \, dt \right\} d\eta \\
& - \left[\frac{p(x) + 1}{p^2(x)} \int_0^T a_x(x, t) \, dt \right] \left[\int_0^T \int_0^T \gamma(\xi, \eta, u, u_x, u_y) \frac{s(\xi, \eta; x, y)}{q(\eta)} \, d\xi \, d\eta \right. \\
& \left. + \int_0^T \int_0^x \gamma(\xi, \eta, u, u_x, u_y) s(\xi, \eta; x, y) \, d\xi \, d\eta \right],
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
(Ku)_y(x, y) = {} & -a(x, y)(Ku)(x, y) + \frac{1}{q(y)} \int_0^T \gamma(\xi, y, u, u_x, u_y) \exp\left\{ -\int_\xi^x b(s, y) \, ds \right\} d\xi \\
& + \int_0^x \gamma(\xi, y, u, u_x, u_y) \exp\left\{ -\int_\xi^x b(s, y) \, ds \right\} d\xi .
\end{aligned} \tag{2.13}
$$

Now we state our main result as the following theorem.

THEOREM 2.1. *Suppose*

(i) *Hypotheses* H_1, H_2 *hold*;

(ii) *the functions a, b, c and the constant L are such that* $[L + C]\mu < 1$.

Then there exists a unique function $u(x, y)$ *in the entire plane such that*

$$Lu = f(x, y, u, u_x, u_y), \qquad u(x + T, y) = u(x, y) = u(x, y + T).$$

Proof. From the properties of the space C_p, the periodicity of the functions a, b, c and f and the equations (2.11), (2.12) and (2.13) it follows that K maps C_p into itself. For $u, v \in C_p$ from the definition of γ we have

$$
\begin{aligned}
|\gamma(\xi, \eta, u(\xi, \eta), u_x(\xi, \eta), u_y(\xi, \eta)) - \gamma(\xi, \eta, v(\xi, \eta), v_x(\xi, \eta), v_y(\xi, \eta))| \\
\leqq [L + C]\|u - v\| .
\end{aligned} \tag{2.14}
$$

Using (2.14) from (2.11), (2.12) and (2.13) we obtain

$$|(Ku)(x, y) - (Kv)(x, y)| \leqq \mu_1[L + C]\|u - v\|, \tag{2.15}$$

$$|(Ku)_x(x, y) - (Kv)_x(x, y)| \leqq [\mu_1\mu_4 + \mu_2 + B\mu_5][L + C]\|u - v\|, \tag{2.16}$$

$$|(Ku)_y(x, y) - (Kv)_y(x, y)| \leqq [A\mu_1 + \mu_3][L + C]\|u - v\|. \tag{2.17}$$

Hence,

$$\|Ku - Kv\| \leqq [L + C]\mu\|u - v\| = \alpha\|u - v\|. \tag{2.18}$$

Since $\alpha < 1$ from Hypothesis H_2, by the contraction mapping principle we conclude the existence of a unique function $u \in C_p$ such that $u = Ku$. The function whose existence is asserted by the theorem is obtained by extending u periodically to all of E^2.

3. Concluding remarks. In Theorem 2.1 for clarity of exposition and technical convenience we assumed that f satisfies a Lipschitz condition in the last three variables. However, we can show the existence of a solution (not necessarily unique) by merely assuming that f satisfies a Lipschitz condition in the last two variables as in [1]. Moreover, the global Lipschitz condition of Theorem 2.1 may be relaxed. Suppose there is a number d such that the Lipschitz condition of Hypothesis H_1 holds in $D = R \times [-d, d]^3$, with Lipschitz constant L_d. It can be shown that the conclusion of Theorem 2.1 holds if d is sufficiently large. In fact, let

$$\sup_{(x,y,u,p,q)\in D} |f(x, y, u, p, q)| \leqq M_d.$$

Now if we require that, besides $[L_d + C]\mu < 1$, we have

$$d \geqq \mu M_d/(1 - \mu C),$$

then the set

$$U_d = \{u \in C_p : (\sup_R |u(x, y)| + \sup_R |u_x(x, y)| + \sup_R |u_y(x, y)|) \leqq d\}$$

is mapped into itself by the operator K. Using L_d instead of L in the remainder of the proof of Theorem 2.1 leads us to the desired conclusion.

Finally we note that under the transformation $t = x + y$, $\xi = x - y$, the differential equation

$$u_{tt} - u_{\xi\xi} = \bar{a}(t, \xi)u_t + \bar{b}(t, \xi)u_\xi + \bar{c}(t, \xi)u = \bar{f}(t, \xi, u, u_t, u_\xi)$$

changes into (2.1). Moreover, the periodicity and Lipschitz conditions placed on a, b, c and f in H_1 and H_2 are satisfied if the functions $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{f}$ have the corresponding properties.

REFERENCES

[1] A. K. AZIZ AND A. M. MEYERS, *Periodic solutions of hyperbolic partial differential equations in a strip*, Trans. Amer. Math. Soc., 146 (1969), pp. 167–178.

[2] L. CESARI, *Periodic solutions of hyperbolic partial differential equations*, International Symposium on Nonlinear Differential Equations and Nonlinear Mechanics (Colorado Springs, 1961), Academic Press, New York, 1963, pp. 33–57; Intern. Symposium on Nonlinear Oscillations (Kiev, 1961), Izv., Akad. Nauk SSSR, 2 (1963), pp. 440–457.

[3] ———, *A criterion for the existence in a strip of periodic solutions of hyperbolic partial differential equations*, Rend. Circ. Mat. Palermo (2), 14 (1965), pp. 1–24.

[4] ———, *Existence in the large of periodic solutions of hyperbolic partial differential equations*, Arch. Rational Mech. Anal., 20 (1965), pp. 170–190.

[5] ———, *Functional analysis and differential equations*, Advances in Differential and Integral Equations, Studies in Applied Mathematics 5, Society for Industrial and Applied Mathematics, Philadelphia, 1969, pp. 143–155.

[6] J. K. HALE, *Periodic solutions of a class of hyperbolic equations containing a small parameter*, Arch. Rational Mech. Anal., 23 (1966), pp. 380–398.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

RIEMANN–STIELTJES INTEGRATION OF OPERATOR-VALUED FUNCTIONS WITH RESPECT TO VECTOR-VALUED FUNCTIONS IN BANACH SPACES*

EDUARD PRUGOVEČKI†

Abstract. The concept of Riemann–Stieltjes integrals is generalized to operator-valued integrands and with respect to vector-valued functions. Integrals of this type appear in perturbation theory of linear operators and in quantum scattering theory. Different versions of such integrals are defined and their existence for given classes of functions is investigated. Some simple examples are explicitly considered.

1. Introduction. In the course of deriving Hilbert space versions of the Lippmann–Schwinger equations and of the transition operator in quantum scattering theory, operators represented by integrals of the form

$$I = \int_{-\infty}^{+\infty} \Phi(H, \mu) T \, d_\mu F_\mu \tag{1.1}$$

have been encountered [1]; here $\Phi(H, \mu)$ is a function of $\mu \in \mathbb{R}^1$ and of a self-adjoint operator H acting in a Hilbert space $\mathscr{H}$, while F_μ is the spectral function of another operator which does not commute with H, and T is a bounded operator on $\mathscr{H}$. If E_λ is the spectral function of H, then by using the spectral theorem for H, one obtains

$$I = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} \Phi(\lambda, \mu) \, d_\lambda E_\lambda \right\} T \, d_\mu F_\mu. \tag{1.2}$$

After introducing an appropriate notion of iterated Riemann–Stieltjes integrals [1], one can write (1.2) in the form

$$I = \int_{-\infty}^{+\infty} d_\mu \int_{-\infty}^{+\infty} \Phi(\lambda, \mu) \, d_\lambda (E_\lambda T F_\mu). \tag{1.3}$$

A typical example of integral representation of the form (1.1) can be provided for the wave operators $\Omega_\pm$,

$$\Omega_\pm = \operatorname*{s-lim}_{t \to \mp\infty} e^{iHt} e^{-iH_0 t} P_0, \tag{1.4}$$

where $H_0 = H_0^*$ and P_0 is the projector on the absolutely continuous subspace of $\mathscr{H}$ with respect to H_0. It is well known that these operators play an important role in perturbation theory of linear operators [2], [3] and the quantum theory [4]. In [1] it is shown that

$$\Omega_\pm = \operatorname*{s-lim}_{\eta \to \pm 0} \int_{-\infty}^{+\infty} \frac{i\eta}{\mu + i\eta - H} d_\mu F_\mu, \tag{1.5}$$

where F_μ is the spectral measure of H_0.

---

* Received by the editors January 8, 1971.

† Department of Mathematics, University of Toronto, Toronto, Ontario, Canada. This work was supported in part by the National Research Council under Grant A5206.

It is the purpose of this paper to define generalizations of integrals of the type (1.1) and (1.3), to investigate where they exist and to establish some of their basic properties.

Since the publication of [1], other work on integrals of the type (1.1) and (1.3) has been brought to our attention. As far as we could determine, integrals of the type (1.1) were first introduced and discussed by Daletskiĭ and Krein [5], who encountered them in problems on the perturbation theory of linear operators. Later on, integrals of the type (1.1) and (1.3) were encountered by Birman [6], [7] in quantum scattering theory. This has given rise to a systematic study [8], [9] by Birman and Solomyak in which these operators are looked upon as transformers on cross-spaces. Some more recent work has been done by Solomyak [10].

In this paper we intend to define n-dimensional Riemann–Stieltjes integrals of the type (1.1) and (1.3) as operators on a Banach space $\mathscr{X}$. However, since the applications which we have in mind [1], [4], [11] occur in Hilbert spaces, most of the illustrations and many of the results are confined to the case when $\mathscr{X}$ is a Hilbert space. The use of Banach spaces rather than Hilbert spaces in the basic definitions is, however, justified since these definitions and the elementary properties of these integrals are not any more involved in Banach spaces than in Hilbert spaces.

In § 2 we define the concept of strong Riemann–Stieltjes integrals and show that they possess most of the elementary properties expected of any integrals. In § 3 we define the concept of vector-valued functions of bounded norm-variation and prove the existence of strong Riemann–Stieltjes with respect to such functions. We proceed to prove in § 4 the existence of such integrals with respect to a much wider class of vector-valued functions, but we can achieve this only by restricting ourselves to integrands with Bochner-integrable strong derivatives. The concept of weak Riemann–Stieltjes integrals is introduced in § 5; all the properties and basic results on strong integrals are shown to hold in a frequently more general context for weak integrals. Finally, in § 6 we introduce the concept of cross-iterated Riemann–Stieltjes integrals, which covers the case of the integral in (1.3).

2. Strong Riemann–Stieltjes integrals. Consider a vector-valued function $f_{\lambda\cdots\omega}$ on an n-dimensional closed and nondegenerate interval $\Delta \subset \mathbb{R}^n$, assuming values in a complex Banach space $\mathscr{X}$; the intervals considered in this paper will always be closures of intervals of the form $(u_1, v_1) \times \cdots \times (u_n, v_n)$, where $u_1, \cdots, u_n \in \mathbb{R}^1 \cup \{-\infty\}$ and $v_n \in \mathbb{R}^1 \cup \{+\infty\}$. Let $A(\lambda, \cdots, \omega)$ be an operator-valued function with range in the Banach algebra $\mathfrak{B}(\mathscr{X})$ of bounded operators on $\mathscr{X}$. We intend to give a meaning to the symbol

$$I = \int_\Delta A(\lambda, \cdots, \omega) f(d\lambda \cdots d\omega) \tag{2.1}$$

by following the analogy with the definition of Riemann–Stieltjes integrals. In fact, the definition of I in (2.1) will be such that it will represent a Riemann–Stieltjes integral when $f_{\lambda\cdots\omega}$ assumes values in the space $\mathscr{X} = \mathbb{C}^1$ of complex numbers (endowed with the usual norm-topology) and when the operator-valued function $A(\lambda, \cdots, \omega)$ is, in fact, a complex function.

Suppose that $\Delta = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and that

$$a_1 = \lambda_0 < \lambda_1 < \cdots < \lambda_k = b_1, \quad \cdots, \quad a_n = \omega_0 < \omega_1 < \cdots < \omega_k = b_n \tag{2.2}$$

are subdivisions of $[a_1, b_1], \cdots, [a_n, b_n]$, respectively. The above points determine a simple subdivision of Δ into nondegenerate closed subintervals $[\lambda_{k-1}, \lambda_k] \times \cdots \times [\omega_{p-1}, \omega_p]$ of Δ, which have no inner points in common; in this context the term "simple" refers to the geometric configuration of this subdivision in which the intervals of the subdivision have sides parallel to the coordinate hyperplanes, and the common points of any two neighboring intervals of the subdivision constitute a whole side of each one of those two intervals.

In the sequel whenever we talk about subdivisions we shall have in mind simple subdivisions. We shall refer to

$$\delta = \max_{\substack{k=1,\cdots,K \\ p=1,\cdots,P}} [(\lambda_k - \lambda_{k-1})^2 + \cdots + (\omega_p - \omega_{p-1})^2]^{1/2} \tag{2.3}$$

as the norm of the subdivision Γ given in (2.2). For arbitrarily chosen inter-subdivision points $\lambda'_k \in [\lambda_{k-1}, \lambda_k], \cdots, \omega'_p \in [\omega_{p-1}, \omega_p]$ we can build the vector-valued *Riemann–Stieltjes sums*

$$\begin{gathered} Q(\Gamma) = \sum_{k=1}^{K} \cdots \sum_{p=1}^{P} A(\lambda'_k, \cdots, \omega'_p) f(\Delta_{k\cdots p}), \\ \Delta_{k\cdots p} = (\lambda_{k-1}, \lambda_k] \times \cdots \times (\omega_{p-1}, \omega_p], \end{gathered} \tag{2.4}$$

where we have introduced the following vector-valued function on intervals $(\alpha_1, \beta_1] \times \cdots \times (\alpha_n, \beta_n] \subset \mathbb{R}^n$:

$$f((\alpha_1, \beta_1] \times \cdots \times (\alpha_n, \beta_n]) = \sum_{\gamma_1 \in \{\alpha_1, \beta_1\}} \cdots \sum_{\gamma_n \in \{\alpha_n, \beta_n\}} (-1)^{Z(\alpha)} f_{\gamma_1 \cdots \gamma_n}; \tag{2.5}$$

here $Z(\alpha)$ denotes the number of α-symbols appearing in the set $\{\gamma_1, \cdots, \gamma_n\}$ for given $f_{\gamma_1 \cdots \gamma_n}$.

Let us consider now the strong limit (i.e., the limit in the norm in $\mathscr{X}$)

$$\operatorname*{s-lim}_{\delta \to 0} \sum_{k=1}^{K} \cdots \sum_{p=1}^{P} A(\lambda'_k, \cdots, \omega'_p) f(\Delta_{k,\cdots,p}) \tag{2.6}$$

taken for sequences of subdivisions of Δ with norms converging to zero. If this limit exists and is independent of the particular choice of subdivisions and of inter-division points $\lambda'_k, \cdots, \omega'_p$, we shall say that the integral (2.1) exists and has the value (2.6). We shall call this integral the *strong Riemann–Stieltjes integral (in $\mathscr{X}$) over Δ of $A(\lambda, \cdots, \omega)$ with respect to $f_{\lambda\cdots\omega}$.*

In case Δ is an infinite interval we define the corresponding integral in the same manner in which improper Riemann–Stieltjes integrals are defined, i.e., by starting with finite subintervals $\Delta_1 \subset \Delta_2 \subset \cdots$ of Δ ($\Delta_1 \cup \Delta_2 \cup \cdots = \Delta$) and taking the limit:

$$\int_{\Delta} A(\lambda, \cdots, \omega) f(d\lambda \cdots d\omega) = \operatorname*{s-lim}_{\Delta_n \to \Delta} \int_{\Delta_n} A(\lambda, \cdots, \omega) f(d\lambda \cdots d\omega). \tag{2.7}$$

The above integral will be called the *improper strong Riemann–Stieltjes integral of* $A(\lambda, \cdots, \omega)$ with respect to $f_{\lambda\cdots\omega}$, as opposed to the weakly improper strong Riemann–Stieltjes integral in which the "s-lim" in (2.7) would be replaced by a limit in the weak topology of $\mathscr{X}$.

The above definition can be generalized immediately to functions $A(\lambda, \cdots, \omega)$ which are unbounded linear operators. When we drop the requirement that $A(\lambda, \cdots, \omega) \in \mathfrak{B}(\mathscr{X})$ we have to impose the following restriction.

Condition C. For any finite nondegenerate subinterval Δ_0 of Δ there is a $\delta(\Delta_0) > 0$ such that $f((\lambda_{k-1}, \lambda_k] \times \cdots \times (\omega_{p-1}, \omega_p])$ is in the domain of definition of $A(\lambda'_k, \cdots, \omega'_p)$ for all $\lambda'_k \in [\lambda_{k-1}, \lambda_k], \cdots, \omega'_p \in [\omega_{p-1}, \omega_p]$ and for any subinterval $[\lambda_{k-1}, \lambda_k] \times \cdots \times [\omega_{p-1}, \omega_p]$ of diameter smaller than $\delta(\Delta_0)$.

When the above condition is satisfied, we shall say that $A(\lambda, \cdots, \omega)$ is *strongly integrable on* Δ *with respect to* $f_{\lambda\cdots\omega}$ if the strong Riemann–Stieltjes integral (2.1) exists when Δ is finite, or if the improper strong Riemann–Stieltjes integral (2.7) exists when Δ is infinite.

The above defined integrals (2.1) possess most of the elementary properties of Riemann–Stieltjes integrals, in particular, the following property.

PROPOSITION 2.1. *Suppose* $A_i(\lambda, \cdots, \omega)$ *and* $f^{(j)}_{\lambda\cdots\omega} \in \mathscr{X}$ *are defined in* Δ. *If* $B_i, i = 1, 2$, *are bounded operators on* $\mathscr{X}$ *and* $c_j, j = 1, 2$, *are complex numbers, then*

$$\int_\Delta [B_1 A_1(\lambda, \cdots, \omega) + B_2 A_2(\lambda, \cdots, \omega)](c_1 f^{(1)} + c_2 f^{(2)})(d\lambda \cdots d\omega)$$
$$= \sum_{i=1}^{2} \sum_{j=1}^{2} c_j B_i \int_\Delta A_i(\lambda, \cdots, \omega) f^{(j)}(d\lambda \cdots d\omega),$$

and the integral on the left-hand side of the above relation exists if the integrals on the right-hand side of that relation exist.

The statements in the above proposition are a straightforward consequence of the obvious relation

$$Q(\Gamma) = \sum_{i=1}^{2} \sum_{j=1}^{2} c_j B_i Q_{ij}(\Gamma)$$

between the Riemann–Stieltjes sums of the respective integrals.

PROPOSITION 2.2. *Suppose* Δ_1 *and* Δ_2 *are two nondegenerate closed intervals in* $\mathbb{R}^n$ *having no internal points in common, and that* $\Delta = \Delta_1 \cup \Delta_2$ *is also a closed interval in* $\mathbb{R}^n$. *Then*

$$\int_\Delta A(\lambda, \cdots, \omega) f(d\lambda \cdots d\omega)$$
$$= \int_{\Delta_1} A(\lambda, \cdots, \omega) f(d\lambda \cdots d\omega) + \int_{\Delta_2} A(\lambda, \cdots, \omega) f(d\lambda \cdots d\omega) \tag{2.8}$$

if all the integrals appearing in the above relation exist.

Proof. The Riemann–Stieltjes sums (2.4) define a multivalued function of the subdivision Γ, different values being obtained for different inter-subdivision points. If Γ_1 and Γ_2 are simple subdivisions of Δ_1 and Δ_2, then respective simple refinements Γ_1' and Γ_2' can be built in such a manner that by combining Γ_1' and Γ_2' we obtain a simple subdivision Γ of Δ. Let $\Delta_1' \in \Gamma_1'$ and $\Delta_2' \in \Gamma_2'$ be two closed intervals intersecting in an $(n-1)$-dimensional interval Δ''. By choosing for any two such intervals a common inter-subdivision point (lying in Δ''), we shall obtain values $Q'(\Gamma_1')$, $Q'(\Gamma_2')$ and $Q'(\Gamma)$ for the respective Riemann–Stieltjes sums such that

$$Q'(\Gamma) = Q'(\Gamma_1') + Q'(\Gamma_2').$$

In the limit, this relation yields (2.8).

PROPOSITION 2.3. *If T is a bounded operator on $\mathscr{X}$, then $A(\lambda, \cdots, \omega)$ is strongly integrable on Δ with respect to $Tf_{\lambda\cdots\omega}$ if and only if $A(\lambda, \cdots, \omega)T$ is strongly integrable on Δ with respect to $f_{\lambda\cdots\omega}$; moreover,*

$$\int_\Delta [A(\lambda, \cdots, \omega)T]f(d\lambda \cdots d\omega) = \int_\Delta A(\lambda, \cdots, \omega)[Tf(d\lambda \cdots d\omega)].$$

The proof of the above proposition is obvious and it relies on the identity

$$\begin{aligned}[A(\lambda_k', \cdots, \omega_p')T]f(\Delta_{k\cdots p}) &= A(\lambda_k', \cdots, \omega_p') \sum_{\gamma_1 \in \{\lambda_{k-1}, \lambda_k\}} \cdots \sum_{\gamma_n \in \{\omega_{p-1}, \omega_p\}} (-1)^{Z(\alpha)} Tf_{\gamma_1 \cdots \gamma_n} \\ &= A(\lambda_k', \cdots, \omega_p')(Tf)(\Delta_{k\cdots p})\end{aligned}$$

in which (2.5) was used.

PROPOSITION 2.4. *If $G_{\lambda\cdots\omega}$ is a family of linear operators on $\mathscr{X}$, the operator S_Δ mapping any $f \in \mathscr{D}[S_\Delta]$ into*

$$S_\Delta(x) = \int_\Delta A(\lambda, \cdots, \omega)(Gf)(d\lambda \cdots d\omega), \tag{2.9}$$

and with domain of definition $\mathscr{D}[S_\Delta]$ consisting of all $f \in \mathscr{X}$ for which the above (proper or improper) integral is defined, is a linear operator.

The above proposition is a straightforward consequence of Proposition 2.1. For the linear operator S_Δ we shall frequently employ the notation:

$$S_\Delta = \int_\Delta A(\lambda, \cdots, \omega)G(d\lambda \cdots d\omega). \tag{2.10}$$

If the above operator is bounded and $\mathscr{D}[S_\Delta]$ is dense in $\mathscr{X}$, we shall say that $A(\lambda, \cdots, \omega)$ is *strongly integrable with respect to* $G_{\lambda\cdots\omega}$.

In the case of integrals of operator-valued functions with respect to operator-valued rather than vector-valued functions, one can introduce

$$\tilde{S}_\Delta = \int_\Delta G(d\lambda \cdots d\omega)A(\lambda, \cdots, \omega) \tag{2.11}$$

instead of (2.10). The definition of the above integral would proceed in an obvious manner by means of Riemann–Stieltjes sums of the form

$$\sum_{k=1}^{K} \cdots \sum_{p=1}^{P} G(\Delta_{k\cdots p})A(\lambda_k', \cdots, \omega_p').$$

However, in the case of operator-valued functions on a Hilbert space $\mathscr{H}$, we would obviously have that $\tilde{S}_\Delta \subseteq S_\Delta^*$, provided that S_Δ is densely defined. Thus, in this practically most interesting case, (2.11) can be reduced to (2.10).

Other types of integrals, such as

$$\int_\Delta A(\lambda, \cdots, \omega)G(d\lambda \cdots d\omega)B(\lambda, \cdots, \omega), \tag{2.12}$$

can be defined by following the same general recipe. It is easy to reformulate many of the results of this paper for the case of integrals of type (2.12).

3. The existence of strong integrals with respect to functions of bounded norm-variation. We define the norm-variation $V_\Delta[f_{\lambda\cdots\omega}]$ of $f_{\lambda\cdots\omega}$ over the finite non-degenerate interval $\Delta = [a_1, b_1] \times \cdots \times [a_n, b_n]$ as

$$V_\Delta[f_{\lambda\cdots\omega}] = \sup \sum_{k=1}^{K} \cdots \sum_{p=1}^{P} \left\| f((\lambda_{k-1}, \lambda_k] \times \cdots \times (\omega_{p-1}, \omega_p]) \right\|, \tag{3.1}$$

where the supremum is taken over all possible simple subdivisions of Δ. If Δ is an infinite interval, then, by definition,

$$V_\Delta[f_{\lambda\cdots\omega}] = \sup_{\Delta_1 \subset \Delta} V_{\Delta_1}[f_{\lambda\cdots\omega}], \tag{3.2}$$

where Δ_1 varies over the set of all finite subintervals of Δ.

LEMMA 3.1. *Suppose the strong Riemann–Stieltjes integral of $A(\lambda, \cdots, \omega)$ with respect to $f_{\lambda\cdots\omega}$ exists. If $f_{\lambda\cdots\omega}$ is of bounded norm-variation $V_\Delta[f_{\lambda\cdots\omega}]$ and $A(\lambda, \cdots, \omega)$ is uniformly bounded on Δ,*

$$\|A(\lambda, \cdots, \omega)\| \leqq C, \qquad (\lambda, \cdots, \omega) \in \Delta, \tag{3.3}$$

then

$$\left\| \int_\Delta A(\lambda, \cdots, \omega) f(d\lambda \cdots d\omega) \right\| \leqq C V_\Delta[f_{\lambda\cdots\omega}]. \tag{3.4}$$

Proof. For finite Δ, the above result follows immediately from the following estimate on the Riemann–Stieltjes sum $Q(\Gamma)$ defined in (2.4):

$$\|Q(\Gamma)\| \leqq \sum_{k=1}^{K} \cdots \sum_{p=1}^{P} \| A(\lambda_k', \cdots, \omega_p')\| \, \|f(\Delta_{k\cdots p})\| \leqq C V_\Delta[f_{\lambda\cdots\omega}].$$

The result can be immediately extended to infinite Δ by taking the limit (2.7) and noting that $V_{\Delta_0} \leqq V_\Delta$ when $\Delta_0 \subset \Delta$.

THEOREM 3.1. *Let $f_{\lambda\cdots\omega}$ be of bounded norm-variation on Δ and suppose $A(\lambda, \cdots, \omega) \in \mathfrak{B}(\mathscr{X})$ is continuous on Δ in the uniform topology of $\mathfrak{B}(\mathscr{X})$. Then*:

(a) *if Δ is finite, $A(\lambda, \cdots, \omega)$ is strongly integrable on Δ with respect to $f_{\lambda\cdots\omega}$*;

(b) *if Δ is infinite and $\|A(\lambda, \cdots, \omega)\|$ is bounded on Δ, then $A(\lambda, \cdots, \omega)$ is strongly integrable on Δ with respect to $f_{\lambda\cdots\omega}$.*

Proof. (a) Let $\{(\lambda_{kk'}, \cdots, \omega_{pp'})\}$ determine a subdivision Γ' which is a refinement of the subdivision Γ given by (2.2), so that $\lambda_{k0} = \lambda_k, \cdots, \omega_{p0} = \omega_p$. Denote by $\tilde{\lambda}_{kk'}, \cdots, \tilde{\omega}_{pp'}$ the points preceding $\lambda_{kk'}, \cdots, \omega_{pp'}$, respectively. If Γ is so fine that

$$\|A(\lambda', \cdots, \omega') - A(\lambda'', \cdots, \omega'')\| < \varepsilon \tag{3.5}$$

whenever $(\lambda', \cdots, \omega')$ and $(\lambda'', \cdots, \omega'')$ belong to the same interval in Γ, then

$$\begin{aligned}\|Q(\Gamma) - Q(\Gamma')\| &\leqq \sum_{k,k'} \cdots \sum_{p,p'} \|A(\lambda'_k, \cdots, \omega'_p) - A(\lambda'_{kk'}, \cdots, \omega'_{pp'})\| \\ &\quad \cdot \|f((\tilde{\lambda}_{kk'}, \lambda_{kk'}] \times \cdots \times (\tilde{\omega}_{pp'}, \omega_{pp'}])\| \\ &\leqq \varepsilon \sum_{k,k'} \cdots \sum_{p,p'} \|f((\tilde{\lambda}_{kk'}, \lambda_{kk'}] \times \cdots \times (\tilde{\omega}_{pp'}, \omega_{pp'}])\| \leqq \varepsilon\, V_\Delta |f_{\lambda \cdots \omega}|.\end{aligned}$$

Due to the continuity of $A(\lambda, \cdots, \omega)$, for any $\varepsilon > 0$ there is a $\delta(\varepsilon)$ such that the inequality (3.5) holds for all subdivisions Γ of the compact interval Δ with norms not exceeding $\delta(\varepsilon)$.

(b) Suppose Δ is infinite and that $\Delta_1, \Delta_2, \cdots$ is any sequence of closed finite intervals with disjoint interiors and such that $\bigcup \Delta_i = \Delta$. Then by Lemma 3.1,

$$\sum_{i=1}^{\infty} \left\| \int_{\Delta_i} A(\lambda, \cdots, \omega) f(d\lambda \cdots d\omega) \right\| \leqq C V_\Delta(f_{\lambda \cdots \omega}).$$

This implies that the improper strong integral (2.1) exists, thus completing the proof.

We shall apply the above theorem to proving the existence of two types of strong Riemann–Stieltjes integrals in Hilbert spaces.

Let $\mathscr{H}$ be a complex Hilbert space with inner product $\langle \cdot | \cdot \rangle$, which is linear in the *second* argument. For any given $g, h \in \mathscr{H}$, denote by $|g\rangle a \langle h|$ the linear operator which yields $\langle h | af \rangle f$ when it acts on $f \in \mathscr{H}$. It is well known that for any trace-class operator T there are two orthonormal systems $g_1, g_2, \cdots$ and $h_1, h_2, \cdots$ and a sequence of nonnegative numbers $a_1, a_2, \cdots$ such that

$$T = \sum_{i=1}^{\infty} |g_i\rangle a_i \langle h_i|, \tag{3.6}$$

$$\|T\|_1 = \sum_{i=1}^{\infty} a_i, \tag{3.7}$$

where $\|\cdot\|_1$ denotes the trace-class norm.

LEMMA 3.2. *If $E_x, x \in \mathbb{R}^n$, is a spectral function in the complex Hilbert space $\mathscr{H}$ and T is of trace-class, then for any $f \in \mathscr{H}$ the vector-valued function $f_x = TE_x f$ is of bounded norm-variation on $\mathbb{R}^n$, and*

$$V_{\mathbb{R}^n}[TE_x f] \leqq 2\|T\|_1\{\|f\|^2 + \|f\| + 1\}. \tag{3.8}$$

Proof. Let us consider the case when $T = |g\rangle\langle h|$. Denoting by $E(\Delta)$ the spectral measure determined by E_x and using the identity

$$\begin{aligned}\langle h | E(\Delta_{\mathbf{k}}) f\rangle = \tfrac{1}{2}\{&\langle f + h | E(\Delta_{\mathbf{k}})(f + h)\rangle + i\langle f + ih | E(\Delta_{\mathbf{k}})(f + ih)\rangle \\ &- (1 + i)\langle f | E(\Delta_{\mathbf{k}}) f\rangle - (1 + i)\langle h | E(\Delta_{\mathbf{k}}) h\rangle\},\end{aligned}$$

where $\mathbf{k} = (k, \cdots, p)$ and $\bigcup \Delta_{\mathbf{k}} = \Delta$, we easily arrive at the result that

$$\begin{aligned} V_\Delta[\langle h|E_x f\rangle g] &\leqq \sup \sum_{\mathbf{k}} \|g\| \, |\langle h|E(\Delta_{\mathbf{k}})f\rangle| \\ &\leqq \tfrac{1}{2}\|g\|\{\|E(\Delta)f + h)\|^2 + \|E(\Delta)(f + ih)\|^2 + \sqrt{2}\|E(\Delta)f\|^2 \\ &\qquad + \sqrt{2}\|E(\Delta)h\|^2\} \end{aligned} \tag{3.9}$$

for any finite closed interval $\Delta \subset \mathbb{R}^n$.

In the general case with T given by (3.6) we obviously have

$$V_\Delta[TE_x f] \leqq \sum_{j=1}^{\infty} a_j V_\Delta[\langle h_j|E_x f\rangle g_j].$$

Combining this inequality with (3.9) and with estimates like

$$\|E(\Delta)(f + h_j)\| \leqq \|f\| + \|h_j\| = \|f\| + 1,$$

we obtain

$$V_\Delta[TE_x f] \leqq \tfrac{1}{2}\|T\|_1\{(2 + \sqrt{2})\|f\|^2 + 4\|f\| + 2 + \sqrt{2}\}$$

from which (3.8) immediately follows.

THEOREM 3.2. *Suppose $A(x), x \in \Delta$, is continuous in the uniform topology of $\mathfrak{B}(\mathscr{H})$ and that $\|A(x)\| \leqq C$ for all values of x in the closed interval $\Delta \subset \mathbb{R}^n$. If T is a trace-class operator on the Hilbert space $\mathscr{H}$ and $E_x, x \in \mathbb{R}^n$, is a spectral function, then $A(x)T$ is strongly integrable with respect to E_x and*

$$\left\| \int_\Delta A(x)TE(dx) \right\| \leqq 6C\|T\|_1. \tag{3.10}$$

Proof. The existence of

$$\int_\Delta A(x)TE(dx)f = \int_\Delta A(x)(TE)(dx)f, \qquad f \in \mathscr{H},$$

follows from Lemma 3.2 and Proposition 1.4. Furthermore, from Lemma 3.1 and (3.8) we obtain that

$$\left\| \int_\Delta A(x)TE(dx)f \right\| \leqq 2\|T\|_1\{\|f\|^2 + \|f\| + 1\}$$

for all $f \in \mathscr{H}$. By letting f vary over the unit sphere $\|f\| = 1$, we arrive at (3.10).

4. Existence of strong integrals with differentiable integrands. In the preceding sections we have managed to prove the strong integrability of $A(\lambda, \cdots, \omega)$ with respect to $f_{\lambda\cdots\omega}$ by imposing on $f_{\lambda\cdots\omega}$ the very stringent condition of norm-boundedness. If we discard this condition in favor of more relaxed conditions on $f_{\lambda\cdots\omega}$, we find that it is necessary to impose some more stringent conditions on $A(\lambda, \cdots, \omega)$ in order to establish the existence of the considered integrals (2.1) and (2.7). We shall prove next the existence (2.1) by using a method of Daletskiĭ and Krein [5] based on the existence of derivatives of $A(\lambda, \cdots, \omega)$. In this context we define the strong partial derivative

$$\frac{\partial A(\lambda, \cdots, \omega)}{\partial \lambda} = \operatorname*{s-lim}_{\Delta\lambda \to 0} \frac{A(\lambda + \Delta\lambda, \cdots, \omega) - A(\lambda, \cdots, \omega)}{\Delta\lambda} \tag{4.1}$$

with corresponding definitions of partial derivatives with respect to the other variables.

The following lemma and theorem represent a generalization of Theorem 1.1 in [5]; the employed results on Bochner integrals can be found either in Chap. III of [12], or can be easily deduced from the theorems contained in [12].

LEMMA 4.1. *If $dB(\lambda)/d\lambda \in \mathfrak{B}(\mathscr{X})$ is Bochner integrable on $[a, b]$ with respect to the Lebesgue measure, and if f_λ, $\lambda \in [a, b]$, is any function with values in $\mathscr{X}$, then*

$$\left\| \sum_{k=1}^{n} B(\lambda'_k)(f_{\lambda_k} - f_{\lambda_{k+1}}) \right\| \leqq \|B(\lambda'_n) f_b\| + \|B(\lambda'_0) f_a\| + \sup_{a \leqq \lambda \leqq b} \|f_\lambda\| \int_a^b \left\| \frac{dB(\lambda')}{d\lambda'} \right\| d\lambda' \tag{4.2}$$

for any $a = \lambda_0 < \lambda_1 < \lambda_n = b$ and $\lambda_{k-1} \leqq \lambda'_k \leqq \lambda_k$.

Proof. Straightforward algebra yields:

$$\sum_{k=1}^{n} B(\lambda'_k)(f_{\lambda_k} - f_{\lambda_{k-1}}) = \sum_{k=1}^{n} B(\lambda'_k) f_{\lambda_k} - \sum_{k=0}^{n-1} B(\lambda'_{k+1}) f_{\lambda_k}$$

$$= B(\lambda'_n) f_b - B(\lambda'_i) f_a - \sum_{k=1}^{n-1} [B(\lambda'_{k+1}) - B(\lambda'_k)] f_{\lambda_k}.$$

By taking advantage of the existence of the Bochner integral of $dB(\lambda)/d\lambda$ with respect to the Lebesgue measure, we obtain the relation

$$\left\| \sum_{k=1}^{n-1} (B(\lambda'_{k+1}) - B(\lambda'_k)) f_{\lambda_k} \right\| = \left\| \sum_{k=1}^{n-1} \left[\int_{\lambda'_k}^{\lambda'_{k+1}} \frac{dB(\lambda')}{d\lambda'} d\lambda' \right] f_{\lambda_k} \right\|$$

$$\leqq \sum_{k=1}^{n-1} \int_{\lambda'_k}^{\lambda'_{k+1}} \left\| \frac{dB}{d\lambda'}(\lambda') \right\| d\lambda' \|f_{\lambda_k}\|$$

$$\leqq \left\{ \int_a^b \left\| \frac{dB}{d\lambda'}(\lambda') \right\| d\lambda' \right\} \max_{k=1,\cdots,n} \|f_{\lambda_k}\|$$

from which (4.2) immediately follows.

THEOREM 4.1. *Suppose the operator-valued function $A(\lambda) \in \mathfrak{B}(\mathscr{X})$ has a strong derivative $dA(\lambda)/d\lambda$ for all λ in the closed finite interval $[a, b]$ and that $dA(\lambda)/d\lambda$ is Bochner integrable on $[a, b]$ with respect to Lebesgue measure. If*

$$\int_a^b X_S(\lambda) f(d\lambda), \qquad X_S(\lambda) = \int_a^\lambda \chi_S(\lambda')\, d\lambda',$$

exists for the characteristic function $\chi_S(\lambda)$ of any Lebesgue measurable set $S \subset [a, b]$, then $A(\lambda)$ is strongly integrable with respect to f_λ and

$$\left\| \int_a^b A(\lambda) f(d\lambda) \right\| \leqq \|A(a) f_a\| + \|A(b) f_b\| + \sup_{a \leqq \lambda \leqq b} \|f_\lambda\| \int_a^b \left\| \frac{dA(\lambda')}{d\lambda'} \right\| d\lambda'. \tag{4.3}$$

Proof. The Bochner integrability of $dA(\lambda)/d\lambda$ (with respect to Lebesgue measure) on the interval $[a, b]$, which is of finite measure, implies that for given $\varepsilon_1 > 0$ there is a simple operator-valued function

$$D_1(\lambda) = \sum_{k=1}^{n} B_k \chi_k(\lambda), \qquad B_k \in \mathfrak{B}(\mathscr{H}),$$

where $\chi_k(\lambda)$ is the characteristic function of some measurable subset of $[a, b]$, such that

$$\int_a^b \left\| \frac{dA(\lambda)}{d\lambda} - D_1(\lambda) \right\| d\lambda < \varepsilon_1 . \tag{4.4}$$

According to the assumptions stated in the theorem,

$$\int_a^b X_k(\lambda) f(d\lambda), \qquad X_k(\lambda) = \int_a^\lambda \chi_k(\lambda')\, d\lambda',$$

exists. Hence the difference $\|Q_k(\Gamma) - Q_k(\Gamma')\|$ of any two Riemann–Stieltjes sums for $X_k(\lambda)$ can be made arbitrarily small by choosing sufficiently fine subdivisions Γ and Γ' of $[a, b]$. Thus, if $Q_D(\Gamma)$ and $Q_D(\Gamma')$ are the corresponding Riemann–Stieltjes sums of

$$D(\lambda) = \int_a^\lambda D_1(\lambda')\, d\lambda' + A(a) = \sum_{k=1}^{n} B_k X_k(\lambda) + A(a),$$

then we can achieve

$$\|Q_D(\Gamma) - Q_D(\Gamma')\| \leqq \sum_{k=1}^{n} \|B_k\|\, \|Q_k(\Gamma) - Q_k(\Gamma')\| < \varepsilon \tag{4.5}$$

for all sufficiently fine Γ and Γ'.

If $Q_A(\Gamma)$ and $Q_A(\Gamma')$ denote the corresponding Riemann–Stieltjes sums for $A(\lambda)$, we can obviously write

$$\|Q_A(\Gamma) - Q_A(\Gamma')\| \leqq \|Q_A(\Gamma) - Q_D(\Gamma)\| + \|Q_D(\Gamma) - Q_D(\Gamma')\| + \|Q_A(\Gamma') - Q_D(\Gamma')\|.$$

Let Γ be the subdivision $a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$. Applying Lemma 4.1 to $B(\lambda) = A(\lambda) - D(\lambda)$ and $\tilde{f}_\lambda = f_\lambda - f_b$ we obtain

$$\|Q_A(\Gamma) - Q_D(\Gamma)\| \leqq \left\| [A(\lambda_0') - A(a) + \sum_{k=1}^{n} B_k X_k(\lambda_0')](f_a - f_b) \right\|$$

$$+ \sup_{a \leqq \lambda \leqq b} \|f_\lambda - f_b\| \int_a^b \left\| \frac{dA(\lambda)}{d\lambda} - D_1(\lambda) \right\| d\lambda.$$

In the limit of finer and finer subdivisions, $\lambda'_0 \to a$ and $X_k(\lambda_0) \to X_k(a) = 0$. Consequently, the first term can be made arbitrarily small for all sufficiently fine subdivisions. Thus, in view of (4.4) we conclude that for any given $\varepsilon > 0$,

$$\|Q_A(\Gamma) - Q_D(\Gamma)\| < \varepsilon$$

for all sufficiently fine subdivisions of $[a, b]$.

The same argument can be applied to $\|Q_A(\Gamma') - Q_D(\Gamma')\|$, so that after taking into consideration (4.5), we obtain

$$\|Q_A(\Gamma) - Q_A(\Gamma')\| < 3\varepsilon$$

for any sufficiently fine subdivision of $[a, b]$. This establishes the strong integrability of $A(\lambda)$.

By again applying Lemma 4.1, only this time to $A(\lambda)$ and f_λ, and going to the limit of finer and finer partitions (which gives rise to $\lambda'_n \to b$ and $\lambda'_0 \to a$) we obtain (4.3).

THEOREM 4.2. *Suppose the conditions imposed in Theorem* 4.1 *on* $A(\lambda)$ *and* f_λ *are satisfied on any finite closed interval in* $\mathbb{R}^1$. *If* $\|A(\lambda)f_\lambda\| \to 0$ *when* $\lambda \to \pm\infty$, *and if*

$$\sup_{\lambda\in\mathbb{R}^1} \|f_\lambda\| < +\infty,$$

$$\int_{-\infty}^{+\infty} \left\| \frac{dA(\lambda')}{d\lambda'} \right\| d\lambda' < +\infty,$$

then the improper strong Riemann–Stieltjes integral of $A(\lambda)$ *with respect to* f_λ *exists and*

$$\left\| \int_{-\infty}^{+\infty} A(\lambda) f(d\lambda) \right\| \leqq \sup_{\lambda\in\mathbb{R}^1} \|f_\lambda\| \int_{-\infty}^{+\infty} \left\| \frac{dA(\lambda)}{d\lambda} \right\| d\lambda. \tag{4.6}$$

Proof. For arbitrary $b_1 > b$ we obtain from (4.3):

$$\left\| \int_0^{b_1} A(\lambda) f(d\lambda) - \int_0^{b} A(\lambda) f(d\lambda) \right\| = \left\| \int_b^{b_1} A(\lambda) f(d\lambda) \right\|$$

$$\leqq \|A(b_1) f_{b_1}\| + \|A(b) f_b\| + \sup_{\lambda\in\mathbb{R}^1} \|f_\lambda\| \int_b^{b_1} \left\| \frac{dA(\lambda')}{d\lambda'} \right\| d\lambda'.$$

The conditions specified in the statement of the theorem insure that the right-hand side of the above relation can be made arbitrarily small by choosing b sufficiently large and $b_1 > b$. Hence the integrability on $[0, +\infty)$ follows.

A similar argument can be used for $(-\infty, 0]$. Finally, (4.3) leads directly to (4.6).

THEOREM 4.3. *If the strong derivative* $dA(\lambda)/d\lambda$ *of the operator-valued function* $A(\lambda) \in \mathfrak{B}(\mathscr{H})$ *is Bochner integrable on* $\mathbb{R}^1$ *with respect to the Lebesgue measure, then* $A(\lambda)T$, $T \in \mathfrak{B}(\mathscr{H})$, *is strongly integrable on* $\mathbb{R}^1$ *with respect to any spectral function* E_λ *and*

$$\left\| \int_{\infty}^{+\infty} A(\lambda) T E(d\lambda) \right\| \leqq \int_{-\infty}^{+\infty} \left\| \frac{dA(\lambda)}{d\lambda} T \right\| d\lambda. \tag{4.7}$$

Proof. Since $dA(\lambda)/d\lambda$ is Bochner integrable on $\mathbb{R}^1$, $(dA(\lambda)/d\lambda)T$ is also Bochner integrable on $\mathbb{R}^1$. Furthermore, for any $f \in \mathscr{H}$,

$$\int_a^b X(\lambda)E(d\lambda)f = \int_a^b X(\lambda)\, dE_x f$$

exists since $X(\lambda)$ is continuous and E_λ is a spectral measure. In view of the additional fact that

$$\sup_{\lambda \in \mathbb{R}^1} \|E_\lambda f\| = \|f\|,$$

we can say that all the conditions stipulated in Theorem 4.2 for the existence of

$$Sf = \int_{-\infty}^{+\infty} A(\lambda)TE(d\lambda)f$$

are satisfied. Thus the linear mapping $f \to Sf$ is defined for all $f \in \mathscr{H}$, and by (4.6),

$$\|Sf\| \leqq \sup_{\lambda \in \mathbb{R}^1} \|E_\lambda f\| \int_{-\infty}^{+\infty} \left\| \frac{dA(\lambda)}{d\lambda} T \right\| d\lambda = \|f\| \int_{-\infty}^{+\infty} \left\| \frac{dA(\lambda)}{d\lambda} T \right\| d\lambda.$$

This completes the proof.

The above theorem can be used to establish the existence of integrals of the type

$$\int_{-\infty}^{+\infty} \Phi(\lambda, H)E(d\lambda), \tag{4.8}$$

which are exemplified in (1.5), and in which $\Phi(\lambda, H)$ is a function of the self-adjoint operator H. For example, if $\eta \neq 0$ and

$$\Phi(\lambda, H) = \frac{1}{\lambda - H + i\eta}, \tag{4.9}$$

then

$$\frac{d\Phi(\lambda, H)}{d\lambda} = -(\lambda - H + i\eta)^{-2}$$

is Bochner integrable on $\mathbb{R}^1$ if H is bounded:

$$\int_{-\infty}^{+\infty} \left\| \frac{d\Phi(\lambda, H)}{d\lambda} \right\| d\lambda \leqq \int_{-\|H\|}^{+\|H\|} \frac{d\lambda}{\eta^2} + \int_{\|H\|}^{+\infty} \left| \frac{1}{\lambda - \|H\| + i\eta} \right|^2 d\lambda + \int_{-\infty}^{-\|H\|} \left| \frac{1}{\lambda + \|H\| + i\eta} \right|^2 d\lambda.$$

However, if H is unbounded, there are arbitrarily large $|\lambda|$ such that

$$\|(\lambda - H + i\eta)^{-2}\| = \eta^{-2}$$

and $d\Phi(\lambda, H)/d\lambda$ is not Bochner integrable on $\mathbb{R}^1$. To prove in this case the existence and boundedness of integrals of type (4.8) we need the concept of weak integrability, which will be introduced in the next section.

5. Weak Riemann–Stieltjes integrals. We shall say that the vector f in the Banach space $\mathscr{X}$ is the weak limit of a sequence $f_1, f_2, \cdots \in \mathscr{X}$ if and only if $\phi(f) = \lim \phi(f_n)$ for every continuous linear functional $\phi \in \mathscr{X}'$ from the normed conjugate space $\mathscr{X}'$ of $\mathscr{X}$. In that case we shall write $f = \text{w-lim}\, f_n$.

Weak versions of the strong integrals (2.1) and (2.7) can be defined by replacing everywhere in the definitions of these integrals strong limits by weak limits. Thus the following definitions:

$$\text{(5.1)} \quad \int_{\Delta_n} A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega} = \underset{\delta\to 0}{\text{w-lim}} \sum_{k=1}^{K} \cdots \sum_{p=1}^{P} A(\lambda'_k, \cdots, \omega'_p) f(\Delta_{k\cdots p}),$$

$$\text{(5.2)} \quad \int_{\Delta} A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega} = \underset{\Delta_n\to\Delta}{\text{w-lim}} \int_{\Delta_n} A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega}$$

do not require any more detailed clarification beyond the reminder that if the operators $A(\lambda, \cdots, \omega)$ are not defined everywhere in $\mathscr{X}$, then Condition C in § 2 has to be satisfied.

It is useful to note that if $A(\lambda, \cdots, \omega)$ is a multiple of the identity operator, $A(\lambda, \cdots, \omega) = a(\lambda, \cdots, \omega)\mathbf{1}$, then for any $\phi \in \mathscr{X}'$,

$$\phi\left(\int_{\Delta} A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega}\right) = \int_{\Delta} a(\lambda, \cdots, \omega)\, d_{\lambda\cdots\omega}\phi(f_{\lambda\cdots\omega}).$$

All the basic properties of strong integrals are retained in the present case of weak integrals. We state these properties without proof.

PROPOSITION 5.1. *If $B_i, i = 1, 2$, are bounded operators on $\mathscr{X}$ and $c_j, j = 1, 2$, are complex numbers, then*

$$\int_{\Delta} B_1 A_1(\lambda, \cdots, \omega) + B_2 A_2(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega (c_1 f^{(1)}_{\lambda\cdots\omega} + c_2 f^{(2)}_{\lambda\cdots\omega})$$
$$= \sum_{i=1}^{2} \sum_{j=1}^{2} c_j B_i \int_{\Delta} A_i(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f^{(j)}_{\lambda\cdots\omega},$$

and the existence of the integrals on the right-hand side of the above relation imply the existence of the integral on the left-hand side.

PROPOSITION 5.2. *If Δ, Δ_1 and Δ_2 are closed intervals in $\mathbb{R}^n$ and $\Delta = \Delta_1 \cup \Delta_2$, and if $\Delta_1 \cap \Delta_2$ contains only boundary points of Δ_1 and Δ_2, then*

$$\int_{\Delta} A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega}$$
$$= \int_{\Delta_1} A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega} + \int_{\Delta_2} A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega}$$

provided that all the integrals in the above relation exist.

PROPOSITION 5.3. *If T is a bounded operator on $\mathscr{X}$, then $A(\lambda, \cdots, \omega)$ is weakly integrable on Δ with respect to $Tf_{\lambda\cdots\omega}$ if and only if $A(\lambda, \cdots, \omega)T$ is integrable on Δ with respect to $f_{\lambda\cdots\omega}$; moreover,*

$$\int_{\Delta} A(\lambda, \cdots, \omega)T\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega} = \int_{\Delta} A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega T f_{\lambda\cdots\omega}.$$

PROPOSITION 5.4. *If $G_{\lambda\cdots\omega}$ is a family of linear operators in $\mathscr{X}$, the mapping which takes $f \in \mathscr{X}$ into*

$$S_\Delta(f) = \int_\Delta A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega G_{\lambda\cdots\omega} f, \tag{5.3}$$

with domain consisting of all vectors f for which the above integral exists, is a linear operator in $\mathscr{X}$.

The proofs of the above four propositions are completely analogous to the proofs of Propositions 2.1–2.4.

If the operator defined by (5.3) is bounded and defined everywhere in $\mathscr{X}$, then we shall say that $A(\lambda, \cdots, \omega)$ is *weakly integrable* with respect to $G_{\lambda\cdots\omega}$.

The results of §§ 3 and 4 can be expanded considerably when dealing with weak Riemann–Stieltjes integrals on (complex) Hilbert spaces. For example, if we denote by $A'(\lambda, \cdots, \omega)$ the conjugate in $\mathscr{X}'$ of the bounded operator $A(\lambda, \cdots, \omega)$, then Lemma 3.1 can be reformulated under more general conditions as follows.

LEMMA 5.1. *If $A(\lambda, \cdots, \omega)$ is weakly integrable on Δ with respect to the function $f_{\lambda\cdots\omega} \in \mathscr{X}$ of bounded norm-variation $V_\Delta[f_{\lambda\cdots\omega}]$, and for any $\phi \in \mathscr{X}'$ there is a constant C_ϕ such that $\|A'(\lambda, \cdots, \omega)\phi\| < C_\phi$ for all $(\lambda, \cdots, \omega) \in \Delta$, then*

$$\left|\phi\left(\int_\Delta A(\lambda, \cdots, \omega)\, d_\lambda \cdots d_\omega f_{\lambda\cdots\omega}\right)\right| \leqq C_\phi V_\Delta[f_{\lambda\cdots\omega}]. \tag{5.4}$$

The proof of the above lemma runs along the same lines as the proof of Lemma 4.1 once it is observed that

$$|\phi(Q(\Gamma))| \leqq \sum_{k=1}^{K} \cdots \sum_{p=1}^{P} \|A'(\lambda_k', \cdots, \omega_p')\phi\| \, \|f(\Delta_{k\cdots p})\| \leqq C_\phi V_\Delta[f_{\lambda\cdots\omega}].$$

A similar alteration in the proof of Theorem 3.1 yields the following result.

THEOREM 5.1. *Let $f_{\lambda\cdots\omega}$ be of bounded norm-variation on Δ and suppose $A'(\lambda, \cdots, \omega) \in \mathfrak{B}(\mathscr{X}')$ is continuous on Δ in the strong topology of $\mathfrak{B}(\mathscr{X}')$. Then*:

(a) *if Δ is finite, $A(\lambda, \cdots, \omega)$ is weakly integrable on Δ with respect to $f_{\lambda\cdots\omega}$*;

(b) *if Δ is infinite and $\|A(\lambda, \cdots, \omega)\phi\|$ is bounded on Δ for every $\phi \in \mathscr{X}'$, then $A(\lambda, \cdots, \omega)$ is integrable on Δ with respect to $f_{\lambda\cdots\omega}$.*

Theorem 5.1 can be used to prove the existence of weak (but not strong) integrals of the type

$$\int_{\Delta^{(x)} \times \Delta^{(\xi)}} \Phi(\xi, x) B(x)\, d\mu(x)\, d_\xi T E_\xi = \int_{\Delta^{(x)} \times \Delta^{(\xi)}} \Phi(\xi, y) B(y)\, d_y\, d_\xi \sigma_y T E_\xi, \qquad \sigma_y = \int^y d\mu(x), \tag{5.5}$$

on a Hilbert space $\mathscr{H}$; in the above equation the weak Riemann–Stieltjes integral on the right-hand side defines the expression on the left-hand side, and T is a bounded operator, $\Phi(\xi, x)$ is a complex function on $\Delta^{(x)} \times \Delta^{(\xi)}$, μ is a measure on $\Delta^{(x)} \subset \mathbb{R}^n$, E_ξ is a spectral measure on $\Delta^{(\xi)} \subset \mathbb{R}^n$ and $\int^y d\mu(x)$ is the measure of the set $\Delta_x \cap |(-\infty, \tau] \times \cdots \times (-\infty, \omega]|$, where $y = (\tau, \cdots, \omega)$. Integrals of this type have been encountered in [1] (cf. also [4], [11]); e.g.,

$$\int_{[0,+\infty)\times\mathbb{R}^1} e^{-i(H+\lambda-i\varepsilon)t}T\,dt\,dE_\lambda = \int_{[0,\times\infty)\times\mathbb{R}^1} e^{-i(H+\lambda)t}\,d_y\,d_\lambda\sigma_y TE_\lambda,$$

$$\sigma_y = \int_0^y e^{-\xi t}\,dt,$$

where H is some self-adjoint operator in $\mathscr{H}$ and $\varepsilon > 0$. According to Theorem 5.1 and Proposition 5.4, if $TE_\xi g$ is of bounded norm-variation for all $g \in \mathscr{H}$, if $B(x)$ is continuous in the strong topology of $\mathfrak{B}(\mathscr{H})$ and if $\Phi(\xi, x)$ is a continuous function (and if $\|B(x)\|$ is bounded in case $\Delta^{(x)}$ is infinite), then (5.5) exists.

PROPOSITION 5.5. *Suppose the operator-valued function $B(x) \in \mathfrak{B}(\mathscr{H})$ is such that $B(x)g$ is Bochner integrable and $\|B(x)g\|^2$ is μ-integrable on $\Delta^{(x)}$. If $\Delta^{(x)}$ is of finite measure μ and $\Phi(\xi, x)$ is continuous and bounded on $\Delta^{(x)} \times \Delta^{(\xi)}$, then*

$$\int_{\Delta^{(x)}\times\Delta^{(\xi)}} \Phi(x,\xi)\,d_y\,d_\xi\left\{\int^y B(x)\,d\mu(x)\right\}TE_\xi \tag{5.6}$$

exists; furthermore, if $B(x)$ is uniformly strongly continuous on any finite closed subinterval of Δ, and $TE_\xi g$ is of bounded norm-variation for all $g \in \mathscr{H}$, then (5.5) *also exists and is equal to* (5.6).

Proof. For any $f, g \in \mathscr{H}$, the function

$$\left\langle f \,\middle|\, \int^y B(x)\,d\mu(x)TE_\xi g\right\rangle$$

is of bounded norm-variation. This easily follows from the inequality

$$\begin{aligned}\left|\left\langle f\,\middle|\int_{\Delta_1} B(x)\,d\mu(x)TE(\Delta_2)g\right\rangle\right| &\leqq \int_{\Delta_1}\left|\left\langle f\middle|B(x)E(\Delta_2)g\right\rangle\right| d\mu(x)\\ &\leqq \frac{1}{2}\int_{\Delta_1}\{\langle B(x)f + g|TE(\Delta_2)(B(x)f+g)\rangle\\ &\quad+\langle B(x)f + ig|TE(\Delta_2)(B(x)f+g)\rangle\\ &\quad+\sqrt{2}\langle B(x)f|TE(\Delta_2)B(x)f\rangle\\ &\quad+\sqrt{2}\langle g|TE(\Delta_2)g\rangle\}\,d\mu(x)\end{aligned}$$

which holds for any $\Delta_1 \subset \Delta^{(x)}$ and $\Delta_2 \subset \Delta^{(\xi)}$; the assumptions on $B(x)$ insure the existence of the above integrals. Hence, by invoking Theorem 3.1 and taking into consideration that for the complex function $\Phi(x, \xi)$ continuity on a closed finite interval implies uniform continuity on that interval, we conclude that

$$\int_{\Delta^{(x)}\times\Delta^{(\xi)}} \Phi(x,\xi)\,d_y\,d_\xi\left\langle f\middle|\int^y B(x)\,d\mu(x)TE_\xi g\right\rangle$$

$$\cdot\left\langle f\middle|\int_{\Delta^{(x)}\times\Delta^{(\xi)}}\Phi(x,\xi)\,d_y\,d_\xi\left\{\int^y B(x)\,d\mu(x)\right\}TE_\xi g\right\rangle$$

exists.

For the difference in the Riemann–Stieltjes sums for (5.5) and (5.6) we have the following estimate:

$$\left|\left\langle f \middle| \sum_{\mathbf{k}} \sum_{\mathbf{l}} \Phi(x'_{\mathbf{k}}, \xi'_{\mathbf{l}}) \left\{ B(x'_{\mathbf{k}})\mu(\Delta_{\mathbf{k}}^{(x)})E(\Delta_{\mathbf{l}}^{(\xi)}) - \left[\int_{\Delta_{\mathbf{k}}^{(x)}} B(x)\, d\mu(x)\right] TE(\Delta_{\mathbf{l}}^{(\xi)}) \right\} g \right\rangle\right|$$
$$\leqq C \left\{ \sum_{\mathbf{k}} \int_{\Delta_{\mathbf{k}}^{(x)}} \|(B(x'_k) - B(x)f\|\, d\mu(x) \right\} \left\{ \sum_{\mathbf{l}} \| TE(\Delta_{\mathbf{l}}^{(\xi)}) \right\} \|,$$

provided that $|\Phi(x, \xi) \leqq C$ for all $x \in \Delta^{(x)}$ and $\xi \in \Delta^{(\xi)}$. The equality of (5.5) and (5.6) is a consequence of the fact that due to the assumptions made in the statement of the proposition, the above difference can be made arbitrarily small for sufficiently fine subdivisions of $\Delta^{(x)} \times \Delta^{(\xi)}$. The proof of the proposition is complete.

Let us introduce the weak counterpart of (4.1):

$$\partial_\lambda A(\lambda, \cdots, \omega) = \operatorname*{w-lim}_{\Delta\lambda \to 0} \frac{1}{\Delta\lambda}[A(\lambda + \Delta\lambda, \cdots, \omega) - A(\lambda, \cdots, \omega)].$$

If $\partial_\lambda A$ is a bounded operator, we shall denote by $\partial_\lambda A'$ its conjugate in $\mathscr{X}'$:

$$(\partial_\lambda A'\phi)(f) = \phi(\partial_\lambda Af) = \lim_{\Delta\lambda \to 0} \frac{1}{\Delta\lambda}[\phi(A(\lambda + \Delta\lambda, \cdots, \omega)) - \phi(A(\lambda, \cdots, \omega))].$$

Now, in analogy with Lemma 4.1 and Theorems 4.1–4.3, we can state the following lemma.

LEMMA 5.2. *If for some $\phi \in \mathscr{X}'$ the vector-valued function $\partial_\lambda B'(\lambda)\phi$ is Bochner integrable on $[a, b]$ with respect to Lebesgue measure, then*

$$\left| \sum_{k=1}^{n} \phi(B(\lambda'_k)(f_{\lambda_k} - f_{\lambda_{k-1}})) \right| \leqq |\phi(B(\lambda'_n)f_b)| + |\phi(B(\lambda'_0)f_a)|$$
$$+ \sup_{a \leqq \lambda \leqq b} \|f_\lambda\| \int_a^b \|\partial_{\lambda'} B'(\lambda')\phi\|\, d\lambda'$$

for that $\phi \in \mathscr{X}'$.

THEOREM 5.2. *Suppose that for every $\phi \in \mathscr{X}'$ the vector-valued function $\partial_\lambda A'(\lambda)\phi$ is Bochner integrable on $[a, b]$, with respect to Lesbesgue measure and that*

$$\int_a^b X_S(\lambda)\, d_\lambda f_\lambda$$

exists for every measurable set $S \subset [a, b]$. Then $A(\lambda)$ is weakly integrable with respect to f_λ and

$$\left|\phi\left(\int_a^b A(\lambda)\, d_\lambda f_\lambda\right)\right| \leqq |\phi(A(a)f_a)| + |\phi(A(b)f_b)|$$
$$+ \left\{\sup_{a \leqq \lambda \leqq b} \|f_\lambda\|\right\} \int_a^b \|\partial_\lambda A'(\lambda)\phi\|\, d\lambda \tag{5.7}$$

for all $\phi \in \mathscr{X}'$.

THEOREM 5.3. *Suppose the conditions imposed in Theorem 5.2 on $\partial A'(\lambda)$ and f_λ are satisfied on any compact interval in $\mathbb{R}^1$ and that* w-lim $A(\lambda)f_\lambda = \mathbf{0}$ *for* $\lambda \to \pm\infty$. *If* $\|f_\lambda\| \leqq$ const. *for* $\lambda \in \mathbb{R}^1$ *and if*

$$\int_{-\infty}^{+\infty} \|\partial_{\lambda'} A'(\lambda')\phi\| \, d\lambda' < +\infty$$

for all $\phi \in \mathscr{X}'$, then the improper weak Riemann–Stieltjes integral of $A(\lambda)$ with respect to f_λ exists and

$$\left|\phi\left(\int_{-\infty}^{+\infty} A(\lambda) d_\lambda f_\lambda\right)\right| \leqq \sup_{\lambda\in\mathbb{R}^1} \|f_\lambda\| \int_{-\infty}^{+\infty} \|\partial_\lambda A'(\lambda)\phi\| \, d\lambda.$$

Let us recall that when $\mathscr{X}$ is a Hilbert space $\mathscr{H}$, then for every $\phi \in \mathscr{X}'$ we can find a $g \in \mathscr{H}$ such that $\phi(\cdot) = \langle g|\cdot\rangle$, and that the conjugate A' of $A \in \mathfrak{B}(\mathscr{H})$ is identical to the adjoint A^* of A. Hence, the counterpart of Theorem 4.3 is easily seen to be the following.

THEOREM 5.4. *If $\partial_\lambda A^*(\lambda)g$ is Bochner integrable on $\mathbb{R}^1$ with respect to Lebesgue measure for all $g \in \mathscr{H}$, then for any given $T \in \mathfrak{B}(\mathscr{H})$ the operator-valued function $A(\lambda)T$ is weakly integrable on $\mathbb{R}^1$ with respect to any spectral measure E_λ and*

$$\left|\left\langle g \middle| \int_{-\infty}^{+\infty} A(\lambda) T \, d_\lambda E_\lambda f \right\rangle\right| \leqq \|f\| \int_{-\infty}^{+\infty} \|T^* \partial_\lambda A^*(\lambda) g\| \, d\lambda$$

for all $f, g \in \mathscr{H}$.

The proofs of the above lemma and theorems can be easily obtained from the proofs of Lemma 4.1 and Theorems 4.1–4.3, respectively, by minor modifications of those proofs; these modifications consist mainly in replacing estimates for $\|\cdot\|$ by estimates on $|\phi(\cdot)|$ (with ϕ varying over $\mathscr{X}'$) and in substituting weak limits for strong limits.

Theorem 5.4 can be used to establish the existence of the weak integral

$$\int_{-\infty}^{+\infty} \Phi(\lambda, H) \, d_\lambda E_\lambda \tag{5.8}$$

in cases where the existence of its strong counterpart (4.8) cannot be established by using Theorem 4.3 and when

$$\int_{-\infty}^{+\infty} \|\partial_\lambda \Phi^*(\lambda, H) g\| \, d\lambda < +\infty \tag{5.9}$$

for all $g \in \mathscr{H}$. However, when $\Phi(\lambda, H)$ is given by (4.9), and therefore

$$\partial_\lambda \Phi^*(\lambda, H) = -(\lambda - H - i\eta)^{-2},$$

the convergence of (5.9) can be readily verified only for a dense set of vectors g, rather than for all $g \in \mathscr{H}$. For example, if $F(\Delta)$ is the spectral measure of H, then

$$\left\|\frac{F((k, k+1])}{(\lambda - H - i\eta)^2} g\right\|^2 = \int_k^{k+1} \frac{1}{|\lambda - \mu - i\eta|^4} d_\mu \|F_\mu g\|^2 \leqq [\psi(\lambda, k, \eta)]^2 \|F((k, k+1])g\|^2$$

$$\psi(\lambda, k, \eta) = \begin{cases} \dfrac{1}{(\lambda - k)^2 + \eta^2} & \text{for } \lambda \leqq k, \\ 1/\eta^2 & \text{for } k < \lambda \leqq k + 1, \\ \dfrac{1}{(\lambda - k - 1)^2 + \eta^2} & \text{for } \lambda > k + 1, \end{cases} \tag{5.10}$$

and consequently,

$$\begin{aligned} \int_{-\infty}^{+\infty} \|\partial_\lambda \Phi^*(\lambda, H)g\| \, d\lambda &\leqq \sum_{k=-\infty}^{+\infty} \|F((k, k+1])g\| \int_{-\infty}^{+\infty} \psi(\lambda, k, \eta) \, d\lambda \\ &\leqq \frac{1 + \pi\eta}{\eta^2} \sum_{k=-\infty}^{+\infty} \|F((k, k+1])g\|. \end{aligned}$$

The series on the right-hand side of the above relation converges only for a dense set of vectors $g \in \mathscr{H}$. To establish the existence of (5.8), we need the following result.

PROPOSITION 5.6. *Suppose that for all* $g \in \mathscr{H}$ *the function* $\partial_\lambda A^*(\lambda)g$ *is Bochner integrable with respect to Lebesgue measure on any compact interval in* $\mathbb{R}^1$. *Let* $\cdots < t_{k-1} < t_k < t_{k+1} < \cdots$ *be a sequence which is such that*

$$\sum_{k=-\infty}^{+\infty} \left\{ \left| \left\langle g \middle| A(t_k)(E_{t_{k+1}} - E_{t_k})f \right\rangle \right| + \|(E_{t_{k+1}} - E_{t_k})f\| \int_{t_k}^{t_{k+1}} \|\partial_\lambda A^*(\lambda)g\| \, d\lambda \right\} \tag{5.11}$$

converges for all $f, g \in \mathscr{H}$. *Then* $A(\lambda)$ *is weakly integrable on* $\mathbb{R}^1$ *with respect to* E_λ *and*

$$\left| \left\langle g \middle| \int_{-\infty}^{+\infty} A(\lambda) \, d_\lambda E_\lambda f \right\rangle \right|$$

does not exceed in value (5.11).

Proof. The proof follows as an easy consequence from the observation that

$$\int_{t_{k-1}}^{t_k} A(\lambda) \, d_\lambda E_\lambda = \int_{t_{k-1}}^{t_k} A(\lambda) \, d_\lambda (E_\lambda - E_{t_{k-1}}).$$

In fact, according to (5.7) the convergence of (5.10) implies that

$$\sum_{k=-\infty}^{+\infty} \left| \left\langle g \middle| \int_{t_{k-1}}^{t_k} A(\lambda) \, d_\lambda E_\lambda f \right\rangle \right|$$

converges for all $f, g \in \mathscr{H}$. This establishes the existence of

$$\operatorname*{w-lim}_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b A(\lambda) \, d_\lambda E_\lambda,$$

i.e., of the improper weak Riemann–Stieltjes integral of $A(\lambda)$ with respect to E_λ.

Let us apply the above proposition to investigating the existence of (5.8) with $\Phi(\lambda, H)$ given by (4.9). Let us take $t_k = k$. Using the estimate

$$\|\Phi^*(k, H)g\|^2 \leqq 2 \sum_{m=0}^{+\infty} \frac{1}{(|k| - m)^2 + \eta^2} \|F((m, m+1])g\|^2,$$

we obtain

$$\sum_{k=-\infty}^{+\infty} \left|\left\langle \Phi^*(k,H)g|E((k,k+1])f\right\rangle\right| \leqq \sum_{k=-\infty}^{+\infty} \|\Phi^*(k,H)g\| \; \|E((k,k+1])f\|$$

$$\leqq \left[\sum_{k=-\infty}^{+\infty} \|\Phi^*(k,H)g\|^2\right]^{1/2}\left[\sum_{k=-\infty}^{+\infty} \|E((k,k+1])f\|^2\right]^{1/2}$$

$$\leqq \sqrt{2}\|f\|\left[\sum_{k=-\infty}^{+\infty}\sum_{m=0}^{+\infty} \frac{1}{(|k|-m)^2+\eta^2}\|F((m,m+1])g\|^2\right]^{1/2}$$

$$\leqq \sqrt{2}\|f\|\left[\sum_{m=0}^{+\infty} \|F((m,m+1])g\|^2 \sum_{k=-\infty}^{+\infty} \frac{1}{(|k|-m)^2+\eta^2}\right]^{1/2}$$

$$\leqq 2\|f\| \; \|g\| \sum_{k=0}^{+\infty} \frac{1}{k^2+\eta^2}.$$

In a similar manner, by using (5.10) we have

$$\sum_{k=-\infty}^{+\infty} \|E((k,k+1])f\| \int_k^{k+1} \|(\lambda - H - i\eta)^{-2}g\| \, d\lambda$$

$$\leqq \sum_{k=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} \|E((k,k+1])f\| \; \|F((m,m+1])g\| \int_k^{k+1} \psi(\lambda,m,\eta)\, d\lambda$$

$$\leqq \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} \|F((m,m+1])g\|$$

$$\cdot \|E((m+n,m+n+1])f\| \sup_{m+n \leqq \lambda \leqq m+n+1} \psi(\lambda,m,\eta)$$

$$\leqq \sum_{n=-\infty}^{+\infty} \sup_{m+n\leqq\lambda\leqq m+n+1} \psi(\lambda,m,\eta) \sum_{m=-\infty}^{+\infty} \|F((m-1,m])g\|$$

$$\cdot \|E((m+n,m+n+1])f\|$$

$$\leqq 2\|f\| \; \|g\| \sum_{k=0}^{+\infty} \frac{1}{k^2+\eta^2}.$$

Thus, in the present case, (5.11) converges and consequently the existence of (5.8) with the integrand (4.9) is established.

6. Cross-iterated Riemann–Stieltjes integrals. In [1] we have encountered integrals of a more general nature than weak or strong Riemann–Stieltjes integrals. These integrals are of the general form

$$I_c = \int_{\Delta^{(1)}} A(\lambda, \cdots, \rho)\, d_\lambda \cdots d_\rho \int_{\Delta^{(2)}} B(\lambda, \cdots, \rho, \sigma, \cdots, \omega)\, d_\sigma \cdots d_\omega f_{\lambda\cdots\rho\sigma\cdots\omega}. \tag{6.1}$$

A more specific example is provided by (1.3).

In order to avoid too many technicalities in defining (6.1), let us assume that $A(\lambda, \cdots, \rho) \in \mathfrak{B}(\mathscr{X})$ for all $(\lambda, \cdots, \rho) \in \Delta^{(1)}$. If $\Delta^{(1)}$ is a finite closed interval of the form

$$\Delta^{(1)} = [a_1, b_1] \times \cdots \times [a_s, b_s]$$

and a subdivision Γ of $\Delta^{(1)}$ is given,

$$a_1 = \lambda_0 < \lambda_1 < \cdots < \lambda_K = b_1, \quad \cdots, \quad a_s = \rho_0 < \rho_1 < \cdots < \rho_N = b_s,$$

then we define Riemann–Stieltjes sums for (6.1) as follows:

$$Q_c(\Gamma) = \sum_{k=1}^{K} \cdots \sum_{n=1}^{N} A(\lambda'_k, \cdots, \rho'_n) \int_{\Delta^{(2)}} B(\lambda'_k, \cdots, \rho'_n, \sigma, \cdots, \omega)\, d_\sigma \cdots d_\omega f_{\cdots\sigma\cdots\omega}(\Delta_{k\cdots n}),$$

(6.2)
$$\lambda_{k-1} \leqq \lambda'_k \leqq \lambda_k, \quad \cdots, \quad \rho_{n-1} \leqq \rho'_n \leqq \rho_n;$$

here, for $\tilde{\Delta} = (\alpha_1, \beta_1] \times \cdots \times (\alpha_s, \beta_s]$ we have introduced

$$f_{\cdots\sigma\cdots\omega}(\tilde{\Delta}) = \sum_{\gamma_1 \in \{\lambda_k, \lambda_{k+1}\}} \cdots \sum_{\gamma_s \in \{\rho_n, \rho_{n+1}\}} (-1)^{Z(\alpha)} f_{\gamma_1 \cdots \gamma_s \sigma \cdots \omega}$$

in analogy to (2.5). The *cross-iterated Riemann–Stieltjes integral* I_c over $\Delta^{(1)}$ is then defined by

$$I_c = \operatorname*{w-lim}_{\delta \to 0} Q_c(\Gamma)$$

in the limit of finer and finer subdivisions Γ of $\Delta^{(1)}$.

If $\Delta^{(1)}$ is infinite, then we define the improper cross-iterated Riemann–Stieltjes integral over $\Delta^{(1)}$ in the expected manner by (in abbreviated notation)

$$\text{(6.3)} \qquad \int_{\Delta^{(1)}} A\, d \int_{\Delta^{(2)}} B\, df = \operatorname*{w-lim}_{\Delta_0^{(1)} \to \Delta^{(1)}} \int_{\Delta_0^{(1)}} A\, d \int B\, df$$

in the limit of monotonic sequences of sets $\Delta_0^{(1)}$ which cover $\Delta^{(1)}$.

It is worthwhile noting that a strong version of (6.1) can be defined:

$$\int_{\Delta^{(1)}} A(\lambda, \cdots, \rho) \left\{ (d\lambda \cdots d\rho) \int_{\Delta^{(2)}} B(\lambda, \cdots, \rho, \sigma, \cdots, \omega)\, d_\sigma \cdots d_\omega f_{\lambda \cdots \rho\sigma \cdots \omega} \right\}$$

(6.4)
$$= \operatorname*{s-lim}_{\delta \to 0} Q_c(\Gamma).$$

Furthermore, by replacing the weak second integral in (6.1) and (6.4) by its strong version, we can also define the integrals

$$\text{(6.5)} \quad \int_{\Delta^{(1)}} A(\lambda, \cdots, \rho)\, d_\lambda \cdots d_\rho \int_{\Delta^{(2)}} B(\lambda, \cdots, \rho, \sigma, \cdots, \omega) f_{\lambda \cdots \rho \cdots}(d\sigma \cdots d\omega),$$

$$\text{(6.6)} \quad \int_{\Delta^{(1)}} A(\lambda, \cdots, \rho)(d\lambda \cdots d\rho) \int_{\Delta^{(2)}} B(\lambda, \cdots, \rho, \sigma, \cdots, \omega) f_{\lambda \cdots \rho \cdots}(d\sigma \cdots d\omega)$$

in the obvious manner.

When $\mathscr{X} = \mathbb{C}^1$ and $f_{\lambda \cdots \omega}$ is a complex function of the form $u_{\lambda \cdots \rho} v_{\sigma \cdots \omega}$, then it is easily seen that (6.1) assumes the form of an ordinary iterated Riemann–Stieltjes integral:

$$\int_{\Delta^{(1)}} A(\lambda, \cdots, \rho)\, d_\lambda \cdots d_\rho u_{\lambda \cdots \rho} \int B(\lambda, \cdots, \omega)\, d_\rho \cdots d_\omega v_{\sigma \cdots \omega}.$$

It is easy to see that the above definitions imply that

$$\int A(\lambda, \cdots, \rho)\, d_\lambda \cdots d_\rho \int B(\lambda, \cdots, \omega)\, d_\sigma \cdots d_\omega f_{\lambda\cdots\omega}$$
$$= \int d_\lambda \cdots d_\rho \int A(\lambda, \cdots, \rho) B(\lambda, \cdots, \omega)\, d_{\sigma\cdots\omega} f_{\lambda\cdots\omega}$$

with analogous relations holding for (6.4)–(6.6). Hence, we can state that

$$\int d_\lambda \cdots d_\rho \int D(\lambda, \cdots, \omega)\, d_\sigma \cdots d_\omega f_{\lambda\cdots\omega}$$

is of the same degree of generality as (6.1).

All the basic properties of weak and strong Riemann–Stieltjes integrals, which are stated in Propositions 2.1–2.4 and 5.1–5.5, are preserved by the cross-iterated Riemann–Stieltjes integrals.

PROPOSITION 6.1. *For* $B_1, B_1 \in \mathcal{N}(\mathscr{X})$,

$$\int d_x \int [B_1 D_1(x, y) + B_2 D_2(x, y)]\, d_y f_{xy}$$
$$= B_1 \int d_x \int D_1(x, y)\, d_y f_{xy} + B_2 \int d_x \int D_2(x, y)\, d_y f_{xy}$$

and the existence of the integrals on the right-hand side imply the existence of the integrals on the left-hand side.

PROPOSITION 6.2. *For closed intervals* $\Delta_1, \Delta_2, \Delta = \Delta_1 \cup \Delta_2$ *such that* $\Delta_1 \cap \Delta_2$ *contains only boundary points of* Δ_1 *and* Δ_2,

$$\int_\Delta d_x \int D(x, y)\, d_y f_{xy} = \int_{\Delta_1} d_x \int D(x, y)\, d_y f_{xy} + \int_{\Delta_2} d_x \int D(x, y)\, d_y f_{xy},$$

$$\int d_x \int_\Delta D(x, y)\, d_y f_{xy} = \int d_x \int_{\Delta_1} D(x, y)\, d_y f_{xy} + \int d_x \int_{\Delta_2} D(x, y)\, d_y f_{xy}$$

provided that the above integrals exist.

PROPOSITION 6.3. *If* $G_{xy} \in \mathfrak{B}(\mathscr{X})$, *then the mapping taking* $f \in \mathscr{X}$ *into*

$$Wf = \int d_x \int D(x, y)\, d_y G_{xy} f,$$

whenever the above integral exists, is a linear operator.

Naturally, we expect that some cross-iterated integrals can be written as weak Riemann–Stieltjes integrals. For example, it is easily seen that the integral in (5.5) is equal to

$$\int_{\Delta^{(x)}} B(y)\, d_y \int_{\Delta^{(\xi)}} \Phi(\xi, y) T \sigma_y\, d_\xi E_\xi.$$

PROPOSITION 6.4. *Suppose* $D'(x, y) \in \mathfrak{B}(\mathscr{X}')$ *is strongly continuous on* $\Delta^{(x)} \times \Delta^{(y)}$, *and that in case* $\Delta^{(x)} \times \Delta^{(y)}$ *is an infinite interval,* $\|\phi D'(x, y)\| \leqq C_\phi$ *for all* $(x, y) \in \Delta^{(x)}$

$\times\ \Delta^{(y)}$ *and any* $\phi \in \mathscr{X}'$. *If* f_{xy} *is of bounded norm-variation on* $\Delta^{(x)} \times \Delta^{(y)}$, *then*

$$\int_{\Delta^{(x)}} d_x \int_{\Delta^{(y)}} D(x, y)\, d_y f_{xy} = \int_{\Delta^{(x)} \times D^{(y)}} D(x, y)\, d_x\, d_y f_{xy} \tag{6.7}$$

and both of the above integrals exist.

Proof. The integral on the right-hand side of (6.7) exists by Theorem 5.1.

If $\Delta^{(x)} \times \Delta^{(y)}$ is finite and $Q_c(\Gamma)$ and $Q(\Gamma)$ denote the respective Riemann–Stieltjes sums of the first and second integrals in (6.7), then by using Lemma 5.1 we easily obtain that

$$\begin{aligned} |\phi(Q_c(\Gamma) - Q(\Gamma))| &\leqq \sum_{\mathbf{k}} \sum_{\mathbf{l}} \left| \phi\left(\left(\int_{\Delta_{\mathbf{l}}} D(x'_{\mathbf{k}}, y)\, d_y f_{\cdot y}(\Delta_{\mathbf{k}}) - \sum_{\mathbf{l}} D(x'_{\mathbf{k}}, y'_{\mathbf{l}}) f(\Delta_{\mathbf{k}} \times \Delta_{\mathbf{l}}) \right) \right) \right| \\ &\leqq \sum_{\mathbf{k}} \sum_{\mathbf{l}} \sup_{y \in \Delta_{\mathbf{l}}} \|D'(x'_{\mathbf{k}}, y) - D'(x'_{\mathbf{k}}, y')\phi\|\, \|f(\Delta_{\mathbf{k}} \times \Delta_{\mathbf{l}})\|. \end{aligned}$$

Since $D'(x'_{\mathbf{k}}, y)$ is continuous, we can find for every $\varepsilon > 0$ some $\delta(\varepsilon)$ such that for any subdivision Γ of norm smaller than $\delta(\varepsilon)$,

$$\sup_{y \in \Delta_{\mathbf{l}}} \|D'(x'_{\mathbf{k}}, y) - D'(x'_{\mathbf{k}}, y'_{\mathbf{l}})\| < \varepsilon$$

for all **k** and **l**. Hence

$$|\phi(Q_c(\Gamma) - Q(\Gamma))| < \varepsilon \sum_{\mathbf{k}} \sum_{\mathbf{l}} \|f(\Delta_{\mathbf{k}} \times \Delta_{\mathbf{l}})\| \leqq \varepsilon V_\Delta[f_{xy}].$$

This establishes (6.7) as well as the existence of the cross-iterated integral in (6.7).

In the case $\Delta^{(y)}$ is infinite and $\Delta_n^{(y)} \subset \Delta^{(y)}$ is finite, we can always write the closure of $\Delta^{(y)} - \Delta_n^{(y)}$ as a union of a finite number of intervals with disjoint interiors (naturally, we always assume that the faces of the considered intervals are parallel to the coordinate planes). For the respective Riemann–Stieltjes sums $Q_c(\Gamma)$ and $Q_c^{(n)}(\Gamma)$ we easily obtain by using Lemma 5.1:

$$|\phi(Q_c(\Gamma) - Q_c^{(n)}(\Gamma))| \leqq C_\phi(V_{\Delta^{(y)}}[f_{xy}] - V_{\Delta_n^{(y)}}[f_{xy}]).$$

Hence, we deduce the existence of the cross-iterated integral over $\Delta^{(x)} \times \Delta^{(y)}$. Furthermore, since (6.7) has been established for $\Delta^{(x)} \times \Delta_n^{(y)}$, it can be extended immediately to $\Delta^{(x)} \times \Delta^{(y)}$ by letting $\Delta_n^{(y)} \to \Delta^{(y)}$.

The case when $\Delta^{(x)}$ is also infinite can be treated in a similar manner.

Let us consider two spectral functions E_λ and F_μ corresponding to the bounded self-adjoint operators A and B on $\mathscr{H}$. If $\Phi(A, \mu)$ is weakly integrable with respect to TF_μ, $T \in \mathfrak{B}(\mathscr{H})$, then by using the spectral theorem for self-adjoint operators it can be immediately deduced from the definition of cross-iterated integrals that

$$\int_{-\infty}^{+\infty} \Phi(A, \mu)\, d_\mu TF_\mu = \int_{-\infty}^{+\infty} d_\mu \int_{-\infty}^{+\infty} \Phi(\lambda, \mu)\, d_\lambda E_\lambda TF_\mu,$$

For $\Phi(\lambda, \mu) = \lambda^m \mu^n$ we have $\Phi(A, \mu) = A^m \mu^n$ and consequently,

$$A^m T B^n = \int_{-\infty}^{+\infty} A^m \mu^n\, d_\mu TF_\mu = \int_{-\infty}^{+\infty} d_\mu \int_{-\infty} \lambda^m \mu^n\, d_\lambda E_\lambda TF_\mu.$$

If $\sum_{m,n} D_{mn}\lambda^m\mu^n$ is a polynomial with operator-valued coefficients D_{mn}, then by applying Proposition 6.2 we obtain

$$\sum_{m,n} D_{mn}A^mTB^n = \int_{-\infty}^{+\infty} d_\mu \int_{-\infty}^{+\infty} \left(\sum_{m,n} D_{mn}\lambda^m\mu^n\right) d_\lambda E_\lambda TF_\mu.$$

REFERENCES

[1] E. PRUGOVEČKI, *Rigorous derivation of generalized Lippmann–Schwinger equations from time-dependent scattering theory*, Nuovo Cimento, 63 B (1969), pp. 569–592.

[2] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.

[3] K. O. FRIEDRICHS, *Perturbation of Spectra in Hilbert Space*, American Mathematical Society, Providence, R.I., 1965.

[4] E. PRUGOVEČKI, *Quantum Mechanics in Hilbert Space*, Academic Press, New York, 1971.

[5] YU. L. DALETSKII AND S. G. KREIN, *The integration and differentiation of functions of Hermitian operators and application to perburbation theory*, Voronezh. Gos. Univ. Trudy Sem. Funktional Anal., 1956, No. 1, pp. 81–105.

[6] M. SH. BIRMAN, *On the existence of wave operators*, Izv. Akad. Nauk SSSR Ser. Mat., 24 (1963), no. 4, pp. 883–906.

[7] ———, *Local criterion of the existence of wave operators*, Dokl. Akad. Nauk SSSR, 159 (1964), no. 3, pp. 445–488.

[8] M. SH. BIRMAN AND M. Z. SOLOMYAK, *Stieltjes Double-integral operators, I*, Topics in Mathematical Physics, vol. 1, Consultants Bureau, New York, 1967, pp. 25–54.

[9] ———, *Stieltjes double-integral operators, II*, Topics in Mathematical Physics, vol. 2, Consultants Bureau, New York, 1968, pp. 19–46.

[10] M. Z. SOLOMYAK, *Double-integral operators in the ring $\hat{R}$*, Topics in Mathematical Physics, vol. 3, Consultants Bureau, New York, 1969, pp. 79–91.

[11] E. PRUGOVEČKI, *Integral representation of wave and transition operators in non-relativistic scattering theory*, Nuovo Cimento, 4B (1971), pp. 124–134.

[12] E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semi-groups*, American Mathematical Society, Providence, R.I., 1957.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

THE NORM CONTINUITY PROPERTIES OF SQUARE ROOTS*

RICHARD BOULDIN†

Abstract. This paper is concerned with linear operators on a complex Hilbert space. There is no continuous square root function defined on the set of all normal operators. However, this paper defines a square root function on each of several large sets of operators and establishes the continuity of each function in the operator norm.

1. Introduction. The purpose of this note is to show that the square root as a mapping on certain sets of operators is continuous in the operator norm. By "operator" we shall mean a bounded linear map of the Hilbert space H into itself and the only topology on the operators which will concern us is the metric topology induced by the operator norm.

Many difficulties arise from the ambiguity of the phrase "square root." Some operators do not have any square roots, in other words, there is no operator B such that $B^2 = A$ (see [1], [4], [5], [9]), and other operators have an infinite number of different square roots. In order to confirm this last assertion let A be a normal operator with an infinite number of reducing subspaces, say $\{H_i : i \in I\}$; then assuming that $A^{1/2}$ is one square root of A defined by the operational calculus for normal operators, we see that $-(A^{1/2}/H_i) \oplus A^{1/2}/H_i^{\perp}$ is a new square root. Clearly, this process gives rise to a different square root for each $i \in I$.

In view of these difficulties one must restrict to a set of operators with the property that each has a square root and one must derive the square roots of different operators in that set by the same clearly defined process. Although we are able to prove the continuity of the square root for enough large sets of operators that most practical situations are handled, there is no definition of square root which is continuous on the set of all normal operators.

2. Theorems showing continuity. For z restricted to an open subset of the complex plane that fails to intersect some branch cut, for example, a ray which begins at the origin, there is a definition of $z^{1/2}$ which is analytic on the given set. If one such branch cut intersects the unit circle in the point $e^{i\theta}$, then the appropriate definition of $z^{1/2}$ is the following: $z^{1/2} = |z|^{1/2} e^{i\alpha/2}$, where $z = |z| e^{i\alpha}$ and $\alpha \in (\theta - 2\pi, \theta]$. Motivated by these basic facts from complex variables, we define $S(\theta)$ to be the set of operators A with the spectrum of each, $\sigma(A)$, contained in the complement of $\{r e^{i\theta} : r \in [0, \infty)\}$ with respect to the complex plane. For $A \in S(\theta)$ we define $A^{1/2}$ by the following:

$$A^{1/2} = \frac{-1}{2\pi i} \int_c z^{1/2} R_A(z)\, dz,$$

where $z^{1/2}$ has the definition given above, $R_A(z)$ is the resolvent operator of A at z, and the integral has its usual meaning in the Taylor–Dunford operational calculus.

* Received by the editors March 30, 1971.

† Department of Mathematics, University of Georgia, Athens, Georgia 30601.

By the spectral mapping theorem for analytic functions we see that $A^{1/2} \in S(\theta)$ and thus we have defined a square root map from $S(\theta)$ into $S(\theta)$. We call this the *principal square root for* $S(\theta)$.

THEOREM 1. *Assume that* $\sigma(A) \cup \sigma(B)$ *does not intersect* $\{r\,e^{i\theta}: r \in [0, \infty)\}$. *For any* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $\|A^{1/2} - B^{1/2}\| < \varepsilon$ *provided* $\|A - B\| < \delta$.

Proof. Using the second resolvent equation we note that

$$\begin{aligned} A^{1/2} - B^{1/2} &= \frac{-1}{2\pi i}\int_c z^{1/2}R_A(z)\,dz - \frac{-1}{2\pi i}\int_c z^{1/2}R_B(z)\,dz \\ &= \frac{-1}{2\pi i}\int_c z^{1/2}[R_A(z) - R_B(z)]\,dz \\ &= \frac{-1}{2\pi i}\int_c z^{1/2}(R_A(z)[B - A]R_B(z))\,dz. \end{aligned}$$

Choose an appropriate parameterization so that $c = \{z(t): t \in [0, 1]\}$ and observe the following inequality:

$$\begin{aligned} &\left\| \int_0^1 z(t)^{1/2}R_A(z(t))[B - A]R_B(z(t))\,dt \right\| \\ &\leqq \int_0^1 \|z(t)^{1/2}R_A(z(t))[B - A]R_B(z(t))\|\,dt \\ &\leqq \|B - A\| \sup_{z\in c} \{|z|^{1/2}\|R_A(z)\|\,\|R_B(z)\|\}. \end{aligned}$$

The first inequality uses the convergence of the integral in the norm topology; see [6]. Since c is compact and all three of the functions $|z|^{1/2}$, $\|R_A(z)\|$, and $\|R_B(z)\|$ are continuous in a neighborhood of c, it follows that the above supremum is finite. Thus the theorem is proved.

COROLLARY 1. *If the sequence of operators* $\{A_k: k = 1, 2, \cdots\}$ *converges in the operator norm to* $A_0 \in S(\theta)$, *then for all* k *sufficiently large, say* $k \geqq p$, *we have* $A_k \in S(\theta)$ *and* $\{A_k^{1/2}: k = p, p + 1, \cdots\}$ *converges in the operator norm to* $A_0^{1/2}$.

Proof. Use Theorem 1, the upper semicontinuity of the spectrum [36, p. 86], and the norm continuity of inverses [3, p. 245], to draw the above conclusion.

If A is a normal operator with spectral measure $E_A(\cdot)$, then regardless of where $\sigma(A)$ is located we can define $A^{1/2}$ to be $\int z^{1/2}\,dE_A(z)$, where $z^{1/2}$ has branch cut $\{r\,e^{i\theta}: r \in [0, \infty)\}$. Since $z^{1/2}$ is continuous on the complement of $\{r\,e^{i\theta}: r \in [0, \infty)\}$ in the complex plane, the Hilbert space operational calculus is certainly applicable. Moreover, it follows from 15 Corollary [2, p. 879] that if A is a member of some $S(\theta)$ then the preceding definition of $A^{1/2}$ agrees with the $A^{1/2}$ determined by the principal square root for $S(\theta)$. In the following, $A^{1/2}$ is defined using $E_A(\cdot)$ and $(-\infty, 0]$ as the branch cut for $z^{1/2}$.

LEMMA 1. *If* A *is a normal operator with* $\sigma(A) \subset \{z: \operatorname{Re} z \geqq 0\}$, *then* $\|(A + \varepsilon)^{1/2} - A^{1/2}\| < \varepsilon^{1/2}$ *for any* $\varepsilon > 0$.

Proof. It is trivial to see that $\operatorname{Re}(z^{1/2}) \geqq (\operatorname{Re} z)^{1/2}$ for z such that $\operatorname{Re} z \geqq 0$. Consequently $\inf\{\operatorname{Re}([z + \varepsilon]^{1/2}): z \in \sigma(A)\} \geqq \varepsilon^{1/2}$ or $\inf\{\operatorname{Re}(z^{1/2}): z \in \sigma(A + \varepsilon)\}$

$\geqq \varepsilon^{1/2}$, and by the spectral mapping theorem for analytic functions, we deduce that $\inf\{\operatorname{Re} z : z \in \sigma([A + \varepsilon]^{1/2})\} \geqq \varepsilon^{1/2}$. Thus if z is in the closed convex hull of $\sigma([A + \varepsilon]^{1/2})$, then $\operatorname{Re} z \geqq \varepsilon^{1/2}$ and by the Hausdorff–Toeplitz theorem we see that $z \in W([A + \varepsilon]^{1/2})^-$ implies $\operatorname{Re} z \geqq \varepsilon^{1/2}$. (Here W^- denotes the closure of W.) Certainly it is true that $z \in W([A + \varepsilon]^{1/2} + A^{1/2})$ implies $\operatorname{Re} z \geqq \varepsilon^{1/2}$. Because the spectrum is contained in the closed numerical range we conclude that $z \in \sigma([A + \varepsilon]^{1/2} + A^{1/2})$ implies $\operatorname{Re} z \geqq \varepsilon^{1/2}$; then

$$1/\inf\{|z| : z \in \sigma([A + \varepsilon]^{1/2} + A^{1/2})\} \leqq \varepsilon^{-1/2}. \tag{1}$$

Because $(A + \varepsilon)^{1/2}$ and $A^{1/2}$ are both functions of A in the Hilbert space operational calculus, these two operators commute, and it follows that $([A + \varepsilon]^{1/2} + A^{1/2})$ is a normal operator. The invertibility of the operator $([A + \varepsilon]^{1/2} + A^{1/2})$ follows from the above inequality for its spectrum, and the spectrum of this inverse is given a trivial algebraic form of the spectral mapping theorem. For any normal operator the norm equals the spectral radius and by (1) above,

$$\|([A + \varepsilon]^{1/2} + A^{1/2})^{-1}\| = r(([A + \varepsilon]^{1/2} + A^{1/2})^{-1}) \leqq \varepsilon^{-1/2}. \tag{2}$$

Because $\varepsilon = [A + \varepsilon] - A = ([A + \varepsilon]^{1/2} + A^{1/2})([A + \varepsilon]^{1/2} - A^{1/2})$ we see that $[A + \varepsilon]^{1/2} - A^{1/2} = \varepsilon([A + \varepsilon]^{1/2} + A^{1/2})^{-1}$ and the lemma now follows from (2).

THEOREM 2. *Let L be a closed half disc bounded by $l = \{r e^{i\theta} : r \in [-m, m]\}$. There is a uniformly continuous square root map on the set of normal operators with spectra contained in L; this means, for $\varepsilon > 0$ there exists $\delta > 0$ such that $\sigma(A) \cup \sigma(B) \subset L$ and $\|A - B\| < \delta$ imply that $\|A^{1/2} - B^{1/2}\| < \varepsilon$.*

Proof. Clearly we can take $\alpha \in (-\pi, \pi]$ such that $\sigma(e^{i\alpha}A) \cup \sigma(e^{i\alpha}B) \subset \{z : \operatorname{Re} z \geqq 0\}$. It is straightforward to see that $(e^{i\alpha}A)^{1/2} = e^{i\alpha/2}A^{1/2}$ and $(e^{i\alpha}B)^{1/2} = e^{i\alpha/2}B^{1/2}$. Consequently we may prove this theorem assuming that $L = \{z : \operatorname{Re} z \geqq 0, |z| \leqq m\}$.

Let $\varepsilon > 0$ be given and choose $\gamma > 0$ such that $\gamma^{1/2} < \varepsilon/3$. By Lemma 1 we have $\|(A + \gamma)^{1/2} - A^{1/2}\| < \gamma^{1/2}$ and $\|(B + \gamma)^{1/2} - B^{1/2}\| < \gamma^{1/2}$ for any normal operators A and B. If $\sigma(C) \subset L$, then clearly $\sigma(C + \gamma) \subset \{z : \operatorname{Re} z \geqq \gamma\}$, and if C is normal, then $\|[(C + \gamma) - z]^{-1}\| \leqq \gamma^{-1}$ for any nonpositive real z. Using this estimate for the norms of the resolvents of A and B in the proof of Theorem 1, one obtains $\delta > 0$ such that $\sigma(A) \cup \sigma(B) \subset L$ and $\|(A + \gamma) - (B + \gamma)\| = \|A - B\| < \delta$ imply that $\|(A + \gamma)^{1/2} - (B + \gamma)^{1/2}\| < \varepsilon/3$. By the triangle inequality the theorem follows.

COROLLARY 2. *For any positive m there is a uniformly continuous square root defined on the set of self-adjoint operators A with $\|A\| \leqq m$.*

Proof. Take L to be $\{z : \operatorname{Im} z \geqq 0, |z| \leqq m\}$.

Although there is no analogue to Corollary 1 for normal operators with spectrum contained in L, if the spectrum of each operator is sufficiently sparse, then a result is possible.

THEOREM 3. *If $\{A_k : k = 1, 2, \cdots\}$ is a sequence of compact normal operators which converges in norm to the normal operator A_0, then $\{A_k^{1/2} : k = 1, 2, \cdots\}$ converges in norm to $A_0^{1/2}$.*

Proof. Let E_k and E_0 denote the spectral measures for A_k and A_0, respectively. Let the spectra of A_k and A_0 be written $\{z_{kj}:j = 1, 2, \cdots\}$ and $\{z_{0j}:j = 1, 2, \cdots\}$, respectively; further let the modulus of these points be nonincreasing as j increases. Define P_{kj} to be $E_k(\{z_{kj}\})$ and note that the following formula holds by 15 Corollary [2, p. 879]:

$$P_{kj} = \frac{-1}{2\pi i}\int_c (A_k - z)^{-1}\,dz.$$

We can apply Theorem 3.16 [8, p. 212] to conclude that $\{P_{kj}:k = 1, 2, \cdots\}$ converges in norm to $P_{0j} = E_0(\{z_{0j}\})$. Clearly it follows that $\{A_kP_{kj}:k = 1, 2, \cdots\}$ converges in norm to A_0P_{0j} and so $\{z_{kj}:k = 1, 2, \cdots\}$ converges to z_{0j}.

Since there are an uncountable number of rays originating at the origin, we can choose one, say R, which intersects $\sigma(A_0)$ in no nonzero point. For an appropriate definition of $z^{1/2}$ we can conclude that $\{(z_{kj})^{1/2}:k = 1, 2, \cdots\}$ converges to $(z_{0j})^{1/2}$ for $j = 1, 2, \cdots$ and consequently $\{\sum_{j=1}^{p}(z_{kj})^{1/2}P_{kj}:k = 1, 2, \cdots\}$ converges in norm to $\sum_{j=1}^{p}(z_{0j})^{1/2}P_{0j}$.

One notes that $A_k^{1/2} = \sum_{j=1}^{\infty}(z_{kj})^{1/2}P_{kj}$ and $A_0^{1/2} = \sum_{j=1}^{\infty}(z_{0j})^{1/2}P_{0j}$. From this, the triangle inequality, the indexing scheme for the eigenvalues, and the normality of the operators one concludes that

$$(3)\quad \|A_k^{1/2} - A_0^{1/2}\| \leqq \Big\|\sum_{j=1}^{p}(z_{kj})^{1/2}P_{kj} - \sum_{j=1}^{p}(z_{0j})^{1/2}P_{0j}\Big\| + |z_{kp+1}|^{1/2} + |z_{0p+1}|^{1/2}$$

for any positive integer p that we choose. By choosing p sufficiently large we get that $|z_{0p+1}|^{1/2} < \varepsilon/3$, and by the continuity of isolated eigenvalues there is some m_1 such that $|z_{kp+1}|^{1/2} < \varepsilon/3$ provided $k \geqq m_1$. The conclusion of our first paragraph was that we can choose m_2 such that the first term on the right of (3) is not greater than $\varepsilon/3$ provided that $k \geqq m_2$. Clearly this suffices to prove the theorem.

The next theorem is further evidence that we can handle operators with a thin spectrum.

THEOREM 4. *If $\{A_k:k = 1, 2, \cdots\}$ is any sequence of operators which converges in norm to A_0 (an invertible operator with countable spectrum), then for some positive integer p the sequence $\{A_k^{1/2}:k = p, p + 1, \cdots\}$ converges in norm to $A_0^{1/2}$.*

Proof. There are an uncountable number of rays originating at the origin and consequently one of them, call it R, must not intersect $\sigma(A_0)$. Because this spectrum is compact there is some $\varepsilon > 0$ such that $\inf\{|z - w| : z \in \sigma(A_0), w \in R\} = \varepsilon$. By the semicontinuity of the spectrum there exists some positive integer p such that R does not intersect $\sigma(A_k)$ for $k \geqq p$. Thus we deduce this theorem from Theorem 1.

Although we have restricted our attention to bounded operators we would like to give a brief indication of how results can be derived for closed linear transformations which may fail to be bounded. For a discussion of convergence in the generalized sense the reader is directed to [8, pp. 200–213].

THEOREM 5. *If $\{A_k:k = 1, 2, \cdots\}$ is a sequence of nonnegative linear transformations from H into H and if the sequence converges in the generalized sense to the nonnegative linear transformation A_0, then $\{A_k^{1/2}:k = 1, 2, \cdots\}$ converges in the generalized sense to $A_0^{1/2}$.*

Proof. By Theorem 2.25 [8, p. 206] for any $\varepsilon > 0$ we know that $\{(A_k + \varepsilon)^{-1}: k = 1, 2, \cdots\}$ converges in norm to $(A_0 + \varepsilon)^{-1}$. By our Theorem 2 we see that $\{(A_k + \varepsilon)^{-1/2}: k = 1, 2, \cdots\}$ converges in norm to $(A_0 + \varepsilon)^{-1/2}$ and by Theorem 2.25 of [8] this implies that $\{(A_k + \varepsilon)^{1/2}: k = 1, 2, \cdots\}$ converges in the generalized sense to $(A_0 + \varepsilon)^{1/2}$.

Although our Lemma 1 was proved for bounded operators, the proof is the same, step for step, if one assumes that A is a nonnegative linear transformation from H to H. Consequently $\|A_k^{1/2} - (A_k + \varepsilon)^{1/2}\| < \varepsilon^{1/2}$ for $k = 1, 2, \cdots$ and $\|A_0^{1/2} - (A_0 + \varepsilon)^{1/2}\| < \varepsilon^{1/2}$. By part (a) of 2.23 of [8] convergence in the norm implies convergence in the generalized sense; thus as ε goes to 0 we have $(A_k + \varepsilon)^{1/2}$ for $k = 1, 2, \cdots$ converging in the generalized sense to $A_k^{1/2}$ and, similarly, $(A_0 + \varepsilon)^{1/2}$ converges to $A_0^{1/2}$. This conclusion, the conclusion of the first paragraph, and the triangle inequality for the generalized metric prove this theorem. This triangle inequality follows from the first paragraph on p. 202 and the fourth paragraph on p. 198 of [8].

3. Some remarks. We conclude by proving a negative result and discussing other possible theorems.

THEOREM 6. *Let $N(H)$ be the set of normal operators on H. There is no square root map on $N(H)$ which is continuous.*

Proof. The transformation from the complex numbers into $N(H)$ defined by $z \to zI$ is obviously isometric. Therefore the continuous square root on $N(H)$ induces a continuous square root on the complex numbers. This contradicts elementary complex variable theory.

If one is willing to restrict to sets of operators which commute pairwise, then one can prove strong conclusions, just as one would expect. It is easy to see that our work extends straightforwardly to nth roots other than square roots.

REFERENCES

[1] D. DECKARD AND C. PEARCY, *Another class of invertible operators without square roots*, Proc. Amer. Math. Soc., 14 (1963), pp. 445–449.
[2] N. DUNFORD AND J. SCHWARTZ, *Linear Operations*, vol. II, Interscience, New York, 1963.
[3] P. R. HALMOS, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, N.J., 1967.
[4] P. R. HALMOS, G. LUMER AND J. J. SCHAFFER, *Square roots of operators*, Proc. Amer. Math. Soc., 4 (1963), pp. 142–149.
[5] P. R. HALMOS AND G. LUMER, *Square roots of operators. II*, Ibid., 5 (1954), pp. 589–595.
[6] E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semi-groups*, Colloquium Publications, American Mathematical Society, Providence, R.I., 1957.
[7] R. V. KADISON, *Strong continuity of operator functions*, Pacific J. Math., 26 (1968), pp. 121–129.
[8] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.
[9] J. J. SCHAFFER, *More about invertible operators without roots*, Proc. Amer. Math. Soc., 16 (1965), pp. 213–219.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

THE DUAL POISSON–LAGUERRE TRANSFORM OF A CLASS OF GENERALIZED FUNCTIONS*

J. N. PANDEY†

Abstract. The inversion formula of F. M. Cholewinski and D. T. Haimo for the dual Poisson–Laguerre transform is extended to a class of generalized functions interpreting convergence in the weak distributional sense. It is proved that this class of generalized functions contains tempered distributions defined over $(0, \infty)$. A structure formula for a class of generalized functions whose dual Poisson–Laguerre transform exists is also determined.

Further, it is shown that the dual Poisson–Laguerre transform of the class of generalized functions satisfies the heat equation $[xD^2 + (\alpha + 1 - x)D]u = \partial u/\partial t$, $\alpha > -1$.

1. Introduction. In a series of papers [1]–[4] amongst many others F. M. Cholewinski and D. T. Haimo have developed inversion and representation theories for integral transforms whose kernels are functions associated with the fundamental solutions of various generalized heat equations. Also in the series of papers [5]–[8] the author has extended the inversion formulas of Cholewinski and Haimo to various classes of generalized functions interpreting convergence in the weak distributional sense. The present goal is to carry out such a study for the dual Poisson–Laguerre transform studied by Cholewinski and Haimo in [1].

For $\alpha > -1$, let $L_n^\alpha(x)$ denote the Laguerre polynomial of degree n given by

$$L_n^\alpha(x) = \frac{x^{-\alpha} e^x}{n!}\left[\frac{d}{dx}\right]^n (x^{n+\alpha} e^{-x}), \qquad n = 0, 1, 2, \cdots.$$

We define functions $\Omega(x)$ and $\Lambda(x)$ by

$$d\Omega(x) = e^{-x}x^\alpha\, dx,$$

$$d\Lambda(x) = \frac{1}{\Gamma(\alpha + 1)} d\Omega(x).$$

Let ∇_x stand for the differentiation operator $[xD^2 + (\alpha + 1 - x)D]$. For any complex s and real y and t, define

$$(1) \qquad g_\alpha(s, y; t) = \left(\frac{e^t}{e^t - 1}\right)^{\alpha+1} \exp\left[\frac{-(s + y)}{e^t - 1}\right] I\left(\frac{2(sy\, e^t)^{1/2}}{e^t - 1}\right),$$

where

$$I(z) = 2^\alpha \Gamma(\alpha + 1) z^{-\alpha} I_\alpha(z),$$

$I_\alpha(z)$ being the Bessel function of imaginary argument.

Remark. At first it appears that for fixed real values of y and $t (t \neq 0)$, $g_\alpha(s, y; t)$ has branch cut in the complex s-plane. But by expanding $I(z)$ in powers of z one can see that only even powers of z appear. Therefore, $g_\alpha(s, y; t)$ is an entire function of s for fixed real values of y and t.

* Received by the editors February 11, 1971.

† Department of Mathematics, Carleton University, Ottawa 1, Ontario, Canada. This work was supported in part by the National Research Council under Grant A 5298 and a Summer Research Fellowship of the Canadian Mathematical Congress held in 1970 at Queen's University.

It is well known that for real s, and real y and t, $t > 0$,

$$g_\alpha(s, y; t) = \sum_{n=0}^{\infty} e^{-nt} L_n^\alpha(s) L_n^\alpha(y) \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)} \qquad [16, \text{p. } 189\,(19)]. \tag{1}$$

Therefore, one can easily check that for real s (hence also for complex s),

$$\nabla_y^n g_\alpha(s, y; t) = \left(\frac{\partial}{\partial t}\right)^n g_\alpha(s, y; t) \quad \text{for } n = 1, 2, 3, \cdots.$$

The following result has been proved by Cholewinski and Haimo [1, Thm. 6.1].

Let ϕ be integrable in every finite interval and let

$$\int_0^\infty g_\alpha(x_0, y; 1)\phi(y)\, d\Lambda(y)$$

converge for some $x_0 \geqq 0$. If

$$\lim_{h \to 0+} \frac{1}{h} \int_x^{x+h} [\phi(y) - \phi(x)]\, d\Lambda(y) = 0,$$

then

$$\lim_{t \to 0+} \int_0^\infty g_\alpha(x, y; t)\phi(y)\, d\Lambda(y) = \phi(x). \tag{2}$$

Heuristically one can say that

$$e^{-t\nabla_x} g_\alpha(x, y; 1) = g_\alpha(x, y; 1 - t), \qquad 0 < t < 1,$$

where the differentiation operator $e^{-t\nabla_x}$ is defined by

$$e^{-t\nabla_x} = \sum_{r=0}^{\infty} \frac{(-t\nabla_x)^r}{(r)!}.$$

In fact, a formal manipulation leads to

$$\sum_{r=0}^{\infty} \frac{(-t\nabla_x)^r}{r!} g_\alpha(x, y; 1) = \sum_{r=0}^{\infty} \frac{(-t)^r \nabla_x^r g_\alpha(x, y; 1)}{r!}$$
$$= \sum_{r=0}^{\infty} \frac{(-t)^r g_\alpha^{(r)}(x, y; 1)}{r!} = g_\alpha(x, y; 1 - t)$$

for

$$\nabla_x^r g_\alpha(x, y; 1) = \left(\frac{\partial}{\partial t}\right)^r g_\alpha(x, y; t)|_{t=1}.$$

Therefore (formally),

$$\lim_{t \to 0+} \int_0^\infty g_\alpha(x, y; t)\phi(y)\, d\Lambda(y) = \lim_{t \to 1-} \int_0^\infty g_\alpha(x, y; 1 - t)\phi(y)\, d\Lambda(y)$$
$$= \lim_{t \to 1-} e^{-t\nabla_x} \int_0^\infty g_\alpha(x, y; 1)\phi(y)\, d\Lambda(y)$$
$$= e^{-\nabla_x} \int_0^\infty g_\alpha(x, y; 1)\phi(y)\, d\Lambda(y),$$

where

$$e^{-\nabla_x} f(x) = \lim_{t \to 1-} e^{-t\nabla_x} f(x) \quad \text{for appropriate } f(x).$$

Thus, the inversion formula (2) as established by Cholewinski and Haimo in [1] can now be stated (formally) in the following way.

Let ϕ be integrable in every finite interval and let

$$\int_0^\infty g_\alpha(x_0, y; 1)\phi(y)\, d\Lambda(y)$$

converge for some $x_0 \geqq 0$. If

$$\lim_{h \to 0+} \int_x^{x+h} [\phi(y) - \phi(x)]\, d\Lambda(y) = 0,$$

then

$$e^{-\nabla_x} F(x) = \phi(x), \tag{3}$$

where

$$F(x) = \int_0^\infty g_\alpha(x, y; 1)\phi(y)\, d\Lambda(y).$$

Here, $F(x)$ is known as the Poisson–Laguerre transform of the function $\phi(y)$ with respect to the kernel $g_\alpha(x, y; 1)$.

In this paper we shall extend the inversion formula (2) of Cholewinski and Haimo to a class of generalized functions interpreting convergence in the weak distributional sense and this established inversion formula will be expressed rigorously in the form (3). A structure formula for a class of dual Poisson–Laguerre transformable generalized functions used in this paper will also be established and it will be shown that this class of generalized functions also contains the class of generalized functions of slow growth defined over $(0, \infty)$. Some other related results will also be established.

2. The notation and terminology of this work will follow that of [6], [7] and [8]. Unless otherwise mentioned t, x and y will be understood to be real variables assuming values in the set $(0, \infty)$. The set $(0, \infty)$ will also be denoted by the letter I. The symbol $D'(I)$ will stand for distributions defined over the testing function space $D(I)$, where by $D(I)$ we mean the space of infinitely differentiable functions defined over I having compact supports (in I). The topology of $D(I)$ is that which makes its dual the space $D'(I)$ of Schwartz distributions [13, vol. I, p. 65].

The testing function space $G_{\alpha,\delta}(I)$. For fixed real α and fixed real $\delta \neq -1$ an infinitely differentiable complex-valued function $\phi(x)$ defined over I belongs to $G_{\alpha,\delta}(I)$ if

$$\gamma_k(\phi) \equiv \sup_{0 < x < \infty} |e^{x/(e^{1+\delta}-1)} \nabla_x^k \phi(x)| < \infty \tag{4a}$$

for any k assuming values $0, 1, 2, 3, \cdots$. Clearly $G_{\alpha,\delta}(I)$ is a vector space with respect to the field of complex numbers. The zero element of the vector space $G_{\alpha,\delta}(I)$ is the function defined over I which is identically zero. Since γ_0 is a norm, the collection of seminorms $\{\gamma_k\}_{k=0}^{\infty}$ is separating [11, p. 8]. The topology in $G_{\alpha,\delta}(I)$ is generated by the collection of seminorms $\{\gamma_k\}_{k=0}^{\infty}$ [12, p. 8]. We say that a sequence $\{\phi_v\}_{v=1}^{\infty}$, where each $\phi_v(x)$ belongs to $G_{\alpha,\delta}(I)$, converges in $G_{\alpha,\delta}(I)$ to $\phi(x)$ if for each fixed k, $\gamma_k(\phi_v - \phi)$ tends to zero as v tends to ∞. A sequence $\{\phi_v(x)\}_{v=1}^{\infty}$, where each $\phi_v(x)$ belongs to $G_{\alpha,\delta}(I)$, is a Cauchy sequence in $G_{\alpha,\delta}(I)$ if $\gamma_k(\phi_\mu - \phi_v)$ goes to zero for any nonnegative integer k as μ and v both go to ∞ independently of each other. It can be readily seen that $G_{\alpha,\delta}(I)$ is a sequentially complete, locally convex, Hausdorff topological vector space. The space $D(I)$ is a subspace of $G_{\alpha,\delta}(I)$ and the topology of $D(I)$ is stronger than the topology induced on $D(I)$ by $G_{\alpha,\delta}(I)$ and as such the restriction of any member of $G'_{\alpha,\delta}(I)$ to $D(I)$ is in $D'(I)$.

The testing function space $P_{\alpha,\delta}(I)$. For fixed real $\alpha, \delta, \delta \neq -1$, an infinitely differentiable complex-valued function $\phi(x)$ defined over I is said to belong to the space $P_{\alpha,\delta}(I)$ if

$$\beta_k(\phi) \equiv \gamma_k\left\{\frac{\phi}{\Lambda'(x)}\right\} < \infty \tag{4b}$$

for all $k = 0, 1, 2, \cdots$. The topology on $P_{\alpha,\delta}(I)$ is generated by the sequence of seminorms $\{\beta_k\}_{k=0}^{\infty}$. The concepts of convergence and completeness in $P_{\alpha,\delta}(I)$ are defined in a way similar to those defined in $G_{\alpha,\delta}(I)$. The space $P_{\alpha,\delta}(I)$ is also a sequentially complete, locally convex Hausdorff topological vector space. The restriction of any member of $P'_{\alpha,\delta}(I)$ to $D(I)$ is in $D'(I)$.

LEMMA 1. *For fixed* $\alpha > -1$, *let* $g_\alpha(s, y; t)$ *be the function defined in* (1) *and* ∇_y *be the differentiation operator* $[yD_y^2 + (\alpha + 1 - y)D_y]$, *where* $D_y = d/dy$. *Then for fixed* $t > 0$ *and complex* s,

$$\nabla_y^k g_\alpha(s, y; t) = \sum_{r=0}^{k} P_{r,\alpha}\left(s, y, e^t, \frac{1}{e^t - 1}\right) g_{\alpha+r}(s, y; t), \tag{5}$$

where $P_{r,\alpha}(a, b, c, d)$ *are polynomials in* a, b, c *and* d *with degree depending upon* r.

Proof. For $k = 1$, we have

$$\nabla_y g_\alpha(s, y; t) = \left[\frac{(s + y)e^t}{(e^t - 1)^2} - \frac{\alpha + 1}{e^t - 1}\right] g_\alpha(s, y; t) - \frac{sy}{e^t - 1} g_{\alpha+1}(s, y; t) \tag{6}$$

[1, p. 279]. Operating on both sides of (6) by the operator ∇_y repeatedly we get the result (5) for $k = 2, 3, 4, \cdots$. A rigorous proof can be given by the method of induction.

LEMMA 2. *For complex* $s, y > 0$, $0 < t \leqq 1$, *define the function* $g_\alpha(s, y; t)$ *as in* (1), *where* $\alpha > -1$. *Then for fixed* $\delta > 0$ *we have, for* $k = 0, 1, 2, 3, \cdots$,

$$\sup_{0 < y < \infty} |e^{y/(e^{1+\delta} - 1)} \nabla_y^k g_\alpha(s, y; t)| \leqq M, \tag{7}$$

uniformly for all s *lying in any compact set* Ω *of the complex plane and* $0 < r < t \leqq 1$, *the bound* M *depending upon* k, α, Ω *and* r.

Proof. For fixed complex s and $t > 0$ we have

$$g_\alpha(s, y; t) \sim \frac{\Gamma(\alpha + 1)(e^t)^{\alpha/2 + 3/4}}{2\{\pi(e^t - 1)\}^{1/2}} e^{|s|}(|s|y)^{-\alpha/2 - 1/4} \exp\left[\frac{-(|s|^{1/2} e^{t/2} - y^{1/2})^2}{e^t - 1}\right], \tag{8}$$

$y \to \infty$ [1, p. 280]. Further, it is well known that

$$(9)\quad -\frac{[(xe^t)^{1/2} - y^{1/2}]^2}{e^t - 1} \leqq \frac{Axe^t}{1 - A(e^t - 1)} - Ay \quad \text{for } t > 0 \quad \text{and} \quad A < 1/(e^t - 1).$$

Now fix A in (9) so that

$$1/(e^{1+\delta} - 1) < A < 1/(e - 1).$$

Let

$$A - 1/(e^{1+\delta} - 1) = \eta > 0.$$

Therefore, for $y > 0$ there exists a constant $Q_\alpha > 0$ satisfying

$$(10)\quad | e^{y/(e^{1+\delta}-1)} g_\alpha(s, y; t)| \leqq Q_\alpha e^{-\eta y}$$

uniformly for s lying in Ω and t lying in $[r, 1]$.

Hence, in view of the relation (5) we have, for $y > 0$,

$$| e^{y/(e^{1+\delta}-1)} \nabla_y^k g_\alpha(s, y; t)| \leqq \sum_{r=0}^{k} \left| P_{r,\alpha}\left(s, y, e^t, \frac{1}{e^t - 1}\right)\right| Q_{\alpha+r} e^{-\eta y}.$$

Therefore, there exists a positive constant M depending upon k, α, Ω and r satisfying the relation

$$(11)\quad | e^{y/(e^{1+\delta}-1)} \nabla_y^k g_\alpha(s, y; t)| \leqq M$$

uniformly for all $y > 0$, $r \leqq t \leqq 1$ and $s \in \Omega$. Thus the proof of Lemma 2 is complete.

COROLLARY. *For a complex s and fixed t in* $(0, 1]$, $\alpha > -1$, *the function $g_\alpha(s, y; t)$ as defined by* (1) *belongs to the space $G_{\alpha,\delta}(I)$*, $\delta > 0$.

Proof. The proof is trivial.

From now on we shall choose δ to be a fixed number greater than zero and $\alpha > -1$.

LEMMA 3. *For complex $s, y > 0$ and fixed t in* $(0, 1]$, $\alpha > -1$, *define the function $g_\alpha(s, y; t)$ as in* (1). *Then the function $(\partial/\partial s)^m g_\alpha(s, y; t)$ is a member of $G_{\alpha,\delta}(I)$ for fixed s and t.*

Proof. Let C be a closed circle with radius r and center at s, contained in the compact set Ω of the complex z-plane. Then clearly,

$$(12)\quad g_\alpha(s, y; t) = \frac{1}{2\pi i} \int_C \frac{g_\alpha(z, y; t)}{z - s} dz,$$

where the integration in (12) is taken in the positive direction. Therefore,

$$\left(\frac{\partial}{\partial s}\right)^m g_\alpha(s, y; t) = \frac{m!}{2\pi i} \int_C \frac{g_\alpha(z, y; t)}{(z - s)^{m+1}} dz.$$

Hence

$$e^{y/(e^{1+\delta}-1)} \nabla_y^k \left(\frac{\partial}{\partial s}\right)^m g_\alpha(s, y; t) = \frac{m!}{2\pi i} \int_C \frac{e^{y/(e^{1+\delta}-1)} \{\nabla_y^k g_\alpha(s, y; t)\}}{(z - s)^{m+1}} dz.$$

Assuming that t lies in $[r, 1]$, $r > 0$, we have in view of Lemma 2,

$$\sup_{0<y<\infty} \left| e^{y/(e^{1+\delta}-1)} \nabla_y^k \left\{ \left(\frac{\partial}{\partial s}\right)^m g_\alpha(s, y; t) \right\} \right| \leqq \frac{m!M}{r^m}.$$

COROLLARY. *Let $g_\alpha(s, y; t)$ be the function defined in Lemma 3. Then for fixed t lying in any compact subset of $(0, 1]$ and complex s,*

$$\left(\frac{\partial}{\partial t}\right)^m g_\alpha(s, y; t)$$

is an element of $G_{\alpha,\delta}(I)$.

Proof. We can easily see that

$$(\nabla_y)^m g_\alpha(s, y; t) = \left(\frac{\partial}{\partial t}\right)^m g_\alpha(s, y; t) \quad \text{for } m = 1, 2, 3, \cdots.$$

Therefore, $(\partial/\partial t)^m g_\alpha(s, y; t)$ belongs to $G_{\alpha,\delta}(I)$ as $\{\nabla_y^m g_\alpha(s, y; t)\}$ does.

LEMMA 4. *Let f be an element of $P'_{\alpha,\delta}(I)$ and $y > 0$, $0 < t \leqq 1$. Let $g_\alpha(s, y; t)$ be the function as defined by (1). Then the function $u(s, t)$ defined by*

$$u(s, t) = \langle \Lambda'(y) f(y), g_\alpha(s, y; t) \rangle \tag{13}$$

is an entire function of s and

$$\left(\frac{\partial}{\partial s}\right)^m u(s, t) = \left\langle \Lambda'(y) f(y), \left(\frac{\partial}{\partial s}\right)^m g_\alpha(s, y; t) \right\rangle, \qquad m = 1, 2, 3, \cdots. \tag{14}$$

Proof. We shall prove (14) for $m = 1$, as the proof for any m can be given similarly by the method of induction. It can be easily seen that the expressions in the right-hand side of (13) and (14) are meaningful. Therefore,

$$\frac{u(s + \Delta s, t) - u(s, t)}{\Delta s} - \left\langle \Lambda'(y) f(y), \frac{\partial}{\partial s} g_\alpha(s, y; t) \right\rangle = \left\langle \Lambda'(y) f(y), \theta(s, \Delta s) \right\rangle, \tag{15}$$

where

$$\theta(s, \Delta s) = \frac{g_\alpha(s + \Delta s, y; t) - g_\alpha(s, y; t)}{\Delta s} - \frac{\partial}{\partial s} g_\alpha(s, y; t).$$

Now let C_1 and C_2 be two concentric circles with center at $z = s$ and radii r_1 and r_2 respectively in the complex z-plane such that the points $z = s$ and $z = s + \Delta s$ both lie in the inner circle C_1 $(r_2 > r_1)$. In view of Cauchy's integral formula it can be readily seen that

$$\theta(s, \Delta s) = \frac{\Delta s}{2\pi i} \int_{C_2} \frac{g_\alpha(z, y; t)\, dz}{(z - s)^2 (z - s - \Delta s)}, \tag{16}$$

the integration along C_2 being taken in the positive direction.

If Ω is a compact set of the complex z-plane containing the circle C_2, with an appeal to Lemma 2 we can have a positive constant M satisfying the condition

$$| e^{y/(e^{1+\delta}-1)} \nabla_y^k g_\alpha(s, y; t)| \leqq M \tag{17}$$

uniformly for all $s \in \Omega$ and $y > 0$.

Therefore,

$$| e^{y/(e^{1+\delta}-1)}\nabla_y^k \theta(s, \Delta s)| = \frac{|\Delta s| M}{(r_2 - r_1) r_2}.$$

Hence, $\theta(s, \Delta s) \to 0$ in $G_{\alpha,\delta}(I)$ as $\Delta s \to 0$.

This completes the proof of Lemma 4.

LEMMA 5. *Let f be an element of $P'_{\alpha,\delta}(I)$, $y > 0$ and $0 < t \leqq 1$, and let $g_\alpha(s, y; t)$ be the function as defined by* (1). *Assume also that $u(s, t)$ is the function defined by*

$$u(s, t) = \langle \Lambda'(y) f(y), g_\alpha(s, y; t)\rangle.$$

Then

$$\left(\frac{\partial}{\partial t}\right)^m u(s, t) = \left\langle \Lambda'(y) f(y), \left(\frac{\partial}{\partial t}\right)^m g_\alpha(s, y; t)\right\rangle \quad \textit{for } m = 1, 2, 3, \cdots. \tag{18}$$

Proof. We shall prove the result only for $m = 1$, as the result (18) can be proved for any m in an analogous way by using the technique of induction.

Proof. Let t belong to the set $(r, 1]$, $r > 0$. Let Δt be chosen so small that $t + \Delta t \in (r, 1]$. Note that in the event that $t = 1$ we have to take $\Delta t < 0$. Now

$$\frac{u(s, t + \Delta t) - u(s, t)}{\Delta t} - \left\langle \Lambda'(y) f(y), \frac{\partial}{\partial t} g_\alpha(s, y; t)\right\rangle = \left\langle \Lambda'(y) f(y), \theta_{\Delta s}\right\rangle,$$

where

$$\theta_{\Delta t} = \frac{g_\alpha(s, y; t + \Delta t) - g_\alpha(s, y; t)}{\Delta t} - \frac{\partial}{\partial t} g_\alpha(s, y; t).$$

Therefore,

$$\nabla_y^k \theta_{\Delta t} = \left[\frac{g_\alpha^{(k)}(s, y; t + \Delta t) - g_\alpha^{(k)}(s, y; t)}{\Delta t} - \frac{\partial}{\partial t} g_\alpha^{(k)}(s, y; t)\right],$$

where

$$g_\alpha^{(k)}(s, y; t) = \nabla_y^k g_\alpha(s, y; t) = \left(\frac{\partial}{\partial t}\right)^k g_\alpha(s, y; t),$$

or

$$\nabla_y^k\{\theta_{\Delta t}\} = \begin{cases} g_\alpha^{(k+1)}(s, y; t + \theta\Delta t) - g_\alpha^{(k+1)}(s, y; t), & 0 < \theta < 1, \\ \theta\Delta t g_\alpha^{(k+2)}(s, y; t + \phi\theta\Delta t), & 0 < \phi < 1, \\ \theta\Delta t g_\alpha^{(k+2)}(s, y; t + \alpha'\Delta t), & 0 < \alpha' < 1. \end{cases}$$

Therefore,

$$| e^{y/(e^{1+\delta}-1)}\nabla_y^k \theta_{\Delta t}| < |\Delta t| \, | e^{y/(e^{1+\delta}-1)} g_\alpha^{(k+2)}(s, y; t + \alpha'\Delta t)|.$$

Again, Δt is so chosen that for all possible variations of Δt, $t + \Delta t$ lies in $(r, 1]$, $r > 0$. Taking s to be a fixed complex number and appealing to Lemma 2 we get an obvious constant M depending upon r satisfying the relation

$$| e^{y/(e^{1+\delta}-1)}\nabla_y^k \theta_{\Delta t}| < M|\Delta t|,$$

uniformly for all $y > 0$. Therefore,

$$\theta_{\Delta t} \to 0 \quad \text{in } G_{\alpha,\delta}(I) \quad \text{as } \Delta t \to 0.$$

This completes the proof of Lemma 5.

LEMMA 6. *Let f be an element of $P'_{\alpha,\delta}(I)$, $y > 0$, $\alpha > -1$, $0 < t \leqq 1$ and let $g_\alpha(s, y; t)$ be the function defined in* (1). *Then the function $u(s, t)$ defined by*

$$u(s, t) = \langle \Lambda'(y) f(y), g_\alpha(x, y; t) \rangle$$

is a continuous function of s and t.

Proof. Assume Δt to be small enough so that if t lies in $[r, 1]$, $r > 0$, $t + \Delta t$ also lies in $[r, 1]$. Now,

$$u(x + \Delta x, t + \Delta t) - u(x, t) = \langle \Delta'(y) f(y), \theta(\Delta x, \Delta t) \rangle, \tag{19}$$

where

$$\theta(\Delta x, \Delta t) = g_\alpha(x + \Delta x, y; t + \Delta t) - g_\alpha(x, y; t)$$

or

$$\begin{aligned} \theta(\Delta x, \Delta t) = {} & \{g_\alpha(x + \Delta x, y; t + \Delta t) - g_\alpha(x, y; t + \Delta t)\} \\ & + \{g_\alpha(x, y; t + \Delta t) - g_\alpha(x, y; t)\}. \end{aligned} \tag{20}$$

In view of Lemma 2 it can be readily shown that each of the bracketed expressions in the right-hand side of (20) goes to zero in $G_{\alpha,\delta}(I)$ as Δx, $\Delta y \to 0$ independently of each other. This completes the proof of the lemma.

In the next lemma we shall determine an asymptotic order of $u(s, t)$ as $|s| \to \infty$.

LEMMA 7. *Let $u(s, t)$ be defined as in Lemma 6 for complex s and $0 < t \leqq 1$. Then for $1/(e^{1+\delta} - 1) < A < 1/(e - 1)$,*

$$u(s, t) = O\left[P(|s|) \exp\left[\frac{A|s|e^t}{1 - A(e^t - 1)} \right] e^{|s|} \right], \qquad s \to \infty, \tag{21}$$

where $P(|s|)$ is a polynomial in $|s|$.

Proof. Using the boundedness property of generalized functions, we get

$$|u(s, t)| \leqq C \max_{0 \leqq m \leqq r} \gamma_m[g_\alpha(s, y; t)]$$

for appropriate constant C and a nonnegative integer r. Therefore,

$$|u(s, t)| \leqq C \sum_{k=0}^{r} \left| P_k\left(s, y, e^t, \frac{1}{(e^t - 1)}\right) g_{\alpha+k}(s, y; t)\, e^{y/(e^{1+\delta} - 1)} \right|.$$

Again,

$$|g_\alpha(s, y; t)| \leqq \frac{\Gamma(\alpha + 1)(e^t)^{\alpha/2 + 3/4}}{2\{\pi(e^t - 1)\}^{1/2}} e^{|s|}(|s|y)^{-\alpha/2 - 1/4} \exp\left[\frac{-(|s|^{1/2} e^{t/2} - y^{1/2})^2}{e^t - 1} \right],$$

$y \to \infty$, [1, p. 280] and

$$-\frac{\{(|s|e^t)^{1/2} - y^{1/2}\}^2}{e^t - 1} \leqq \frac{A|s|e^t}{1 - A(e^t - 1)} - Ay, \quad t > 0, \quad A < \frac{1}{e^t - 1}.$$

The result (21) follows immediately by choosing A in $(1/(e^{1+\delta} - 1), 1/(e - 1))$ and observing that $I(2\sqrt{sy}\, e^t/(e^t - 1)) \to 1$ as $y \to 0+$ (for any fixed s).

From now on we shall drop the suffix α of $g_\alpha(x, y; t)$ and represent it only by $g(x, y; t)$. We shall also assume that $\alpha > -1$.

LEMMA 8. *For $x, y > 0$ and $0 < t \leqq 1$, define the function $g(x, y; t)$ as in* (1). *Let $\phi(x) \in D(I)$, the space of infinitely differentiable functions defined over I having compact supports. Then*

$$e^{y/(e^{1+\delta}-1)} \int_0^\infty g(x, y; t)[\phi(x) - \phi(y)]\, d\Lambda(x) \to 0 \quad \text{as } t \to 0+ \tag{22}$$

uniformly for $y > 0$.

Proof. Let us represent the expression in (22) by I and split up the integration in I which is over $(0, \infty)$ into integrations on $(0, y - \eta)$, $(y - \eta, y + \eta)$ and $(y + \eta, \infty)$ and denote the corresponding expressions by I_1, I_2 and I_3 respectively. Here η is an (arbitrary) positive number less than r, the infimum of the support of $\phi(x)$;

$$I = I_1 + I_2 + I_3. \tag{23}$$

Now,

$$I_2 = e^{y/(e^{1+\delta}-1)} \int_{y-\eta}^{y+\eta} g(x, y; t)[\phi(x) - \phi(y)]\, d\Lambda(x).$$

Let us now assume that R is the supremum of the support K of $\phi(x)$. Therefore,

$$|I_2| \leqq \eta \sup_{y-\eta<\tau<y+\eta} [e^{y/(e^{1+\delta}-1)} |\phi'(\tau)|]. \tag{24}$$

Hence,

$$|I_2| \leqq M'\eta \exp\left\{\frac{R + r}{e^{1+\delta} - 1}\right\}, \tag{25}$$

where

$$M' = \sup_{r \leqq \tau \leqq R} |\phi'(\tau)|.$$

We shall also use the symbol M for $\sup_{r \leqq x \leqq R} |\phi(x)|$. Now fix η such that

$$M'\eta \exp\left\{\frac{R + r}{e^{1+\delta} - 1}\right\} < \varepsilon,$$

where ε is some arbitrary positive number. Therefore an appeal to (25) leads to

$$|I_2| < \varepsilon, \tag{26}$$

uniformly for $y > \eta$. Next, we consider

$$I_3 = e^{y/(e^{1+\delta}-1)} \int_{y+\eta}^\infty [\phi(x) - \phi(y)] g(x, y; t)\, d\Lambda(x). \tag{27}$$

If $y \geqq R$, $I_3 = 0$. Therefore, we consider the case $\eta < y < R$. Therefore,

$$|I_3| \leqq 2M\, e^{R/(e^{1+\delta}-1)} \int_{y+\eta}^\infty g(x, y; t)\, d\Lambda(x). \tag{28}$$

Since $I_\alpha(z) \simeq e^z/\sqrt{2\pi z}$ as $z \to \infty$ and $I(z) \to 1$ as $z \to 0$, therefore, for any arbitrary and positive quantity β and an obvious constant P, we have

$$|I(z)| \leqq P\, e^{|z|(1+\beta)}$$

uniformly for all z. Hence

$$(29)\quad |g(x,y;t)| \leqq P\left(\frac{e^t}{e^t-1}\right)^\alpha \exp\left\{-\frac{x+y-2\sqrt{xy}\,e^t(1+\beta)}{e^t-1}\right\}, \quad x,y>0.$$

There also exist a constant Q and $0<\gamma<1$ such that

$$(30)\qquad |g(x,y;t)| \leqq Q\left(\frac{e^t}{e^t-1}\right)^\alpha \exp\left\{-\frac{x(1-\gamma)}{e^t-1}\right\}$$

for all $x>0$ and $0<y\leqq R$. Similarly we can show that for $y>0$ and $0<x\leqq R$, we have

$$(31)\qquad |g(x,y;t)| \leqq Q\left(\frac{e^t}{e^t-1}\right)^\alpha \exp\left\{-\frac{y(1-\gamma)}{e^t-1}\right\}$$

(since $g(x,y;t)=g(y,x;t)$). An appeal to (28) and (30) leads to

$$|I_3| \leqq 2MQ\left(\frac{e^t}{e^t-1}\right)^\alpha e^{R/(e^{1+\delta}-1)}\int_0^\infty \exp\left\{-\frac{x(1-\gamma)}{e^t-1}\right\} d\Lambda(x)$$
$$= 2MQ\exp\left\{|\alpha| + \frac{R}{e^{1+\delta}-1}\right\}\frac{e^t-1}{\{1-\gamma+e^t-1\}^{\alpha+1}} \to 0 \quad \text{as } t\to 0+.$$

Therefore, we have completed the proof of the fact that

$$(32)\qquad I_3 \to 0 \quad \text{as } t\to 0+$$

uniformly for $y>0$. Next we consider

$$I_1 = e^{y/(e^{1+\delta}-1)}\int_0^{y-\eta} g(x,y;t)[\phi(x)-\phi(y)]\,d\Lambda(x).$$

Take

$$J_1 = e^{y/(e^{1+\delta}-1)}\int_0^{y-\eta} g(x,y;t)\phi(x)\,d\Lambda(x),$$

$$(33)\qquad |J_1| \leqq M\,e^{y/(e^{1+\delta}-1)}\int_r^R g(x,y;t)\,d\Lambda(x).$$

Appealing to (31) we can find a small positive number t_0 such that for $0<t<t_0$,

$$e^{y/(e^{1+\delta}-1)}g(x,y;t)\to 0$$

as $y\to\infty$ uniformly for $r\leqq x\leqq R$.

Therefore,

$$(34)\qquad J_1 \to 0 \quad \text{as } y\to\infty \quad \text{for } 0<t<t_0.$$

When $0 < y \leqq N$ we have in view of (33) and (30),

$$(35) \qquad |J_1| \leqq \frac{M\, e^{N/(e^{1+\delta}-1)} Q\, e^{t\alpha}(e^t - 1)}{\{1 - \gamma + e^t - 1\}^{\alpha+1}} \to 0 \quad \text{as } t \to 0+ .$$

Joining (34) and (35) we get

$$(36) \qquad J_1 \to 0 \quad \text{as } t \to 0+$$

uniformly for $y > \eta$. Now,

$$J_2 = e^{y/(e^{1+\delta}-1)} \int_0^{y-\eta} g(x, y; t)\phi(y)\, d\Lambda(x),$$

$$|J_2| \leqq M\, e^{R/(e^{1+\delta}-1)} \int_0^{y-\eta} g(x, y; t)\, d\Lambda(x)$$

$$\leqq M \exp\left\{\frac{R}{e^{1+\delta} - 1} + |\alpha|\right\} \frac{e^t - 1}{\{1 - \gamma + e^t - 1\}^{\alpha+1}} \to 0 \quad \text{as } t \to 0+ .$$

But, $I_1 = J_1 - J_2$. Therefore,

$$(37) \qquad I_1 \to 0 \quad \text{as } t \to 0+ ,$$

uniformly for $y > \eta$. Joining (26), (32) and (37) we get

$$(38) \qquad \limsup_{t \to 0+} |I| \leqq \varepsilon$$

uniformly for $y > \eta$.

If $0 < y \leqq \eta$, we split up the integration in I into integrations on $0 < x < y + \eta$ and $y + \eta < x < \infty$ and represent the corresponding expressions by $\bar{I}$ and $\bar{J}$ respectively. Now

$$I = \bar{I} + \bar{J},$$

$$\bar{I} = e^{y/(e^{1+\delta}-1)} \int_0^{y+\eta} g(x, y; t)[\phi(x) - \phi(y)]\, d\Lambda(x).$$

Clearly,

$$(39) \qquad |\bar{I}| \leqq \eta M'\, e^{r/(e^{1+\delta}-1)} < \varepsilon, \quad t > 0,$$

and

$$(40) \qquad \begin{aligned} \bar{J} &= e^{y/(e^{1+\delta}-1)} \int_{y+\eta}^{\infty} g(x, y; t)[\phi(x) - \phi(y)]\, d\Lambda(x) = I_3, \\ |\bar{J}| = |I_3| &\leqq \frac{2MQ \exp\{|\alpha| + R/(e^{1+\delta} - 1)\}}{(1 - \gamma + e^t - 1)^{\alpha+1}}(e^t - 1) \to 0 \quad \text{as } t \to 0+ . \end{aligned}$$

Combining (39) and (40) we get

$$(41) \qquad \limsup_{t \to 0+} |I| \leqq \varepsilon, \qquad 0 < y \leqq \eta .$$

Therefore, in view of (38) and (41) we have

$$\limsup_{t\to 0+} |I| \leqq \varepsilon, \qquad y > 0.$$

Since ε is arbitrary, our lemma is proved.

3. Inversion formula. In this section we shall establish an inversion formula for the class of generalized functions $P_{\alpha,\delta}(I)$ taking δ to be a fixed positive number. It can be readily seen that if $f(y) \in P'_{\alpha,\delta}(I)$, then $\Lambda'(y)f(y) \in G'_{\alpha,\delta}(I)$ and vice versa. We have proved in § 2 that for fixed s (complex) and t lying in $(0, 1]$, $g(s, y; t)$ is an element of $G_{\alpha,\delta}(I)$. Consequently the expression $\langle \Lambda'(y)f(y), g(x, y; t)\rangle$ which we also denote by $u(x, t)$ is meaningful. We define $u(x, t)$ to be the dual Poisson–Laguerre transform of the generalized function $f(y)$ belonging to $G'_{\alpha,\delta}(I)$. Our object is to prove that $u(x, t)$ converges in the weak distributional sense to $f(x)$ as $t \to 0+$.

THEOREM 1. *Let the function $g(x, y; t)$ be defined as in* (1) *for $x, y > 0$ and $0 < t \leqq 1$. Assume that $f(y)$ is an arbitrary element of $P'_{\alpha,\delta}(I)$ and $\phi(x)$ is an infinitely differentiable function defined over I with compact support. Then*

$$\lim_{t\to 0+} \langle \Lambda'(x)\langle \Lambda'(y)f(y), g(x, y; t)\rangle, \phi(x)\rangle = \langle \Lambda' f, \phi\rangle. \tag{42}$$

Proof. The theorem will be proved by justifying the following manipulations:

$$\langle \Lambda'(x)\langle \Lambda'(y)f(y), g(x, y; t)\rangle, \phi(x)\rangle = \int_0^\infty \langle \Lambda'(y)f(y), g(x, y; t)\rangle \phi(x)\, d\Lambda(x), \tag{43}$$

$$\int_0^\infty \langle \Lambda'(y)f(y), g(x, y; t)\rangle \phi(x)\, d\Lambda(x) = \left\langle \Lambda'(y)f(y), \int_0^\infty g(x, y; t)\phi(x)\, d\Lambda(x)\right\rangle, \tag{44}$$

$$\lim_{t\to 0+} \left\langle \Lambda'(y)f(y), \int_0^\infty g(x, y; t)\phi(x)\, d\Lambda(x)\right\rangle = \langle \Lambda' f, \phi\rangle. \tag{45}$$

In view of Lemma 4, it follows that for fixed t lying in $(0, 1]$, $u(x, t)$ is an analytic function of x. Since $\phi(x)$ is smooth and has compact support, (43) is justified. The relation (44) can also be justified by following the technique of Riemann sums in view of the asymptotic order of $g(x, y; t)$ for fixed t and x lying in any compact subset of I. Since the proof is very similar to that followed in proving Lemma 4 of [8, pp. 13–15], the detail is omitted. In order to justify (45) we have to show that

$$\int_0^\infty g(x, y; t)\phi(x)\, d\Lambda(x) \to \phi(y) \quad \text{in } G_{\alpha,\delta}(I) \quad \text{as } t \to 0+.$$

Now

$$\nabla_y \int_0^\infty g(x, y; t)\phi(x)\, d\Lambda(x) = \int_0^\infty \nabla_x g(x, y; t)\phi(x)\, d\Lambda(x)$$

because $\nabla_y g(x, y; t) = \nabla_x g(x, y; t)$. Also an integration by parts leads to

$$\int_0^\infty \nabla_x g(x, y; t)\phi(x)\, d\Lambda(x) = \int_0^\infty g(x, y; t)\{\nabla_x \phi(x)\}\, d\Lambda(x).$$

Therefore,

$$\nabla_y \int_0^\infty g(x, y; t)\phi(x)\, d\Lambda(x) = \int_0^\infty g(x, y; t)\{\nabla_x \phi(x)\}\, d\Lambda(x).$$

Operating by the operator ∇_y successively it can be shown that

$$\nabla_y^k \int_0^\infty g(x, y; t)\phi(x)\, d\Lambda(x) = \int_0^\infty g(x, y; t)\{\nabla_x^k \phi(x)\}\, d\Lambda(x).$$

Therefore,

$$\nabla_y^k\left[\int_0^\infty g(x, y; t)\phi(x)\, d\Lambda(x) - \phi(y)\right] = \int_0^\infty g(x, y; t)[\phi_k(x) - \phi_k(y)]\, d\Lambda(x),$$

where $\phi_k(x) = \nabla_x^k \phi(x)$. Now in view of Lemma 8 we have

$$\sup_{0<y<\infty} e^{y/(e^{1+\delta}-1)}\nabla_y^k\left[\int_0^\infty g(x, y; t)\phi(x)\, d\Lambda(x) - \phi(y)\right] \to 0 \quad \text{as } t \to 0+.$$

This justifies (45).

THEOREM 2. *Let $u(x, t)$ be defined as in Theorem 1 for $0 < t \leqq 1$. Then for $\phi(x) \in D(I)$,*

$$(46) \qquad \langle e^{-\nabla_x}\langle \Lambda'(y) f(y), g(x, y; 1)\rangle, \phi(x)\rangle = \langle \Lambda' f, \phi\rangle,$$

where the operator $e^{-\nabla_x}$ is defined by

$$\lim_{t \to 1-} e^{-t\nabla_x} = e^{-\nabla_x},$$

and for $0 < t < 1$ the operator $e^{-t\nabla_x}$ is interpreted as

$$\sum_{r=0}^{\infty} \frac{(-t\nabla_x)^r}{r!}.$$

Proof.

$$\sum_{r=0}^{N} \frac{(-t\nabla_x)^r}{r!} g(x, y; 1) = \sum_{r=0}^{N} \frac{(-t)^r}{r!} g^{(r)}(x, y; 1),$$

where

$$g^{(r)}(x, y; 1) = \nabla_x^r g(x, y; 1) = \left(\frac{\partial}{\partial t}\right)^r g(x, y; t)|_{t=1}.$$

Now,

$$g(x, y; 1 - t) - \sum_{r=0}^{N} \frac{(-t)^r}{r!} g^{(r)}(x, y; 1) = \frac{(-t)^{N+1} g^{(N+1)}(x, y; \varepsilon)}{(N + 1)!},$$

where ε is some number lying between 1 and $1 - t$. Again, similarly,

$$\nabla_y^k\left[g(x, y; 1 - t) - \sum_{r=0}^{N} \frac{(-t)^r}{r!} g^{(r)}(x, y; 1)\right] = \frac{(-t)^{(N+1)} g^{(N+k+1)}(x, y; \varepsilon)}{(N + 1)!}$$

because

$$\nabla_y g^{(N+1)}(x, y; \varepsilon) = \frac{\partial}{\partial t} g^{(N+1)}(x, y; t)|_{t=\varepsilon}.$$

Therefore,

$$\sup_{0<y<\infty} \left| e^{y/(e^{1+\delta}-1)} \nabla_y^k \left[g(x, y; 1 - t) - \sum_{r=0}^{N} \frac{(-t)^r g^{(r)}(x, y; 1)}{r!} \right] \right|$$

$$\leqq \sup_{1-t \leqq \varepsilon \leqq 1} \sup_{0<y<\infty} \left| \frac{e^{y/(e^{1+\delta}-1)} g^{(N+k+1)}(x, y; \varepsilon)}{(N+1)!} \right|$$

$$\leqq \frac{B}{(N+1)!} \to 0 \quad \text{as } N \to \infty \qquad \text{(see Lemma 2)}.$$

Therefore, for $0 < t < 1$, we have the following result in the weak distributional sense:

$$e^{-t\nabla_x} g(x, y; 1) = g(x, y; 1 - t).$$

Now,

$$\text{(47)} \quad \sum_{r=0}^{N} \frac{(-t\nabla_x)^r}{r!} \langle \Lambda'(y) f(y), g(x, y; 1) \rangle = \left\langle \Lambda'(y) f(y), \sum_{r=0}^{N} \frac{(-t\nabla_x)^r}{r!} g(x, y; 1) \right\rangle$$

by Lemma 4. Letting $N \to \infty$ in (47), we obtain

$$\text{(48)} \qquad e^{-t\nabla_x} \langle \Lambda'(y) f(y), g(x, y; 1) \rangle = \langle \Lambda'(y) f(y), g(x, y; 1 - t) \rangle.$$

Now let $t \to 1-$. Therefore, for $\phi \in D(I)$ we have, by Theorem 1,

$$\langle e^{-\nabla_x} \langle \Lambda'(y) f(y), g(x, y; 1) \rangle, \phi(x) \rangle = \langle \Lambda' f, \phi \rangle.$$

This completes the proof of Theorem 2.

LEMMA 9. *Let $\phi(x)$ be an infinitely differentiable function defined over I such that for c and α both >0,*

$$\text{(49)} \qquad \sup_{0<x<\infty} | e^{cx} \nabla_x^k \phi(x) | < \infty \quad \textit{for } k = 0, 1, 2, 3, \cdots.$$

Then

(i) $\phi^{(k)}(0+)$ *exists*;

(ii) $\phi^{(k)}(x) = O[e^{-cx}]$ *as* $x \to \infty$, $k = 0, 1, 2, 3, \cdots$.

Proof. Since (49) is true for $k = 0$, $\phi(0+)$ exists. Let $x\phi''(x) + (\alpha + 1 - x)\phi'(x) = p(x)$; clearly, $|p(x)| \leqq M e^{-cx}$, $x > 0$,

$$\phi''(x) + \frac{\alpha + 1 - x}{x} \phi'(x) = \frac{p(x)}{x}.$$

An integrating factor of this differential equation is $x^{\alpha+1} e^{-x}$. Therefore,

$$\phi'(x) e^{-x} x^{\alpha+1} = \int_0^x p(x) e^{-x} x^\alpha \, dx + C,$$

$$\text{(50)} \qquad \phi'(x) = e^x x^{-(\alpha+1)} \int_0^x p(x) e^{-x} x^\alpha \, dx + C e^x x^{-(\alpha+1)}.$$

Hence,

$$\phi(x) = \int_0^x q(x)\,dx + C\int_a^x e^x x^{-(\alpha+1)}\,dx = d, \qquad a > 0, \tag{51}$$

where

$$q(x) = e^x x^{-(\alpha+1)} \int_0^x p(x)\,e^{-x} x^\alpha\,dx.$$

Obviously,

$$\lim_{x\to 0+} \int_0^x q(x)\,dx$$

exists, whereas

$$\lim_{x\to 0+} \int_a^x e^x x^{-(\alpha+1)}\,dx$$

does not. Therefore, in view of (51) and the fact that $\phi(0+)$ exists we can immediately conclude that $C = 0$. Consequently,

$$\phi'(x) = e^x x^{-(\alpha+1)} \int_0^x p(x)\,e^{-x} x^\alpha\,dx.$$

Therefore,

$$\begin{aligned}\phi''(x) = {} & -(\alpha+1)\,e^x x^{-(\alpha+2)} \int_0^x p(x)\,e^{-x} x^\alpha\,dx \\ & + \frac{p(x)}{x} + e^x x^{-(\alpha+1)} \int_0^x p(x)\,e^{-x} x^\alpha\,dx.\end{aligned} \tag{52}$$

Integrating the first expression in (52) by parts, we obtain

$$\begin{aligned}\phi''(x) = {} & e^x x^{-(\alpha+2)} \int_0^x x^{\alpha+1}\,e^{-x}\{p'(x) - p(x)\}\,dx \\ & + e^x x^{-(\alpha+1)} \int_0^x p(x)\,e^{-x} x^\alpha\,dx.\end{aligned} \tag{53}$$

By using the technique used in showing the existence of $\phi(0+)$ and $\phi'(0+)$ one can show that $p(0+)$ and $p'(0+)$ both exist.

Therefore, in view of (53) we conclude that $\phi''(0+)$ exists. Proceeding in this way by induction we can show that $\phi^{(k)}(0+)$ exists for $k = 3, 4, 5, \cdots$ as well. This completes the proof of (i). We now proceed to prove (ii).

An appeal to (49) leads to $|e^{cx}\phi(x)| < \infty$. Therefore,

$$\phi(x) = O[e^{-cx}], \qquad x \to \infty.$$

Let

$$x\phi''(x) + (\alpha + 1 - x)\phi'(x) = p(x). \tag{54}$$

Therefore,

$$|e^{cx}\nabla_x \phi(x)| = |e^{cx} p(x)| < \infty .$$

Hence,

$$p(x) = O[e^{-cx}], \qquad x \to \infty .$$

For $x_1, x_2 > 0$ we get by integrating (54):

$$x_2\phi'(x_2) - x_1\phi'(x_1) + \phi(x_1) - \phi(x_2) + (\alpha + 1 - x_2)\phi(x_2) - (\alpha + 1 - x_1)\phi(x_1)$$

$$+ \int_{x_1}^{x_2} \phi(x)\,dx = \int_{x_1}^{x_2} p(x)\,dx. \tag{55}$$

Now, let $x_1, x_2 \to \infty$ independently of each other in (55). Therefore, in view of the asymptotic order of $p(x)$ and $\phi(x)$ as $x \to \infty$ we have

$$\lim_{\substack{x_1 \to \infty \\ x_2 \to \infty}} |x_2\phi(x_2) - x_1\phi(x_1)| = 0.$$

Hence, in view of Cauchy's convergence criteria we conclude that $x\phi'(x)$ converges as $x \to \infty$. Let

$$\lim_{x \to \infty} x\phi'(x) = C.$$

Letting $x_1 \to \infty$ and replacing x_2 by x in (55), we have

$$x\phi'(x) = C + \phi(x) - (\alpha + 1 - x)\phi(x) + \int_x^\infty [\phi(x) - p(x)]\,dx. \tag{56}$$

Integrating (56) again between (x_1, x_2), we obtain

$$\begin{aligned} &x_2\phi(x_2) - x_1\phi(x_1) - \int_{x_1}^{x_2} \phi(x)\,dx \\ &\quad = C(x_2 - x_1) - \int_{x_1}^{x_2} (\alpha - x)\phi(x)\,dx + \int_{x_1}^{x_2} g(x)\,dx, \end{aligned} \tag{57}$$

where

$$g(x) = \int_x^\infty [\phi(x) - p(x)]\,dx.$$

Letting $x_1, x_2 \to \infty$ independently of each other in (57) we can show that C must be zero in (57). Therefore, from (56) we obtain

$$\phi'(x) = \frac{-(\alpha - x)\phi(x)}{x} + \frac{1}{x}\int_x^\infty [\phi(x) - p(x)]\,dx. \tag{58}$$

It is immediate from (58) that

$$\phi'(x) = O[e^{-cx}], \qquad x \to \infty .$$

Now it is clear from (54) that

$$\phi''(x) = O[e^{-cx}], \qquad x \to \infty .$$

Proceeding this way we can also prove by induction that

$$\phi^{(k)}(x) = O[e^{-cx}], \quad x \to \infty \quad \text{for } k = 3, 4, 5, \cdots.$$

This completes the proof of (ii).

4. The testing function space $S(I)$ of rapid descent. We say that an infinitely differentiable function $\phi(x)$ defined over I belongs to $S(I)$ if

$$\beta_{m,k}(\phi) = \sup_{0<x<\infty} |x^m \phi^{(k)}(x)| < \infty$$

for $m, k = 0, 1, 2, 3, \cdots$. The topology on $S(I)$ is generated by the sequence of seminorms $\{\beta_{m,k}\}_{m,k=0}^{\infty}$. We say that a sequence $\{\phi_\nu(x)\}_{\nu=1}^{\infty}$ converges to $\phi(x)$ in $S(I)$ if $\beta_{m,k}(\phi_\nu - \phi)$ goes to zero as $\nu \to \infty$ for each fixed m and k. We denote by $S'(I)$ the space of continuous linear functionals defined over $S(I)$. An element f of $S'(I)$ is known as the distribution of slow growth or tempered distribution. In view of Lemma 9 it can be readily seen that for $\alpha, \delta > 0$, $G_{\alpha,\delta}(I) \subset S(I)$ and that the convergence of a sequence in $G_{\alpha,\delta}(I)$ implies convergence in $S(I)$. Consequently the restriction of an element $f \in S'(I)$ to $G_{\alpha,\delta}(I)$ is in $G'_{\alpha,\delta}(I)$. Therefore, for $\alpha, \delta > 0$,

$$G'_{\alpha,\delta}(I) \supset S'(I).$$

The testing function space $G_\alpha^z(I)$. It can be readily seen that for $\delta_2 > \delta_1 > 0$ and $\alpha > -1$, $G_{\alpha,\delta_1}(I)$ is a subspace of $G_{\alpha,\delta_2}(I)$ and that the topology of $G_{\alpha,\delta_1}(I)$ is stronger than the topology induced on $G_{\alpha,\delta_1}(I)$ by $G_{\alpha,\delta_2}(I)$. Now let δ_ν be an increasing sequence of positive numbers tending to a positive quantity δ from the left as $\nu \to \infty$. Assuming

$$G_\alpha^\delta(I) = \bigcup_{\nu=1}^{\infty} G_{\alpha,\delta_\nu}(I),$$

we say that a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges in $G_\alpha^\delta(I)$ if it converges in some $G_{\alpha,\delta_m}(I)$ and hence in $G_{\alpha,\delta_{m+1}}(I), G_{\alpha,\delta_{m+2}}(I), \cdots$. We call $G_\alpha^\delta(I)$ the countable union space of $G_{\alpha,\delta_\nu}(I)$, $\delta_\nu \to \delta-$ [11, p. 15]. Clearly $G_\alpha^\delta(I) \subset G_{\alpha,\delta}(I)$.

We can also verify quite easily that if a sequence $\{\phi_\nu\}_{\nu=1}^{\infty}$ converges to ϕ in $G_\alpha^\delta(I)$, it also converges to ϕ in $G_{\alpha,\delta}(I)$. Consequently the restriction of any $f \in G'_{\alpha,\delta}(I)$ to $G_\alpha^\delta(I)$ is in $G_\alpha'^\delta(I)$. In the next theorem we determine a structure formula of a restriction of $f \in G'_{\alpha,\delta}(I)$ to $G_\alpha^\delta(I)$ when $\alpha, \delta > 0$.

We observe that if $\phi \in G_\alpha^\delta(I)$ for α, $\delta > 0$, then in view of Lemma 9, $\phi^{(k)}(0+)$ exists and $\phi^{(k)}(x) = O[e^{-x/(e^{1+\delta}-1)}]$, $x \to \infty$, for $k = 0, 1, 2, \cdots$.

THEOREM 3. *Let f be an arbitrary element of $G'_{\alpha,\delta}(I)$ and ϕ, an arbitrary element of $G_\alpha^\delta(I)$. Then there exist bounded and measurable functions $g_r(x)$ and polynomials $Q_r(x)$ for $r = 0, 1, 2, \cdots, 2k + 1$ satisfying the relation*

$$\langle f, \phi \rangle = \left\langle \sum_{r=0}^{2k+1} (-1)^r D^{r+1} \int_0^x g_r(x)\, e^{cx} Q_r(x)\, dx, \phi(x) \right\rangle. \tag{59}$$

Here $c = 1/(e^{1+\delta} - 1)$ and $\alpha > 0$.

Proof. Using the boundedness property of generalized functions, we obtain

$$|\langle f, \phi \rangle| \leqq C \sup_{0 \leqq r \leqq k} \gamma_r(\phi) \tag{60}$$

for appropriate constants C and k, or

$$\begin{aligned}|\langle f, \phi\rangle| &\leqq C \sup_{0<x<\infty} \sum_{r=0}^{k} |e^{cx}\nabla_x^r \phi(x)| \\ &\leqq C \sup_{0<x<\infty} \sum_{r=0}^{2k} |e^{cx}P_r(x)\phi^{(r)}(x)| \\ &\leqq C \sup_{0<x<\infty} \left| \int_x^\infty \sum_{r=0}^{2k} D(e^{cx}P_r(x)\phi^{(r)}(x)) \right| dx \\ &\leqq C \sup_{0<x<\infty} \int_x^\infty \sum_{r=0}^{2k+1} |e^{cx}Q_r(x)\phi^{(r)}(x)|\, dx.\end{aligned}$$

$P_r(x)$ and $Q_r(x)$ are obvious polynomials. Thus we have

$$\langle f, \phi\rangle| \leqq C \sum_{r=0}^{2k+1} \int_0^\infty |e^{cx}Q_r(x)\phi^{(r)}(x)|\, dx. \tag{61}$$

Therefore, in view of the Riesz representation theorem and Hahn–Banach theorem, we get bounded measurable functions $g_r(x), r = 0, 1, 2, \cdots, 2k+1$, satisfying

$$\langle f, \phi\rangle = \sum_{r=0}^{2k+1} \langle g_r(x), e^{cx}Q_r(x)\phi^{(r)}(x)\rangle. \tag{62}$$

But the regular distribution corresponding to the function $\int_0^x g_r(x)\, e^{cx}Q_r(x)\, dx$ belongs to $G_\alpha'^\delta(I)$. Therefore (62) can now be written as

$$\langle f, \phi\rangle = \left\langle \sum_{r=0}^{2k+1} (-1)^r D^{r+1} \int_0^x g_r(x)\, e^{cx}Q_r(x)\, dx, \phi(x)\right\rangle, \tag{63}$$

where the differentiation in (63) is done in the distributional sense. Observe that when $\phi(x) \in D(I)$, (59) is true for $f \in G'_{\alpha,\delta}(I)$, with $\alpha > -1$, $\delta > 0$.

THEOREM 4. *Let $u(x, t)$ be the dual Poisson–Laguerre transform of $f \in G'_{\alpha,\delta}(I)$, $\alpha > -1$, $\delta > 0$, with respect to the kernel $g_\alpha(x, y; t)$. Then $u(x, t)$ satisfies the (heat) equation*

$$\nabla_x u = \partial u/\partial t, \qquad 0 < t \leqq 1.$$

Proof. We have

$$u(x, t) = \langle \Lambda'(y)f(y), g(x, y; t)\rangle.$$

Therefore, by Lemmas 4 and 5,

$$\left(\nabla_x - \frac{\partial}{\partial t}\right)u(x, t) = \left\langle \Lambda'(y)f(y), \left(\nabla_x - \frac{\partial}{\partial t}\right)g(x, y; t)\right\rangle = 0.$$

This completes the proof of Theorem 4.

REFERENCES

[1] F. M. Cholewinski and D. T. Haimo, *The dual Poisson–Laguerre transform*, Trans. Amer. Math. Soc., 144 (1969), pp. 271–300.

[2] ———, *The Weierstrass–Hankel convolution transform*, J. Analyse Math., 17 (1966), pp. 1–58.

[3] F. M. Cholewinski, *Hankel convolution complex inversion theory*, Mem. Amer. Math. Soc., 58 (1965), 67 pp.

[4] D. T. Haimo, *Integral equations associated with Hankel convolutions*, Trans. Amer. Math. Soc., 116 (1965), pp. 330–375.

[5] J. N. Pandey and A. H. Zemanian, *Complex inversion for the generalized convolution transformation*, Pacific J. Math., 25 (1968), pp. 147–157.

[6] ———, *An extension of Haimo's form of Hankel convolutions*, Ibid., 28 (1969), pp. 641–651.

[7] ———, *Complex inversion for the generalized Hankel convolution transformation*, SIAM J. Appl. Math., 17 (1969), pp. 835–847.

[8] ———, *The generalized Weierstrass–Hankel convolution transform*, Ibid., (20) 1971, pp. 110–123.

[9] ———, *On the Stieltjes transform of generalized functions*, Proc. Cambridge Philos. Soc., to appear.

[10] ———, *A representation theorem for a class of convolution transformable generalized functions*, this Journal, 2 (1971), pp. 286–289.

[11] A. H. Zemanian, *Generalized Integral Transformations*, Interscience, New York, 1968.

[12] ———, *Distribution Theory and Transform Analysis*, McGraw-Hill, New York, 1965.

[13] L. Schwartz, *Théorie des distributions*, vols. I and II, Hermann, Paris, 1957 and 1959.

[14] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, vol. II, Academic Press, New York.

[15] S. Karlin and J. McGregor, *Classical diffusion process and total positivity*, J. Math. Anal. Appl., 1 (1960), pp. 163–183.

[16] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1958.

[17] K. Yosida, *Functional Analysis*, Springer-Verlag, New York, 1968.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

HARMONIC ANALYSIS OF A CLASS OF DISTRIBUTIONS ON $\mathbb{R}^n$, $n \geqq 2$*

YNGVE DOMAR†

Abstract. Let I be a compact subinterval of R_+. M. Gatesoupe proved in [2] that a complex-valued function $\hat{f}$ on I is the restriction, to the subset I of a radius, of a radial Fourier transform of a (radial) function in $L^1(\mathbb{R}^n)$ if and only if $\hat{f}$ is the restriction to I of the Fourier transform of a function $f \in L^1(\mathbb{R})$, satisfying

$$\int_{\mathbb{R}} |f(x)||x|^{(n-1)/2}\,dx < \infty.$$

He has, moreover, various extensions of his theorem. This paper is inspired by Gatesoupe's results, but gives a different and more general approach. We start from an open set $U \subset \mathbb{R}^n$, and a smooth one-parameter family of $(n-1)$-dimensional submanifolds of U with nonvanishing Gaussian curvature and consider distributions in $\mathscr{D}'(U)$ which are constant on each manifold, using a natural extension of the corresponding notion for functions. Localizing the support of the distribution by multiplying it by a function in $\mathscr{D}(U)$, we can characterize the asymptotic behavior at infinity of its Fourier transform by the behavior of the Fourier transform of the dilations of the corresponding one-dimensional distribution localized in a similar way. Restricting the discussion to the case in which the manifolds are concentric spheres we obtain a generalized version of Gatesoupe's result.

1. Introduction. Let Φ be a real-valued infinitely differentiable function on an open set $U \subset \mathbb{R}^n$, $n \geqq 1$. We assume that grad Φ does not vanish in U. For every $f \in \mathscr{D}'(\mathbb{R})$ it is then possible to define $f \circ \Phi \in \mathscr{D}'(U)$ in a way which is a natural extension of the ordinary composition of two functions. The precise procedure for this is described in § 2. For every $\varphi \in \mathscr{D}(U)$, $\varphi(f \circ \Phi)$ is well-defined as an element in $\mathscr{D}'(U)$, but we prefer to interpret it as an element in $\mathscr{D}'(\mathbb{R}^n)$ by extending it to $\mathbb{R}^n$ in such a way that it vanishes outside the support of φ. This is of course possible, and we obtain then a distribution $\varphi(f \circ \Phi) \in \mathscr{D}'(\mathbb{R}^n)$ with compact support.

Our aim is to investigate the Fourier transform of $\varphi(f \circ \Phi)$. We denote the elements in $\mathbb{R}^n$ by $x = (x_1, \cdots, x_{n-1}, t)$ and restrict the Fourier transform to the set of characters on $\mathbb{R}^n$ which take the value 1 on the set $\{x \in \mathbb{R}^n : t = 0\}$. The function $\hat{a}_\varphi$ on $\mathbb{R}$ which we obtain by this restriction can be regarded as the Fourier transform of the projection $a_\varphi \in \mathscr{D}'(\mathbb{R})$ of $\varphi(f \circ \Phi)$ onto the t-axis.

a_φ is compactly supported and hence $\hat{a}_\varphi$ has high differentiability properties and is of at most polynomial growth at infinity. We now assume that $n \geqq 2$ and denote by D the set of all $x \in U$ for which $\partial\Phi/\partial x_i = 0$ for every i, $1 \leqq i \leqq n-1$. Our first result is Theorem 2.7 which shows that the condition $D \cap \text{supp}(\varphi) = \varnothing$ implies that $a_\varphi = \mathscr{D}(\mathbb{R})$. Hence for functions φ of this kind, $\hat{a}_\varphi$ tends to 0 at infinity faster than any power of the norm of the variable.

The situation is more complicated when $D \cap \text{supp}(\varphi)$ is not empty. In order to explain our results in this case we need some further definitions. Let $x_0 \in D$ have the property that the determinant $|\partial^2\phi/\partial x_i \partial x_j|$, $1 \leqq i \leqq n-1$, $1 \leqq j \leqq n-1$, does not vanish at x_0. Classical results from the theory of implicit functions show then that D can be represented, locally at x_0, as a curve $\{(\xi(t), t), t \in J_0\}$, where J_0 is an open interval and where ξ is an infinitely differentiable function from J_0 to

* Received by the editors May 6, 1971.

† Department of Mathematics, Uppsala University, Sysslomansgatan 8, S-752 23 Uppsala, Sweden.

$\mathbb{R}^{n-1}$. We define Ψ as the function with values $\Phi(\xi(t), t)$, $t \in J_0$. It can be proved that Ψ' does not vanish on J_0. Hence we can define $f \circ \Psi \in \mathscr{D}'(J_0)$ and $b_\psi = \psi(f \circ \Psi) \in \mathscr{D}'(\mathbb{R})$, for $\psi \in \mathscr{D}(J_0)$, by applying the procedure described above, this time with $n = 1$. Since b_ψ has compact support it has a Fourier transform $\hat{b}_\psi$.

The discussion in § 3 concerns the asymptotic behavior at infinity of $\hat{a}_\varphi$, when the support of $\varphi \in \mathscr{D}(U)$ is included in a sufficiently small neighborhood of a point $x_0 \in D$ where $|\partial^2\Phi/\partial x_i \partial x_j|$ does not vanish. We prove Theorem 3.5 which states that $\hat{a}_\varphi$ can be estimated at infinity by means of finite sequences of associated functions $\hat{b}_\psi$, and that the remainder term can be forced to tend to 0 at infinity faster than any prescribed power of the norm of the variable. The pertinent relation is (3.3), where λ denotes the index of the characteristic values of the matrix

$$\left(-\frac{\partial\Phi}{\partial t}(x_0)\frac{\partial^2\Phi}{\partial x_i \partial x_j}(x_0)\right), \qquad 1 \leqq i \leqq n-1, \quad 1 \leqq j \leqq n-1,$$

nonsingular by our assumptions, and where δ represents a nonvanishing function in $C^\infty(\mathbb{R})$, such that $\delta(u) = \sqrt{iu}$ (principal value) if $u \in \mathbb{R}$, $|u| \geqq 1$.

Theorem 3.5 contains the essence of the paper, expressed however in a rather implicit form. Section 4 is devoted to the establishing of more explicit connections between the classes of $\hat{a}_\varphi$ and $\hat{b}_\psi$ which are associated with a given $f \in \mathscr{D}'(\mathbb{R})$ and a given $x_0 \in \mathscr{D}$. The results are expressed in terms of certain classes K of functions on $\mathbb{R}$, described in Definition 4.2. K can, for instance, be chosen to be $L^p(\mathbb{R})$, $1 \leqq p \leqq \infty$, as well as polynomially weighted L^p-spaces.

Theorem 4.3 states that if $\psi \in \mathscr{D}(J_0)$, $\psi(t_0) \neq 0$, where t_0 is the t-component of x_0, then x_0 has a neighborhood $U_0 \subset U$ such that

$$\hat{b}_\psi \in K \quad \text{implies} \quad \hat{a}_\varphi \delta^\lambda \bar{\delta}^{n-1-\lambda} \in K$$

if $\varphi \in \mathscr{D}(U)$ has support in U_0. Theorem 4.4 is a result of the converse type: If $\varphi \in \mathscr{D}(U)$, $\varphi(x_0) \neq 0$, and if x_0 is the only point in $D \cap \operatorname{supp}(\varphi)$ with t-coordinate t_0, then t_0 has a neighborhood $J_1 \subset J_0$ such that

$$\hat{a}_\varphi \delta^\lambda \bar{\delta}^{n-1-\lambda} \in K \quad \text{implies} \quad \hat{b}_\psi \in K$$

if $\psi \in \mathscr{D}(J_0)$ has support in J_1. Theorems 4.3 and 4.4 are of a local type, but it is very easy to construct global theorems out of them and Theorem 2.7, using the fact that $\hat{a}_\varphi$ and $\hat{b}_\psi$ are linear in φ and ψ, respectively. One such global result is given in Theorem 4.6. It concerns radial distributions and generalizes results of M. Gatesoupe [2]–[5]. It should be pointed out here that the origin of this investigation was a desire to discover whether Gatesoupe's results could be obtained in a more general setting. As mentioned in a concluding remark, there are many further possible applications for the methods and results of this paper.

2. Preliminaries. In this section we shall give some preliminary definitions and introduce notational conventions. We shall also formulate and prove Theorem 2.7.

Our assumption is that Φ is real-valued and infinitely differentiable in the open set $U \subset \mathbb{R}^n$, $n \geqq 1$, and that $\operatorname{grad} \Phi(x) \neq 0$ if $x \in U$. The following discussion is primarily applicable to the case $n \geqq 2$, but with a proper interpretation of the concepts involved it applies to the case $n = 1$ as well.

For every $c \in \mathbb{R}$ we put $E_c = \{x \in U : \Phi(x) = c\}$. From the assumptions on Φ and simple results from the theory of implicit functions we have that E_c is either empty or an $(n-1)$-dimensional infinitely differentiable submanifold of U, not necessarily connected but without multiple points and closed when considered as a subset of U. We have at the same time a dependence on c of very regular type. The precise properties of the family $\{E_c\}_{c \in \mathbb{R}}$ are expressed in the following well-known lemma, where we have adopted the notations:

$$x = (x_1, \cdots, x_n), \quad \xi_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),$$
$$(\xi_i, v) = (x_1, \cdots, x_{i-1}, v, x_{i+1}, \cdots, x_n), \qquad 1 \leqq i \leqq n.$$

LEMMA 2.1. *Let* $x_0 \in U$ *and* $(\partial\Phi/\partial x_i)(x_0) \neq 0$, *for a certain* i, $1 \leqq i \leqq n$. *Then there exist open sets* $V \subset \mathbb{R}^{n-1}$ *and* $I \subset \mathbb{R}$, *and an infinitely differentiable real-valued function* β_i *on* $V \times I$, *satisfying* $\Phi(\xi_i, \beta_i(\xi_i, c)) = c$ *on* $V \times I$, *and such that* $(\xi_i, c) \to (\xi_i, \beta_i(\xi_i, c))$ *is a bijection of* $V \times I$ *onto a neighborhood* $U_0 \subset U$ *of* x_0.

DEFINITION 2.2. For every $c \in \mathbb{R}$, ν_c denotes the uniform unit measure on E_c with respect to the Euclidean metric induced by the coordinates in $\mathbb{R}^n$. For every $\varphi \in \mathscr{D}(U)$, F_φ denotes the function on $\mathbb{R}$ with values

$$F_\varphi(c) = \int_U \varphi(x)|\operatorname{grad} \Phi(x)|^{-1}\, d\nu_c(x), \qquad c \in \mathbb{R}. \tag{2.1}$$

We shall show that $\varphi \to F_\varphi$ maps $\mathscr{D}(U)$ continuously into $\mathscr{D}(\mathbb{R})$ and we shall then need the following lemma.

LEMMA 2.3. *Let* $x_0 \in U$ *and let* i, V, I, β_i *and* U_0 *satisfy the conditions in Lemma* 2.1. *Then there exists a strictly positive function* $\gamma_i \in C^\infty(V \times I)$ *such that for every* $\varphi \in \mathscr{D}(U)$, *with* $\operatorname{supp}(\varphi) \subset U_0$,

$$F_\varphi(c) = \int_V \varphi(\xi_i, \beta_i(\xi_i, c))\gamma_i(\xi_i, c)\, d\xi_i \tag{2.2}$$

if $c \in I$, *while* $F_\varphi(c) = 0$ *if* $c \notin I$.

Proof. By the assumption, the support of φ is included in the image of $V \times I$ under the bijection mentioned in Lemma 2.1. Hence $\operatorname{supp}(\varphi)$ is disjoint from E_c, when $c \notin I$, and it follows therefore from (2.1) that $F_\varphi(c) = 0$ when $c \notin I$.

In the case $c \in I$, we use Lemma 2.1 which shows that we can express the right-hand side of (2.1) as an $(n-1)$-dimensional integral with ξ_i as integration variable. This gives (2.2), where

$$\gamma_i(\xi_i, c) = \left|\frac{\partial\Phi}{\partial x_i}(\xi_i, \beta_i(\xi_i, c))\right|^{-1} = \left|\frac{\partial\beta_i}{\partial c}(\xi_i, c)\right|$$

for $(\xi_i, c) \in V \times I$; hence $\gamma_i \in C^\infty(V \times I)$ and γ_i is strictly positive.

LEMMA 2.4. $\varphi \to F_\varphi$ *maps* $\mathscr{D}(U)$ *linearly and continuously into* $\mathscr{D}(\mathbb{R})$, *such that* $\operatorname{supp}(F_\varphi) \subset \Phi(\operatorname{supp}(\varphi))$. *Let* $f \in \mathscr{D}'(\mathbb{R})$. *The relation*

$$\langle f \circ \Phi, \varphi \rangle = \langle f, F_\varphi \rangle, \tag{2.3}$$

where φ *varies in* $\mathscr{D}(U)$, *defines a distribution* $f \circ \Phi \in \mathscr{D}'(U)$.

Proof. The second assertion follows obviously from the first by the definition of distributions in U as continuous linear functionals on $\mathscr{D}(U)$.

The linearity is evident, and the relations between the supports follow directly from (2.1). Hence it remains to show the continuity properties of the mapping $\varphi \to F_\varphi$. Using the linearity and a standard compactness argument, we see that it suffices to prove the following.

Every $x_0 \in U$ has a neighborhood $U_0 \subset U$ such that $\varphi \to F_\varphi$ is continuous when restricted to functions φ with $\operatorname{supp}(\varphi) \subset U_0$.

But grad Φ does not vanish. Hence there exist for every $x_0 \in U$ an index i and $V, I, \beta_i, U_0, \gamma_i$ satisfying the conditions in Lemma 2.1 and Lemma 2.3. Then the desired continuity of the restricted $\varphi \to F_\varphi$ follows from (2.2).

We shall now introduce some useful conventions in our notations. If W is an open subset in $\mathbb{R}^m$, $m \geqq 1$, we do not distinguish in our notation between a function in $\mathscr{D}(W)$, and the function in $\mathscr{D}(\mathbb{R}^m)$, which we obtain by extending the former function, defining it as 0 on $\mathbb{R}^m \setminus W$. For every $\varphi \in \mathscr{D}(W)$ and $g \in \mathscr{D}'(W)$, φg is a distribution in $\mathscr{D}'(W)$, but we use the notation φg as well for the distribution on $\mathscr{D}'(W)$ which we obtain by assigning the value 0 to points in $\mathbb{R}^m \setminus W$. Formally this extension is achieved by the defining relation

$$\langle \varphi g, \varphi_0 \rangle = \langle g, \varphi \varphi_0 \rangle, \qquad \varphi_0 \in \mathscr{D}(\mathbb{R}^m), \tag{2.4}$$

a relation which then can be interpreted to hold for $\varphi_0 \in C^\infty(\mathbb{R}^m)$ as well. In the following we shall also take the liberty of giving $\varphi(x)g(x)$ a meaning for every $x \in \mathbb{R}^m$, if $\varphi \in \mathscr{D}(W)$ and $g \in C^\infty(W)$, by interpreting it as 0 if $x \notin W$. Using this last convention we have that (2.2) holds for every $c \in \mathbb{R}$, due to the fact that $(\xi_i, c) \to \varphi(\xi_i, \beta_i(\xi_i, c))$ is a function in $\mathscr{D}(V \times I)$.

In the following we shall let the nth coordinate x_n have a special standing, and it is convenient to put $x_n = t$. We always use the same notation for a function $\varphi_0 \in C^\infty(\mathbb{R})$ as for the function in $C^\infty(\mathbb{R}^n)$ with values $\varphi_0(t)$ for every $x \in \mathbb{R}^n$.

In all that follows, except for the last section, we shall keep $f \in \mathscr{D}'(\mathbb{R})$ fixed. For every $\varphi \in \mathscr{D}(U)$, $\varphi(f \circ \Phi)$ is of course compactly supported.

LEMMA 2.5. *For every $\varphi \in \mathscr{D}(U)$, the relation*

$$\langle a_\varphi, \varphi_0 \rangle = \langle \varphi(f \circ \Phi), \varphi_0 \rangle, \tag{2.5}$$

where the left-hand φ_0 varies in $C^\infty(\mathbb{R})$, defines a compactly supported distribution a_φ in $\mathscr{D}'(\mathbb{R})$. For every $\psi \in C^\infty(\mathbb{R})$,

$$a_{\varphi\psi} = \psi a_\varphi. \tag{2.6}$$

Proof. The right-hand φ_0 in (2.5) should here be interpreted as a function in $C^\infty(\mathbb{R}^n)$. $\varphi_0 \to 0$ in $C^\infty(\mathbb{R})$ implies that $\varphi_0 \to 0$ in $C^\infty(\mathbb{R}^n)$, and this proves the first statement. Furthermore, if $\psi \in C^\infty(\mathbb{R})$ and $\varphi_0 \in C^\infty(\mathbb{R})$, (2.5) shows that

$$\begin{aligned} \langle a_{\varphi\psi}, \varphi_0 \rangle &= \langle \psi\varphi(f \circ \Phi), \varphi_0 \rangle = \langle \varphi(f \circ \Phi), \psi\varphi_0 \rangle \\ &= \langle a_\varphi, \psi\varphi_0 \rangle = \langle \psi a_\varphi, \varphi_0 \rangle, \end{aligned}$$

and this proves (2.6).

Remark. The distribution a_φ can be described as the projection of $\varphi(f \circ \Phi)$ onto the t-axis.

We assume in the following that $n \geqq 2$ and introduce a certain exceptional subset D of U.

DEFINITION 2.6. D is the set of all $x \in U$ for which $(\partial\Phi/\partial x_i)(x)$ vanishes for every i, $1 \leqq i \leqq n-1$.

THEOREM 2.7. *Let* $f \in \mathscr{D}'(\mathbb{R})$, $\varphi \in \mathscr{D}(U)$, $D \cap \operatorname{supp}(\varphi) = \varnothing$. a_φ *is defined by Lemma* 2.5. *Then* $a_\varphi \in \mathscr{D}(\mathbb{R})$.

Proof. In the following the notations are adopted for the case $n \geqq 3$, but the arguments can, properly interpreted, be applied to the case $n = 2$ as well.

By (2.5) and (2.3) we have, for every $\varphi \in \mathscr{D}(U)$, $\varphi_0 \in C^\infty(\mathbb{R})$,

$$\langle a_\varphi, \varphi_0 \rangle = \langle f \circ \Phi, \varphi\varphi_0 \rangle = \langle f, F_{\varphi\varphi_0} \rangle. \tag{2.7}$$

Take $x_0 \in D$. Then there exists an index i, $1 \leqq i \leqq n-1$, such that $(\partial\Phi/\partial x_i)(x_0) \neq 0$. We choose V, I, β_i, U_0 and γ_i so that all conditions in Lemma 2.1 and Lemma 2.3 are satisfied, and we assume that $\varphi \in \mathscr{D}(U_0)$. As mentioned above our conventions give that (2.2), with $\varphi\varphi_0$ instead of φ, holds for every $c \in \mathbb{R}$; thus

$$F_{\varphi\varphi_0}(c) = \int_V \varphi_0(t)\varphi((\eta_i, t), \beta_i((\eta_i, t), c))\gamma_i((\eta_i, t), c)\, d\eta_i\, dt, \qquad c \in \mathbb{R},$$

where

$$\eta_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n-1})$$

and

$$(\eta_i, t) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n-1}, t).$$

We thus obtain

$$F_{\varphi\varphi_0}(c) = \int_{\mathbb{R}^{n-1}} \varphi_0(t)\varphi_1(\eta_i, t, c)\, d\eta_i\, dt, \qquad c \in \mathbb{R}, \tag{2.8}$$

where $\varphi_1 \in \mathscr{D}(\mathbb{R}^n)$. Equation (2.8) can be written

$$F_{\varphi\varphi_0}(c) = \int_{\mathbb{R}} \varphi_0(t)\varphi_2(t, c)\, dt,$$

where $\varphi_2 \in \mathscr{D}(\mathbb{R}^2)$. Hence (2.7) gives

$$\langle a_\varphi, \varphi_0 \rangle = \int_{\mathbb{R}} \langle f, \varphi_2(t, \cdot) \rangle \varphi_0(t)\, dt.$$

The function φ_3 on $\mathbb{R}$ with values $\langle f, \varphi_2(t, \cdot)\rangle$, $t \in \mathbb{R}$, belongs to $\mathscr{D}(\mathbb{R})$. Varying φ_0, we find that $a_\varphi = \varphi_3$.

This proves Theorem 2.7 in the case when $\varphi \in \mathscr{D}(U_0)$. We had chosen x_0 quite arbitrarily in the complement of D. Since a_φ is linear in φ, the usual compactness argument proves the theorem in the general situation.

Remark. Theorem 2.7 implies that under the assumptions of that theorem, the Fourier transform $\hat{a}_\varphi$ of a_φ tends to 0 at infinity faster than any power of the norm of the variable.

3. An asymptotic expansion. We assume in this and the following sections that $n \geqq 2$ and use the notations and conventions introduced in § 2. For a given $f \in \mathscr{D}'(\mathbb{R})$ we continue the investigation of the associated functions a_φ, $\varphi \in \mathscr{D}(U)$, but we now restrict the attention to points in D and to corresponding functions φ with support in prescribed neighborhoods of these points.

It is convenient to put $\xi = (x_1, \cdots, x_{n-1})$, $x = (\xi, t)$, thus writing ξ instead of ξ_n. We also write $x_0 = (\xi_0, t_0)$, $c_0 = \Phi(x_0)$.

LEMMA 3.1. *Let $x_0 \in D$ have the property that the determinant*

$$\left|\frac{\partial^2\Phi}{\partial x_i \partial x_j}(x_0)\right|, \qquad 1 \leqq i \leqq n-1, \quad 1 \leqq j \leqq n-1,$$

does not vanish. Then there exist an open interval I_0, containing c_0, and an infinitely differentiable function $\alpha = (\alpha_0, \alpha_n)$ from I_0 to $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, where α_n corresponds to the t-component in $\mathbb{R}^n$, and with the following properties:

α_n has nonvanishing derivative. For every sufficiently small neighborhood V of $\xi_0 \in \mathbb{R}^{n-1}$ there exists an open interval I with $c_0 \in I \subset \bar{I} \subset I_0$, such that if we put $J = \alpha_n(I)$, then

$$(V \times J) \cap D \cap E_c = \varnothing \qquad \text{if} \quad c \notin I,$$

$$(V \times J) \cap D \cap E_c = \{\alpha(c)\} \quad \text{if} \quad c \in I.$$

Proof. The points in $D \cap E_c$ are obtained by solving the system of equations

$$\begin{cases} \Phi(x) = c, & \\ \dfrac{\partial\Phi}{\partial x_i}(x) = 0, & 1 \leqq i \leqq n-1. \end{cases}$$

By the theory of implicit functions we have, locally at x_0, a unique solution of the form $x = \alpha(c)$, with infinitely differentiable α, if

$$(3.1) \qquad \begin{vmatrix} \dfrac{\partial\Phi}{\partial x_1} & \dfrac{\partial\Phi}{\partial x_2} & \cdots & \dfrac{\partial\Phi}{\partial x_n} \\ \dfrac{\partial^2\Phi}{\partial x_1\,\partial x_1} & \dfrac{\partial^2\Phi}{\partial x_1\,\partial x_2} & \cdots & \dfrac{\partial^2\Phi}{\partial x_1\,\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial^2\Phi}{\partial x_{n-1}\,\partial x_1} & \dfrac{\partial^2\Phi}{\partial x_{n-1}\,\partial x_2} & \cdots & \dfrac{\partial^2\Phi}{\partial x_{n-1}\,\partial x_n} \end{vmatrix} \neq 0$$

at x_0. But

$$\frac{\partial\Phi}{\partial x_1}(x_0) = \frac{\partial\Phi}{\partial x_2}(x_0) = \cdots = \frac{\partial\Phi}{\partial x_{n-1}}(x_0) = 0, \quad \frac{\partial\Phi}{\partial x_n}(x_0) \neq 0.$$

Hence our assumption shows that (3.1) holds at x_0.

From this, all the statements in the lemma follow by elementary arguments. The nonvanishing of α_n' is thus a consequence of the relation

$$\frac{\partial\Phi}{\partial x_n}(\alpha(c))\alpha_n'(c) = 1,$$

obtained by differentiating the first equation in the system.

We associate in the following with each $x_0 \in D$, where $|\partial^2\Phi/\partial x_i\,\partial x_j| \neq 0$, a fixed interval I_0, satisfying the properties of Lemma 3.1. For every such x_0 we give the following definition.

DEFINITION 3.2. Ψ denotes the inverse of the function α_n on I_0, defined in Lemma 3.1. Ψ is by the lemma an infinitely differentiable function on $J_0 = \alpha_n(I_0)$.

For every $f \in \mathscr{D}'(\mathbb{R})$, $f \circ \Psi$ is the distribution in $\mathscr{D}'(J_0)$ which we obtain by applying Lemma 2.4 in the case $n = 1$. For every $\psi \in \mathscr{D}(J_0)$ we put $b_\psi = \psi(f \circ \Psi)$.

We need a remark on this definition due to the circumstance that the discussion leading up to Lemma 2.4 is formulated with regard to the case $n \geqq 2$. In analogy to (2.3) the formal definition of $f \circ \Psi$ is given by

$$\langle f \circ \Psi, \psi \rangle = \langle f, G_\psi \rangle, \tag{3.2}$$

$\psi \in \mathscr{D}(J_0)$, where, in analogy with (2.1),

$$G_\psi(c) = \psi(\alpha_n(c))|\alpha_n'(c)| \tag{3.3}$$

for every c, if according to our conventions we interpret the right-hand side of (3.3) as 0 if $c \notin I_0$.

As was mentioned in the Introduction, our main object in this investigation is to discuss the asymptotic behavior at infinity of the Fourier transform of the distribution $\varphi(f \circ \Phi)$, where $\varphi \in \mathscr{D}(U)$. The Fourier transform is of course a function on $\mathbb{R}^n$, but we restrict the study to the one-dimensional subspace of characters which take the value 1 for points $x = (\xi, 0)$, $\xi \in \mathbb{R}^{n-1}$. Thus we discuss only the values of the Fourier transform in the t-direction. The Fourier transform, restricted in this way, is by definition the function with values

$$\langle \varphi(f \circ \Phi), \chi_u \rangle, \qquad u \in \mathbb{R},$$

where for every $u \in \mathbb{R}$,

$$\chi_u(x) = e^{-2\pi i t u}, \qquad x \in \mathbb{R}^n. \tag{3.4}$$

Hence, by (2.5), we have the following lemma.

LEMMA 3.3. *The restriction of the Fourier transform of $\varphi(f \circ \Phi)$ to the characters with value 1 at the points $(\xi, 0)$, $\xi \in \mathbb{R}^{n-1}$, coincides with the Fourier transform $\hat{a}_\varphi$ of a_φ.*

Lemma 3.3 gives the main motivation for our study of a_φ and its Fourier transform. We remind the reader of the remark after Theorem 2.7, which states that if $D \cap \operatorname{supp}(\varphi) = \varnothing$, then $\hat{a}_\varphi$ tends to 0 at infinity faster than any inverted polynomial. We shall now formulate and prove Theorem 3.5, which shows that if $x_0 \in D$ satisfies the conditions in Lemma 3.1, and if the support of φ is included in a sufficiently small neighborhood of x_0, then $\hat{a}_\varphi$ can be asymptotically estimated in terms of functions $\hat{b}_\psi$, $\psi \in \mathscr{D}(J_0)$. In order to describe the asymptotic relation, we introduce a fixed function δ, given by the following definition.

DEFINITION 3.4. δ is a nonvanishing function in $C^\infty(\mathbb{R})$ such that it satisfies, for $|u| \geqq 1$, the condition

$$\delta(u) = \sqrt{iu},$$

where we take the principal value of the square root. Thus

$$\delta(u) = \sqrt{u}\, e^{i\pi/4} \quad \text{if} \quad u \geqq 1,$$
$$\delta(u) = \sqrt{|u|}\, e^{-i\pi/4} \quad \text{if} \quad u \leqq 1.$$

THEOREM 3.5. *Let $x_0 \in D$, and assume that the matrix*

$$\left(-\frac{\partial \Phi}{\partial x_n} \frac{\partial^2 \Phi}{\partial x_i \, \partial x_j}\right), \qquad 1 \leqq i \leqq n-1, \quad 1 \leqq j \leqq n-1,$$

is nonsingular at x_0. Let λ denote the index of its characteristic roots.

Then there exist a neighborhood $J_1 \subset J_0$ of t_0, a neighborhood V_1 of ξ_0, $V_1 \times J_1 \subset U$, and a nonvanishing function $\omega \in C^\infty(J_1)$ such that, given any open set $J \subset J_1$ and any $\varphi \in \mathscr{D}(V_1 \times J)$, there exist $(\psi_p)_0^\infty$, $\psi_p \in \mathscr{D}(J)$, with $\psi_0 = \omega(\varphi \circ \alpha \circ \Psi)$, and with the following properties:

We define, for every nonnegative integer q the function S_q on $\mathbb{R}$ by

$$\hat{a}_\varphi \delta^\lambda \bar{\delta}^{n-1-\lambda} = \sum_{p=0}^{q-1} \hat{b}_{\psi_p} \delta^{-2p} + S_q. \tag{3.5}$$

Then, for every given $m > 0$, $\quad S_q(u) = O(u^{-m})$

as $|u| \to \infty$, if q is chosen large enough.

Remark. Since $(\partial\Phi/\partial x_n)(x_0) \neq 0$ if $x_0 \in D$, the nonsingularity of the matrix is equivalent to the assumption in Lemma 3.1.

In the proof of Theorem 3.5 we need the following lemma, essentially due to M. Morse and J. Milnor.

LEMMA 3.6. *Let A be a real-valued function in $C^\infty(W_1 \times I_1)$, where $W_1 \times I_1$ is a neighborhood of $(0, c_0) \in \mathbb{R}^{n-1} \times \mathbb{R}$, such that $(\partial A/\partial x_i)(0, c) = 0, c \in I_1$, for every i with $1 \leqq i \leqq n-1$. Furthermore, we assume that $(\partial^2 A/\partial x_i\, \partial x_j)$, $1 \leqq i \leqq n-1$, $1 \leqq j \leqq n-1$, is nonsingular at $(0, c_0)$ and the index of its set of characteristic roots is denoted by λ.*

Then there is a neighborhood $V_2 \times I_2$ of $(0, c_0)$, where we can change the coordinates by a bijective mapping $(x, c) \to (y, c)$, infinitely differentiable as well as its inverse, such that the c-coordinates are unchanged and the points $(0, c)$ are fixed, and such that the values of A in $V_2 \times I_2$ can be expressed as $A(0, c) + B(y)$, where B is a nondegenerate quadratic form of index λ.

Proof of Lemma 3.6. The only difference between this lemma and Lemma 2.1 in Milnor [7] is the presence of the parameter c and the requirement that the mapping and its inverse be infinitely differentiable functions of (x, c) and (y, c) respectively, instead of differentiable functions of x and y, respectively. Milnor's method to construct new coordinates can, however, be applied also in our case, and it gives the desired differentiability.

Proof of Theorem 3.5. By (2.7) we have for $\varphi \in \mathscr{D}(U)$, $u \in \mathbb{R}$,

$$\hat{a}_\varphi(u) = \langle a_\varphi, \chi_u \rangle = \langle f, F_{\varphi\chi_u} \rangle. \tag{3.6}$$

Here χ_u is defined by (3.4). We now apply Lemma 2.3 in the case when $i = n$, and put $\beta_n = \beta$, $\gamma_n = \gamma$. If U_0 is a sufficiently small neighborhood of x_0 and $\varphi \in \mathscr{D}(U_0)$,

$$F_{\varphi\chi_u}(c) = \int_{\mathbb{R}^{n-1}} e^{-2\pi i u \beta(\xi, c)} \varphi(\xi, \beta(\xi, c)) \gamma(\xi, c)\, d\xi \tag{3.7}$$

for every $c \in \mathbb{R}$. We recall here the conventions by which we interpret the integrand as 0 for those $(\xi, c) \in \mathbb{R}^n$ which are not contained in the support of $(\xi, c) \to \varphi(\xi, \beta(\xi, c))$. With the notation taken from Lemma 3.1 we can write (3.7) in the form

$$F_{\varphi\chi_u}(c) = \int_{\mathbb{R}^{n-1}} e^{-2\pi i u \beta(\alpha_0(c) + \xi_0, c)} \varphi(\alpha_0(c) + \xi_0, \beta(\alpha_0(c) + \xi_0, c)) \gamma(\alpha_0(c) + \xi_0, c)\, d\xi_0, \tag{3.8}$$

$c \in \mathbb{R}$, assuming that U_0 was chosen so small that $\alpha_0(c)$ is well-defined for all (ξ, c) in the support of $(\xi, c) \to \varphi(\xi, \beta(\xi, c))$.

We now denote by A the function $(\xi, c) \to \beta(\xi + \alpha_0(c))$, defined in a neighborhood of $(0, c_0) \in \mathbb{R}^n$. By Lemma 2.1 we have in this neighborhood

$$\Phi(\xi + \alpha_0(c), A(\xi, c)) = c.$$

Differentiating, we obtain in this neighborhood

$$\frac{\partial \Phi}{\partial x_i}(\xi + \alpha_0(c), A(\xi, c)) + \frac{\partial \Phi}{\partial x_n}(\xi + \alpha_0(c), A(\xi, c))\frac{\partial A}{\partial x_i}(\xi, c) = 0, \qquad 1 \leqq i \leqq n - 1. \tag{3.9}$$

But $A(0, c) = \alpha_n(c)$, and hence (3.9) gives that

$$\frac{\partial A}{\partial x_i}(0, c) = 0, \qquad 1 \leqq i \leqq n - 1,$$

due to the fact that $\alpha(c) \in D$. Differentiating (3.9) once more we easily obtain

$$\frac{\partial^2 \Phi}{\partial x_i \, \partial x_j}(\alpha(c)) + \frac{\partial \Phi}{\partial x_n}(\alpha(c))\frac{\partial^2 A}{\partial x_i \, \partial x_j}(0, c) = 0, \qquad 1 \leqq i, j \leqq n - 1.$$

Hence $(\partial^2 A/\partial x_i \, \partial x_j)(0, c_0)$ is a nonsingular matrix with index λ, with λ defined as in the formulation of the theorem.

We are therefore in a position to apply Lemma 3.6 in order to perform a transformation of (3.8). We see from this lemma that there exist a neighborhood $V_1 \times I_1$ of $(0, c_0)$ in $\mathbb{R}^{n-1} \times \mathbb{R}$, with $I_1 \subset I_0$ (I_0 defined by Lemma 3.1), an infinitely differentiable bijection ρ of $V_1 \times I_1$ onto a neighborhood of $x_0 \in U$, and a positive function $\tau \in C^\infty(V_1 \times I_1)$, such that, for $\varphi \in \mathscr{D}(\rho(V_1 \times I_1))$,

$$F_{\varphi\chi_u}(c) = e^{-2\pi i u \alpha_n(c)} \int_{\mathbb{R}^{n-1}} e^{-2\pi i u B(y)} \varphi(\rho(y, c))\tau(y, c) \, dy, \qquad u \in \mathbb{R}, \quad c \in \mathbb{R}, \tag{3.10}$$

where $B(y)$ is a nonsingular quadratic form of index λ, and such that

$$\rho(0, c) = (\alpha_0(c), \beta(\alpha_0(c), c)) = \alpha(c) \tag{3.11}$$

if $c \in I_1$. The representation of $F_{\varphi\chi_u}$ by means of (3.10) instead of (3.7) facilitates to a considerable extent the discussion of the behavior of $F_{\varphi\chi_u}$ for large $|u|$, due to the fact that we have an explicit expression of the Fourier transform of the function with values $e^{-2\pi i B(y)}$, $y \in \mathbb{R}^{n-1}$.

We assume in the following that $u \in \mathbb{R}$, $|u| \geqq 1$. It is well known from the elementary theory of distributions, that there exist positive constants a and b, both independent of u, such that the function $z \to e^{-2\pi i u z^2}$ on $\mathbb{R}$ has as Fourier transform the function $z \to a(\delta(u))^{-1} e^{ibz^2 u^{-1}}$ on $\mathbb{R}$. It follows from this that there exist a constant $C_0 \neq 0$ and a real quadratic form $C(\eta)$ on $\mathbb{R}^{n-1}$, both independent of u, such that the function

$$y \to e^{-2\pi i u B(y)}, \qquad y \in \mathbb{R}^{n-1},$$

has the Fourier transform

$$\eta \to C_0(\delta(u))^{-\lambda}(\bar{\delta}(u))^{-(n-1-\lambda)} e^{iC(\eta)u^{-1}}, \qquad \eta \in \mathbb{R}^{n-1}.$$

For every $(\eta, c) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we define

$$h(\eta, c) = \int_{\mathbb{R}^{n-1}} e^{-2\pi i\langle \eta, y\rangle} C_0 \varphi(\rho(y, c))\tau(y, c)\, dy. \tag{3.12}$$

h is then a function in $\mathscr{S}(\mathbb{R}^{n-1} \times \mathbb{R})$, vanishing on $\{(\eta, c) | \eta \in \mathbb{R}^{n-1}, c \notin I_1\}$. Parseval's relation applied to (3.10) gives

$$F_{\varphi\chi_u}(c)\delta(u)^{\lambda}\bar{\delta}(u)^{n-1-\lambda} = e^{-2\pi i u \alpha_n(c)} \int_{\mathbb{R}^{n-1}} e^{iC(\eta)u^{-1}} h(\eta, c)\, d\eta, \qquad c \in \mathbb{R}. \tag{3.13}$$

Let us put

$$h_p(c) = \int_{\mathbb{R}^{n-1}} \frac{(iC(\eta))^p}{p!} h(\eta, c)\, d\eta$$

for every $c \in \mathbb{R}$ and for every nonnegative integer p. Using the inverse relation to (3.12) we see that $h_p(c)$ can be expressed as a finite linear combination of partial derivatives of $\varphi(\rho(y, c))\tau(y, c)$, c treated as a constant, at $y = 0$. By (3.11) we see that the support of h_p is contained in the union of the supports of $c \to \psi_0(\alpha(c))$, where ψ_0 runs through the set of all partial derivatives of φ. Hence, if $J \subset J_1 = \alpha_n(I_1)$ and $\varphi \in \mathscr{D}(V_1 \times J)$, we can conclude that the functions ψ_p defined by

$$\psi_p(t) = h_p(\Psi(t))\Psi'(t) \tag{3.14}$$

on J_1, as 0 outside J_1, belong to $\mathscr{D}(J)$. We have, moreover, by (3.10) and (3.12) the relation

$$\psi_0(t) = h_0(\Psi(t))\Psi'(t) = C_0\varphi(\alpha(\Psi(t)))\tau(0, \Psi(t))\Psi'(t)$$

for $t \in J$. These results show that the sequence $(\psi_p)_0^\infty$ fulfills all the requirements in the theorem except for the asymptotic relation (3.5) and we shall now prove that relation.

We turn back to (3.13) which can be written, for every nonnegative integer q,

$$F_{\varphi\chi_u}(c)\delta(u)^{\lambda}\bar{\delta}(u)^{n-1-\lambda} = e^{-2\pi i u \alpha_n(c)} \sum_{p=0}^{q-1} \delta(u)^{-2p} h_p(c) + R_q(u, c), \tag{3.15}$$

$|u| \geqq 1$, $c \in \mathbb{R}$, where

$$R_q(u, c) = e^{-2\pi i u \alpha_n(c)} \int_{\mathbb{R}^{n-1}} \left(e^{iC(\eta)u^{-1}} - \sum_{p=0}^{q-1} \frac{(iC(\eta)u^{-1})^p}{p!} \right) h(\eta, c)\, d\eta. \tag{3.16}$$

Equation (3.12) shows that, for every nonnegative integer s,

$$\partial^s h/\partial c^s \in \mathscr{S}(\mathbb{R}^{n-1} \times \mathbb{R})$$

and $\partial^s h/\partial c^s$ vanishes on $\{(\eta, c)/\eta \in \mathbb{R}^{n-1}, c \notin I_1\}$, and applying this to (3.16) we easily find that there exists, for every positive integer r, a constant C_r, not depending on u and c, such that

$$\left| \frac{\partial^s}{\partial c^s} R_{2r}(u, c) \right| \leqq C_r |u|^{-r}$$

if $0 \leqq s \leqq r$, $c \in \mathbb{R}$, $|u| \geqq 1$, and such that R_{2r} vanishes on $\{(u, c)\ |u| \geqq 1,\ c \notin I_1\}$. We obtain from this, (3.6) and (3.15) that

$$
\begin{aligned}
\hat{a}_\varphi(u)\delta(u)^\lambda\bar{\delta}(u)^{n-1-\lambda} &= \langle f, F_{\varphi\chi_u}\delta(u)^\lambda\bar{\delta}^{n-1-\lambda}\rangle \\
&= \sum_{p=0}^{q-1} \langle f, e^{-2\pi i u\alpha_n(\cdot)}h_p\rangle\delta(u)^{-2p} + S_q(u), \qquad |u| \geqq 1,
\end{aligned}
\tag{3.17}
$$

where, for every positive m, $S_q(u)|u|^m$ is bounded in $\{u \| u| > 1\}$, if q is chosen large enough. By (3.14) we have for every p,

$$\langle f, e^{-2\pi i u\alpha_n(\cdot)}h_p\rangle = \langle f, e^{-2\pi i u\alpha_n(\cdot)}\psi_p(\alpha_n(\cdot))|\alpha_n'(\cdot)|\rangle,$$

and by (3.2) and (3.3) the right-hand side equals

$$
\begin{aligned}
\langle f, e^{-2\pi i u\alpha_n(\cdot)}G_{\psi p}(\cdot)\rangle &= \langle f \circ \Psi, \psi_p\chi_u\rangle \\
&= \langle \psi_p(f \circ \Psi), \chi_u\rangle = \langle b_{\psi_p}, \chi_u\rangle = \hat{b}_{\psi_p}(u), \qquad |u| \geqq 1.
\end{aligned}
$$

Hence we see from (3.17) that (3.5) holds, where S_q has the desired properties, and this concludes the proof of the theorem.

Remark. Theorem 3.5 can be regarded as a generalization of asymptotic estimates made by W. Littman [6]. Littman's result is obtained by choosing the distribution f as the Dirac measure.

4. Applications of the asymptotic formula. In this section we shall continue the discussion of the classes of functions $\hat{a}_\varphi$ and $\hat{b}_\psi$ which we obtain, starting from a fixed $f \in \mathscr{D}'(\mathbb{R})$. The relations between the classes will be expressed by a family of function spaces which is introduced by the following two definitions.

Definition 4.1. K_0 is the space of all functions in $C^\infty(\mathbb{R})$ which have all their derivatives of at most polynomial growth.

Definition 4.2. K is a linear space of complex-valued functions on $\mathbb{R}$ with the following properties:

1. Let $*$ denote ordinary convolution and let φ be the Fourier transform of a function in $\mathscr{D}(\mathbb{R})$. Then $k \in K \cap K_0$ implies that $k * \varphi \in K$.
2. $k \in K \cap K_0$ implies that $k\delta^{-2} \in K$.
3. There exists a positive integer l such that $k \in K_0$, $k\delta^l \in L^\infty(\mathbb{R})$ implies that $k \in K$.

Remark. The operation $k * \varphi$ in Property 1 of Definition 4.2 makes sense due to the fact that all functions in K_0 are of at most polynomial growth. Obviously $k * \varphi \in K_0$.

In order to indicate the applicability of our results we give here some examples of spaces K.

(a) $K = L^p(\mathbb{R})$, where $1 \leqq p \leqq \infty$.

(b) Given real numbers α and p, $1 \leqq p \leqq \infty$, K is the space of functions k such that $k|\delta|^\alpha \in L^p(\mathbb{R})$.

(c) K is the space of functions on $\mathbb{R}$ for which the restrictions to $]-\infty, 0]$ and $[0, \infty[$ belong to $L^1(-\infty, 0)$ and $L^\infty(0, \infty)$, respectively.

(d) Given a real number α and a nonnegative integer q, K is the space of measurable functions k, for which there exist complex coefficients $(c_p)_0^{q-1}$, depending on k, such that

$$k(u) = \sum_{p=0}^{q-1} c_p|\delta(u)|^\alpha \delta(u)^{-2p} + O(|\delta(u)|^{\alpha-2q})$$

as $|u| \to \infty$.

LEMMA 4.3. *Let α be real and p an integer. Then, for every Fourier transform φ of a function in $\mathscr{D}(\mathbb{R})$, $k \in K \cap K_0$ implies that*

$$|\delta|^\alpha \delta^p(\varphi * (|\delta|^{-\alpha}\delta^{-p}k)) \in K. \tag{4.1}$$

Furthermore, the space of all k such that $|\delta|^\alpha \delta^p k \in K$ is itself a space of type K, that is, it satisfies Definition 4.2.

Proof. The only problem concerning the second statement is the verification of Definition 4.2, Property 1, for the space in question, but this is a direct consequence of the first statement. The latter will now be proved.

Let us choose the positive integer m so large that $k\delta^{-2m}$ is a bounded function. This is possible since $k \in K_0$; hence it is of at most polynomial growth. q is a positive integer to be determined later. A Maclaurin expansion gives

$$(1 - z)^{-(\alpha+p)/2} = \sum_{\nu=0}^{q-1} c_\nu z^\nu + O(z^q)$$

if $z \in \mathbb{R}$, $|z| \leqq 1/2$, where $(c_\nu)_0^{q-1}$ are coefficients independent of z and q. From this we obtain for $u \in \mathbb{R}$, $v \in \mathbb{R}$, putting $z = iv/iu$,

$$|\delta(u - v)|^{-\alpha}\delta(u - v)^{-p} = |\delta(u)|^{-\alpha}\delta(u)^{-p}\left(\sum_{\nu=0}^{q-1} c_\nu(iv)^\nu \delta(u)^{-2\nu}\right) + R_q(u, v), \tag{4.2}$$

where, for some constant C_q,

$$|R_q(u, v)| \leqq C_q|v|^q|u|^{-q-(\alpha+p)/2} \tag{4.3}$$

if $|u| \geqq 2$, $|v| \leqq \frac{1}{2}|u|$. It is easily seen from (4.2) that

$$|R_q(u, v)| \leqq C_q|v|^{q+|\alpha+p|/2-1} \tag{4.4}$$

if $|u| \geqq 2$, $|v| > \frac{1}{2}|u|$, and if the constant C_q is chosen large enough.

We now take $u \in \mathbb{R}$, $|u| \geqq 2$, and form

$$\begin{aligned} A(u) &= |\delta|^\alpha \delta^p(\varphi * (|\delta|^{-\alpha}\delta^{-p}k))(u) \\ &= |\delta(u)|^\alpha \delta(u)^p \int_{\mathbb{R}} \varphi(v)|\delta(u - v)|^{-\alpha}\delta(u - v)^{-p}k(u - v)\, dv. \end{aligned} \tag{4.5}$$

By (4.2) we obtain

$$\begin{aligned} A(u) &= \sum_{\nu=0}^{q-1} c_\nu(\delta(u))^{-2\nu} \int_{\mathbb{R}} \varphi(v)(iv)^\nu k(u - v)\, dv \\ &+ |\delta(u)|^\alpha \delta(u)^p \int_{\mathbb{R}} \varphi(v)R_q(u, v)k(u - v)\, dv = A_1(u) + A_2(u). \end{aligned} \tag{4.6}$$

A_1 is well-defined, for every $u \in \mathbb{R}$, by the relation (4.6). The function $v \to \varphi(v)(iv)^\nu$ is obviously a Fourier transform of a function in $\mathscr{D}(\mathbb{R})$. Hence by Properties 1 and 2 of Definition 4.2, $A_1 \in K$. By the remark, $A_1 \in K_0$ and $A \in K_0$. Hence if we define A_2 on $\mathbb{R}$ by the relation $A = A_1 + A_2$, we see that $A_2 \in K_0$. Hence it remains to show, due to Property 3 of Definition 4.2, that $A_2\delta^l \in L^\infty(\mathbb{R})$ if q is properly chosen.

By (4.3) and (4.4) we have, for $|u| \geqq 2$,

$$|A_2(u)| \leqq C_q|u|^{(\alpha+p)/2}\left\{\int_{|v|\leqq|u|/2} |\varphi(v)|\,|v|^q|u|^{-q-(\alpha+p)/2}|k(u-v)|\,dv + \int_{|v|\geqq|u|/2} |\varphi(v)|\,|v|^{q-1+|\alpha+p|/2}|k(u-v)|\,dv\right\}.$$

Using the inequalities

$$1+|u-v| \leqq 1+|u|+|v| \leqq 2|u| \quad \text{for} \quad |v| \leqq |u|/2, \qquad |u| \geqq 2,$$

$$1+|u-v| \leqq 1+|u|+|v| \leqq 4|v| \quad \text{for} \quad |v| > |u|/2, \qquad |u| \geqq 2,$$

we obtain

$$\begin{aligned}|A_2(u)| &\leqq C_q|u|^{-q+m}2^m \int_{|v|\leqq|u|/2} |\varphi(v)|\,|v|^q(1+|u-v|)^{-m}|k(u-v)|\,dv \\ &\quad + C_q|u|^{|\alpha+p|/2}2^{2m}\int_{|v|\leqq|u|/2} |\varphi(v)|\,|v|^{q+m-1+|\alpha+p|/2} \cdot(1+|u-v|)^{-m}|k(u-v)|\,dv \\ &\leqq C_q|u|^{-q+m}2^m\int_{\mathbb{R}} |\varphi(v)|\,|v|^q(1+|u-v|)^{-m}|k(u-v)|\,dv \\ &\quad + C_q 2^{2m+|\alpha+p|/2+l/2}|u|^{-l/2}\int_{\mathbb{R}} |\varphi(v)|\,|v|^{q+|\alpha+p|+m-1+l/2} \cdot(1+|u-v|)^{-m}|k(u-v)|\,dv.\end{aligned}$$

Both integrals can be interpreted as convolutions between functions in $L^1(\mathbb{R})$ and $L^\infty(\mathbb{R})$; hence they are bounded in u. Choosing q so large that $-q+m \leqq -l/2$, we see that the relation in Property 3 of Definition 4.2 is satisfied for A_2; hence $A_2 \in K$.

We are now in a position to formulate and prove the two main theorems, Theorem 4.4 and Theorem 4.5, which establish explicit relations between the functions $\hat{a}_\varphi$ and $\hat{b}_\psi$ which are obtained from a given $f \in \mathscr{D}'(\mathbb{R})$.

THEOREM 4.4. *Let K satisfy Definition* 4.2 *and let $x_0 \in D$ satisfy the assumptions of Theorem* 3.5. *Let $\psi \in \mathscr{D}(J_0)$, $\psi(t_0) \neq 0$. Then there exists a neighborhood $U_0 \subset U$ of x_0 such that $\hat{b}_\psi \in K$ implies that $\hat{a}_\varphi\delta^\lambda\bar{\delta}^{n-1-\lambda} \in K$ for $\varphi \in \mathscr{D}(U_0)$.*

Proof. Let $U = V_1 \times J_1$ be a neighborhood of x_0, so small that the properties of Theorem 3.5 hold. Then let $J \subset J_1$ be chosen such that $t_0 \in J$ and $\psi \neq 0$ on $\bar{J}$. Put $U_0 = V_1 \times J$. To prove the theorem, let us first observe that if $\varphi \in \mathscr{D}(U_0)$, all terms in (3.5), except possibly the last one, belong to K_0, hence the last one, too, belongs to that class. Choosing q large enough, we see from Property 3 of

Definition 4.2 that $S_q \in K$. By Property 2 of Definition 4.2 it suffices to show that $\hat{b}_{\psi_p} \in K$ for every $p \geqq 0$. But $\psi_p \in \mathscr{D}(J)$ by Theorem 3.5, and hence we can write $\psi_p = \theta_p \psi$, where $\theta_p \in \mathscr{D}(J)$. Definition 3.2 then gives

$$b_{\psi_p} = \psi_p(f \circ \Psi) = \theta_p \psi(f \circ \Psi) = \theta_p b_\psi,$$

and hence $\hat{b}_{\psi_p}$ is a convolution between $\hat{b}_\psi \in K \cap K_0$ and the Fourier transform of θ_p. Hence $\hat{b}_{\psi_p} \in K$ by Property 1 of Definition 4.2.

THEOREM 4.5. *Let K satisfy Definition 4.2 and let $x_0 \in D$ satisfy the assumptions of Theorem 3.5. Let $\varphi \in \mathscr{D}(U)$ satisfy $\varphi(x_0) \neq 0$, and assume that x_0 is the only point in* supp $(\varphi) \cap D$ *with t-coordinate t_0. Then there exists a neighborhood $J_2 \subset J_0$ of t_0 such that $\hat{a}_\varphi \delta^\lambda \bar{\delta}^{n-1-\lambda} \in K$ implies that $\hat{b}_\psi \in K$, for $\psi \in \mathscr{D}(J_2)$.*

Proof. Let us assume that the neighborhood $V \times J$ of x_0 satisfies the conditions in Lemma 3.1. By the assumptions we can assume as well, by reducing J if necessary, that the intersection of $\{x | t \in J\}$ and supp $(\varphi) \cap D$ is contained in $V \times J$, that is, it is the set $D_0 = \{x | x = \alpha(c),\ c \in \Psi(J)\}$. We choose $\psi_0 \in \mathscr{D}(\mathbb{R})$ so that $\psi_0(t_0) \neq 0$, supp $(\psi_0) \subset J$, and due to this choice we can find $\varphi_0 \in \mathscr{D}(V \times J)$ such that φ_0 coincides with 1 in a neighborhood of $D_0 \cap$ supp (ψ_0), where supp (ψ_0) denotes the support of ψ_0, considered as a function on $\mathbb{R}^n$. By linearity,

$$\hat{a}_{\varphi\psi_0\varphi_0} = \hat{a}_{\varphi\psi_0} - \hat{a}_{\varphi\psi_0 - \varphi\psi_0\varphi_0}. \tag{4.7}$$

The function $\varphi\psi_0 - \varphi\psi_0\varphi_0$ belongs to $\mathscr{D}(U)$, and its support does not intersect D. Hence, by Theorem 2.7, $a\varphi\psi_0 - \varphi\psi_0\varphi_0 \in \mathscr{D}(\mathbb{R})$, and thus the second term in (4.7) is a function in K_0, which tends to 0 at infinity faster than any power of the norm of the variable. By Lemma 2.5 we have $\hat{a}_{\varphi\psi_0} = \hat{a}_\varphi * \hat{\psi}_0$. Hence, by Lemma 4.3,

$$\hat{a}_\varphi \delta^\lambda \bar{\delta}^{n-1-\lambda} \in K \text{ implies } \hat{a}_{\varphi\psi_0} \delta^\lambda \bar{\delta}^{n-1-\lambda} \in K.$$

From this discussion we see that

$$\hat{a}_\varphi \delta^\lambda \bar{\delta}^{n-1-\lambda} \in K \text{ implies } \hat{a}_{\varphi\psi_0\varphi_0} \delta^\lambda \bar{\delta}^{n-1-\lambda} \in K.$$

Since the support of $\varphi\psi_0\varphi_0$ is contained in $V \times J$ and $\varphi(x_0)\psi_0(x_0)\varphi_0(x_0) \neq 0$, we conclude that it is no restriction to assume from the outset that the support of φ is contained in an arbitrarily prescribed neighborhood of x_0.

Thus we are free to assume that supp $(\varphi) \subset V_1 \times J_1$, where V_1 and J_1 satisfy the conditions in Theorem 3.5. We introduce a neighborhood $J \subset J_1$ of t_0 such that $\varphi(\alpha(\Psi(t))) \neq 0$ on J. Let $(\varphi^{(r)})_0^\infty$ be functions in $\mathscr{D}(J)$ to be determined later. Applying (3.5) with φ exchanged to $\varphi\varphi^{(r)}$ we obtain

$$\hat{a}_{\varphi\varphi^{(r)}} \delta^\lambda \bar{\delta}^{n-1-\lambda} = \sum_{p=0}^{q-1} \hat{b}_{\psi_{p,r}} \delta^{-2p} + S_{q,r}, \tag{4.8}$$

where $\psi_{p,r} \in \mathscr{D}(J)$, and it is easy to see that $S_{q,r}(u) = O(|u|^{-m})$, as $|u| \to \infty$, for every m, if q is chosen large enough, and that this choice can be made independently of r.

The function $\varphi^{(0)}$ is assumed to satisfy $\varphi^{(0)}(t_0) \neq 0$; otherwise it can be chosen arbitrarily. The remaining functions $\varphi^{(r)}$ are determined recursively by the relations

$$\varphi^{(s)}\omega(\varphi \circ \alpha \circ \Psi) = -\sum_{p=1}^{s} \psi_{p,s-p}, \qquad s \geqq 1, \tag{4.9}$$

and this is possible, since $\omega(t)\varphi(\alpha(\Psi(t))) \neq 0$, if $t \in J$. Here ω is defined in Theorem 3.5. But it is easy to see that

$$\varphi^{(s)}\omega(\varphi \circ \alpha \circ \Psi) = \psi_{0,s},$$

and hence (4.9) can be written

$$\sum_{p=0}^{s} \psi_{p,s-p} = 0. \tag{4.10}$$

Multiplying the relation (4.8) with δ^{-2r}, and adding the relations for $0 \leqq r \leqq q-1$, we obtain

$$\sum_{r=0}^{q-1} \hat{a}_{\varphi\varphi^{(r)}}\delta^{\lambda-2r}\bar{\delta}^{n-1-\lambda} = \hat{b}_{\psi_{0,0}} + \sum_{s=q}^{2q-2} \delta^{-2s}\left(\sum_{p=s-q+1}^{s} \hat{b}_{\psi_{p,s-p}}\right) + S_q,$$

where, for every m, $S_q(u) = O(|u|^{-m})$, if q is chosen large enough. It is obvious that $\hat{b}_{\psi_{p,r}}$ are distributions of bounded order, uniform in p and r, and hence all terms on the right-hand side, except possibly the first one, belong to K, if q is large enough. As for the left-hand side, we can observe that $a_{\varphi\varphi^{(r)}} = \varphi^{(r)}a_\varphi$ for every r, and hence, by Definition 4.2, the left-hand side belongs to K. Thus $\hat{b}_{\psi_{0,0}} \in K$. But by (4.9), $\psi_{0,0}(t_0) \neq 0$; hence $\hat{b}_\psi \in K$ for any $\psi \in \mathscr{D}(J_2)$ if $J_2 \subset J$ is a sufficiently small neighborhood of t_0.

We shall now apply Theorems 4.4 and 4.5 to prove a theorem on radial distributions. The theorem generalizes results of M. Gatesoupe on radial functions (cf. [2], [3], [4] and [5]). We first give two definitions.

Definition 4.6. A radial distribution with compact support in $\mathbb{R}^n\setminus\{0\}$ is a distribution $f \circ \Phi$, defined by Lemma 2.4, where $f \in \mathscr{D}'(\mathbb{R})$ with $\operatorname{supp}(f)$ in a compact subset of $]0, \infty[$, and where $\Phi(x) = |x|$, $x \in \mathbb{R}^n\setminus\{0\}$. $f \circ \Phi$ can be interpreted as a distribution in $\mathscr{D}'(\mathbb{R}^n)$ and by elementary Fourier analysis of distributions, its Fourier transform takes values $\hat{g}(|\xi|)$, $\xi \in \mathbb{R}^n$, where $\hat{g}$ is a complex-valued even function on $\mathbb{R}$. We call $\hat{g}$ the radial Fourier transform of $f \circ \Phi$.

Definition 4.7. We denote by K_1 a class of functions satisfying Definition 4.2 and in addition to that the following two conditions:

4. $k \in K_1$ implies $k\delta/\bar{\delta} \in K_1$.

5. $k \in K_1$ implies that the function $u \to k(-u)$, $u \in \mathbb{R}$, belongs to K_1.

Theorem 4.8. *Let g be a radial distribution with compact support in $\mathbb{R}^n\setminus\{0\}$, in the sense of Definition 4.6. f is the corresponding distribution in $\mathscr{D}'(\mathbb{R})$. $\hat{g}$ is the radial Fourier transform of g, $\hat{f}$ is the Fourier transform of f. Let K_1 satisfy Definition 4.7. Then*

$$\hat{f} \in K_1 \text{ if and only if } \hat{g}|\delta|^{n-1} \in K_1.$$

Proof. With $\Phi(x) = |x|$ on $\mathbb{R}^n\setminus\{0\}$ we have the representation

$$g = f \circ \Phi = \varphi(f \circ \Phi),$$

where φ is an arbitrary function in $\mathscr{D}(\mathbb{R}^n\setminus\{0\})$, taking the value 1 in an open set, containing the support of g. Hence $\hat{g} = \hat{a}_\varphi$ with φ chosen in the mentioned way. Let us first assume that $\hat{f} \in K_1$. It is then enough to prove that every $x_0 \in \mathbb{R}^n\setminus\{0\}$ has a neighborhood U_0 such that $\varphi_0 \in \mathscr{D}(U_0)$ implies that $\hat{a}_{\varphi_0}|\delta|^{n-1} \in K_1$.

For we can then cover the support of φ with a finite set of such neighborhoods and split a_φ into a sum of elements a_{φ_ν}, where every φ_ν has support in such a neighborhood.

We have to consider three separate cases. Let us first assume that $x_0 = (\xi_0, t_0)$ satisfies $\xi_0 \neq 0$. Then $x_0 \neq D$, and Theorem 2.7 shows that $a_\varphi \in \mathscr{D}(\mathbb{R})$, if supp (φ) is contained in a sufficiently small neighborhood of x_0.

The second case occurs when $\xi_0 = 0$, $t_0 > 0$. Then $\Psi(t) = t$ for every t in the corresponding J_0 and λ in Theorem 3.5 is 0. Moreover, $\hat{b}_\psi = \hat{f} * \hat{\psi}$. Hence $\hat{b}_\psi \in K_1$. By Theorem 4.4 there exists a neighborhood U_0 of x_0 such that $\varphi \in \mathscr{D}(U_0)$ implies $\hat{a}_\varphi \bar{\delta}^{n-1} \in K_1$; hence by Definition 4.7, $\hat{a}_\varphi |\delta|^{n-1} \in K_1$.

The third case when $\xi_0 = 0$, $t_0 < 0$, is discussed in a similar way. This time $\lambda = n - 1$, $\Psi(t) = -t$, $\hat{b}_\psi(-u) = \hat{f} * \hat{\psi}(u)$, and due to Property 5 of Definition 4.7 we obtain $\hat{a}_\varphi \delta^{n-1} \in K_1$; hence $\hat{a}_\varphi |\delta|^{n-1} \in K_1$.

To prove the converse, we assume $\hat{g}|\delta|^{n-1} \in K_1$. Then $\hat{a}_\varphi \delta^{n-1} \in K_1$ for a function $\varphi \in \mathscr{D}(\mathbb{R}^n \setminus \{0\})$, which takes the value 1 on the support of g. Since D is a subset of the t-axis, Theorem 4.5 shows that every $t_0 > 0$ has a neighborhood J in $]0, \infty[$ such that $\psi \in \mathscr{D}(J)$ implies that $\hat{f} * \hat{\psi} \in K_1$. The usual compactness argument shows then that $\hat{f} \in K_1$.

There are many possibilities to find further applications of our results. It should first be observed that we have here restricted the discussion of the Fourier transform of $\varphi(f \circ \Phi)$ to the values taken on a particular one-dimensional subspace of $\mathbb{R}^n$. The same discussion can be carried out on any one-dimensional subspace, and this gives a complete picture of the behavior of the Fourier transform in terms of the transforms $\psi(f \circ \Psi)$, where Ψ varies in the class of functions obtained, when we for every direction in the $\mathbb{R}^n$-plane carry out the same discussion as the one which we have given for the direction of the t-axis. By a close examination of the uniformity properties of the remainder term in (3.5) when the direction varies, it is very easy to get theorems which can roughly be described as giving direct connections between the n-dimensional local Fourier transform of $f \circ \omega$ and the one-dimensional local Fourier transform of the functions $f \circ \Psi$. Special results of this kind have been presented in [1], where in particular the case has been studied in which all the functions ψ are affinely equivalent. In this context we only point to the possibility of obtaining results in much more general settings.

REFERENCES

[1] Y. Domar, *Local homomorphisms to $L^1(\mathbb{R}^m)$ of weighted group algebras on $\mathbb{R}^n$*, Uppsala University, Department of Mathematics, Preprint 9, 1969.

[2] M. Gatesoupe, *Caractérisation locale de la sous-algèbre formée des fonctions radiales de $\mathscr{F}L^1(\mathbb{R}^n)$*, Ann. Inst. Fourier, 17 (1967), pp. 93–108.

[3] ———, *Sur les transformations de Hankel et de Fourier,* C.R. Acad. Sci. Paris, 266 (1968), pp. 211–214.

[4] ———, *Sur les éléments de $\mathscr{F}L^p(\mathbb{R}^n)$, $p \geqq 1$, invariants par rotation*, Ibid., 267 (1968), pp. 926–928.

[5] ———, *Sur les transformées de Fourier radiales,* Thèse, Faculté des Sciences, Orsay, Paris, 1970.

[6] W. Littman, *Decay at infinity of solutions to partial differential equations with constant coefficients*, Trans. Amer. Math. Soc., 123 (1966), pp. 449–459.

[7] J. Milnor, *Morse Theory*, Princeton University Press, Princeton, 1963.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

SATURATION THEORY IN CONNECTION WITH MELLIN TRANSFORM METHODS*

W. KOLBE AND R. J. NESSEL†

Abstract. If X denotes one of the spaces L^p, $1 \leqq p < \infty$, or C, where L^p is the set of all f with $[\int_0^\infty |f(r)|^p r^{-1}\, dr]^{1/p} < \infty$ and C the set of all f bounded and continuous on $(0, \infty)$ with $\lim_{a \to 1} [\sup_{0<r<\infty} |f(ar) - f(r)|] = 0$, then $f \in X_{\sigma_1,\sigma_2}$ provided $r^\sigma f(r) \in X$ for every $\sigma \in (\sigma_1, \sigma_2)$. For $f \in X_{\sigma_1,\sigma_2}$ a general approximation process $I_\vartheta(f; r) = \int_0^\infty f(r/\rho) k_\vartheta(\rho)\rho^{-1}\, d\rho$ of Mellin convolution type is considered, the kernel $\{k_\vartheta(r)\}$, $\vartheta > 0$, satisfying suitable conditions such that it constitutes an approximate identity for $\vartheta \to 0+$, i.e.,

$$\lim_{\vartheta \to 0+} \|r^\sigma[I_\vartheta(f; r) - f(r)]\|_X = 0 \quad \text{for each} \quad \sigma \in (\sigma_1, \sigma_2), \quad f \in X_{\sigma_1,\sigma_2}.$$

The main purpose of this paper is the study of saturation phenomena of the process $I_\vartheta(f; r)$. Using Butzer's integral transform method, a complete saturation theorem is obtained under suitable conditions upon the Mellin transform of $k_\vartheta(r)$. As a significant application of the general results, the boundary behavior of the solution $u(r, \vartheta)$ of Dirichlet's problem for the wedge $W = \{(r, \vartheta) | 0 < r < \infty, 0 < \vartheta < \vartheta_0\}$, $0 < \vartheta_0 < 2\pi$, is considered in detail. This problem is first raised in a strong sense (essentially as an abstract Cauchy problem) which allows a rigorous treatment of its solution via the classical Mellin transform method. In particular, boundary values $f_1, f_2 \in X_{-\sigma_0,\sigma_0}$, $\sigma_0 = \pi/\vartheta_0$, are attained in the sense that for each $\sigma \in (-\sigma_0, \sigma_0)$,

$$\lim_{\vartheta \to 0+} \|r^\sigma[u(r, \vartheta) - f_1(r)]\|_X = 0, \quad \lim_{\vartheta \to \vartheta_0-} \|r^\sigma[u(r, \vartheta) - f_2(r)]\|_X = 0.$$

For, for example, symmetric boundary values $f_1 = f_2 = f$, the general saturation theorem then gives that the above quantities cannot tend to zero too rapidly; thus $\|r^\sigma[u(r, \vartheta) - f(r)]\|_X = o(\vartheta)$, $\vartheta \to 0+$, implies $f = 0$.

Integral transform methods such as Mellin transform methods have proven to be of great importance in the solution of initial and boundary value problems for partial differential equations (cf. literature cited in § 4). The aim of this paper is to study such Mellin transform methods in connection with a class of problems in the theory of approximation. To this end, § 1 gives some preliminary results on Mellin transforms. In § 2 a general approximation process $I_\vartheta(f; r)$ of Mellin convolution type is introduced for which the convergence in the X_{σ_1,σ_2}-topology is shown in Theorem 2.2. Section 3 is devoted to a detailed study of saturation phenomena of the integral $I_\vartheta(f; r)$. Finally, § 4 is concerned with Dirichlet's problem for a wedge; this problem is first raised in the strong interpretation (4.3) which allows a rigorous treatment of its solution including uniqueness, and it is shown that the solution may serve as an illustrative example for the general results obtained.

1. Preliminary results on Mellin transforms. Let f be a real-valued[1] function, measurable on $(0, \infty)$. Setting

$$\|f\|_p = \begin{cases} \left[\int_0^\infty |f(r)|^p \frac{dr}{r}\right]^{1/p} & \text{for} \quad 1 \leqq p < \infty, \\ \operatorname*{ess\,sup}_{0<r<\infty} |f(r)| & \text{for} \quad p = \infty, \end{cases} \tag{1.1}$$

* Received by the editors October 19, 1970.

† Lehrstuhl A für Mathematik, T. H. Aachen, Templergraben 55, 51 Aachen, West Germany.
The contribution of the first author was supported by a DFG research grant.

[1] Apart from routine techniques, extensions to complex-valued functions are obvious.

L^p $(=L^p(0,\infty))$ denotes the set of all f for which the norm $\|f\|_p$ is finite (cf. [15, pp. 94, 118]). C is the set of all functions f which are bounded and continuous on $(0,\infty)$ such that (see also Lemma 2.1)

$$\lim_{a\to 1}\,[\sup_{0<r<\infty}|f(ar)-f(r)|]=0. \tag{1.2}$$

Obviously, C becomes a Banach space under the usual sup norm; thus $C\subset L^\infty$. In the following, X always denotes one of the spaces C or L^p, $1\leqq p<\infty$. If f, $g\in X$, we write $f(r)=g(r)$ (a.e.) if equality holds for all $r\in(0,\infty)$ in case $X=C$, and almost everywhere in case $X=L^p$, $1\leqq p<\infty$. A function f is said to belong to $L^p_{\sigma_1,\sigma_2}$ if $r^\sigma f(r)\in L^p$ for every σ of the open interval (σ_1,σ_2); C_{σ_1,σ_2} and X_{σ_1,σ_2} are defined correspondingly.[2] It is an immediate consequence that X_{σ_1,σ_2}, $L^\infty_{\sigma_1,\sigma_2}\subset L^1_{\sigma_1,\sigma_2}$. Indeed (cf. [1]), if $f\in L^p_{\sigma_1,\sigma_2}$ for some $1<p\leqq\infty$, and if for $\sigma\in(\sigma_1,\sigma_2)$, c_1,c_2 are chosen such that $\sigma_1<c_1<\sigma<c_2<\sigma_2$, then by Hölder's inequality,

$$\begin{aligned}\|r^\sigma f(r)\|_1&=\left(\int_0^1+\int_1^\infty\right)r^\sigma|f(r)|\frac{dr}{r}\\&\leqq[(\sigma-c_1)p']^{-1/p'}\|r^{c_1}f(r)\|_p+[(c_2-\sigma)p']^{-1/p'}\|r^{c_2}f(r)\|_p,\end{aligned}\tag{1.3}$$

where p, p' as usual denote conjugate numbers, i.e., $p^{-1}+p'^{-1}=1$. A sequence $\{f_n\}\subset X_{\sigma_1,\sigma_2}$ is said to converge to $f\in X_{\sigma_1,\sigma_2}$ in the X_{σ_1,σ_2}-topology if

$$\lim_{n\to\infty}\|r^\sigma[f_n(r)-f(r)]\|_X=0 \tag{1.4}$$

for each $\sigma\in(\sigma_1,\sigma_2)$; it follows that (1.4) then holds uniformly for σ in any compact subinterval of (σ_1,σ_2). Finally, μ belongs to BV_{σ_1,σ_2} if (i) $\mu\in BV_{\mathrm{loc}}(0,\infty)$, i.e., μ is of bounded variation on every compact subinterval of $(0,\infty)$ and normalized by $\mu(1)=0$, $\mu(r)=[\mu(r+)+\mu(r-)]/2$ for $r\in(0,\infty)$, (ii) $\int_0^\infty r^\sigma|d\mu(r)|<\infty$ for every $\sigma\in(\sigma_1,\sigma_2)$.

Let $s=\sigma+i\tau$ be an arbitrary complex number, and let A be the strip given by $A=\{s|\sigma_1<\mathrm{Re}\,(s)<\sigma_2,\ -\infty<\mathrm{Im}\,(s)<\infty\}$. For $f\in L^1_{\sigma_1,\sigma_2}$ the Mellin transform is defined by

$$f^\wedge(s)=\int_0^\infty r^s f(r)\frac{dr}{r}\quad\left(=\int_0^\infty r^{s-1}f(r)\,dr\right). \tag{1.5}$$

Obviously, one has the estimate $|f^\wedge(s)|\leqq\|r^\sigma f(r)\|_1$, $\sigma=\mathrm{Re}\,(s)$. Correspondingly, the Mellin–Stieltjes transform of $\mu\in BV_{\sigma_1,\sigma_2}$ is given by

$$\mu^\vee(s)=\int_0^\infty r^s\,d\mu(r). \tag{1.6}$$

There is a close connection between Mellin and bilateral Laplace transforms. Indeed, the substitution $r=\exp\{-x\}$ yields (cf. [8, p. 33], [17, p. 246])

$$f^\wedge(s)=\int_{-\infty}^\infty e^{-sx}f(e^{-x})\,dx,\quad \mu^\vee(s)=\int_{-\infty}^\infty e^{-sx}\,d[-\mu(e^{-x})]. \tag{1.7}$$

---

[2] Sometimes we do not actually use the strong hypothesis that, for example, $r^\sigma f(r)\in X$ for every $\sigma\in(\sigma_1,\sigma_2)$. However, possible generalizations are then quite clear from the context.

Hence many of the fundamental properties of Mellin transforms follow from the corresponding one for bilateral Laplace transforms. Thus the Mellin and Mellin–Stieltjes transforms are holomorphic functions of s in the strip A (cf. [17, p. 240]). Furthermore, the uniqueness theorem holds (cf. [17, p. 243 ff]), i.e., if for $f \in L^1_{\sigma_1,\sigma_2}$ $[\mu \in BV_{\sigma_1,\sigma_2}]$ one has $f^\wedge(s) = 0$ $[\mu^\vee(s) = 0]$ for all $s \in A$, then $f(r) = 0$ (a.e.) $[\mu(r) = 0]$ on $(0, \infty)$. Since $X_{\sigma_1,\sigma_2} \subset L^1_{\sigma_1,\sigma_2}$ (cf. (1.3)), the Mellin transform (1.5) is well-defined[3] for every $f \in X_{\sigma_1,\sigma_2}$.

The convolution of $f \in X_{\sigma_1,\sigma_2}$ and $g \in L^1_{\sigma_1,\sigma_2}$ is defined by

$$(f * g)(r) = \int_0^\infty f\left(\frac{r}{\rho}\right) g(\rho) \frac{d\rho}{\rho}. \tag{1.8}$$

It follows (cf. [15, p. 60]) that $f * g$ exists (a.e.) on $(0, \infty)$ and $f * g \in L^1_{\sigma_1,\sigma_2}$. In fact, $f * g \in X_{\sigma_1,\sigma_2}$, and using the generalized Minkowski inequality (cf. [11, p. 148]) and substituting $r = \rho t$, one has

$$\|r^\sigma (f * g)(r)\|_X \leqq \int_0^\infty \left\| r^\sigma f\left(\frac{r}{\rho}\right)\right\|_X |g(\rho)| \frac{d\rho}{\rho} = \|t^\sigma f(t)\|_X \|\rho^\sigma g(\rho)\|_1 \tag{1.9}$$

for each $\sigma \in (\sigma_1, \sigma_2)$. Furthermore, $f * g = g * f$, and the convolution theorem states that

$$[f * g]^\wedge(s) = f^\wedge(s) g^\wedge(s), \qquad s \in A. \tag{1.10}$$

Let us conclude with a result concerning Mellin transforms of derivatives (see also [7, p. 44], [14, p. 291], [20, p. 112]).

LEMMA 1.1. *Let $E = r(d/dr)$ and n be a fixed positive integer. Suppose that all $E^k f$, $0 \leqq k \leqq n - 1$, are absolutely continuous on every finite subinterval of $(0, \infty)$, i.e., $E^{n-1} f \in AC_{\mathrm{loc}}(0, \infty)$, and that f, $E^n f \in X_{\sigma_1,\sigma_2}$. Then*

$$[E^n f]^\wedge(s) = (-s)^n f^\wedge(s), \qquad s \in A. \tag{1.11}$$

Proof. Let $n = 1$. Proceeding as in the Fourier transform case (cf. [6, Theorem 5.1.16]), let

$$m(r) = \begin{cases} 1 & \text{if } r \in [1/e, 1], \\ 0 & \text{otherwise}. \end{cases} \tag{1.12}$$

Obviously, $m \in L^1_{-\infty,\infty}(0, \infty)$ and

$$m^\wedge(s) = \begin{cases} s^{-1}[1 - e^{-s}], & s \neq 0, \\ 1, & s = 0. \end{cases} \tag{1.13}$$

[3] Of course, the definition of the Mellin transform (and correspondingly of the bilateral Laplace transform) may be given under considerably weaker hypotheses upon f. We refer to [17, p. 246 ff] for the "improper integral"-version as well as to [15, pp. 94, 118] where the Mellin transform is defined as the limit of $\int_{1/a}^a r^{\sigma+i\tau-1} f(r)\,dr$ in the mean of order p' over $-\infty < \tau < \infty$ as $a \to \infty$ in case $r^\sigma f(r)$ belongs to L^p, $1 < p \leqq 2$, for just one certain fixed σ; see also [20, p. 106 ff] for the definition in connection with generalized functions. However, in this paper it is sufficient to consider Mellin transforms of functions f satisfying the restrictive properties stated above.

Since $f \in AC_{\mathrm{loc}}(0, \infty)$, it follows by an elementary substitution that

$$f(er) - f(r) = \int_{1/e}^{1} (Ef)\left(\frac{r}{\rho}\right)\frac{d\rho}{\rho} = (m * Ef)(r).$$

Passing to Mellin transforms, since $[f(er)]^{\wedge}(s) = e^{-s}f^{\wedge}(s)$, it follows by (1.10), (1.13) that for $s \neq 0$,

$$(e^{-s} - 1)f^{\wedge}(s) = -s^{-1}(e^{-s} - 1)[Ef]^{\wedge}(s),$$

which proves (1.11) for $n = 1$. In view of the relation

$$\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} f(re^k) = \int_{1/e}^{1} \frac{d\rho_1}{\rho_1} \cdots \int_{1/e}^{1} \frac{d\rho_n}{\rho_n}(E^n f)\left(\frac{r}{\rho_1 \cdots \rho_n}\right), \tag{1.14}$$

which follows by mathematical induction, the proof for arbitrary positive integers n follows along the same lines.

2. Singular integrals of Mellin convolution type. The approximation processes to be discussed in this paper will be given as singular integrals of Mellin convolution type, thus in the form

$$I_\vartheta(f; r) = \int_0^\infty f\left(\frac{r}{\rho}\right) k_\vartheta(\rho)\frac{d\rho}{\rho}. \tag{2.1}$$

Here the parameter ϑ ranges over some interval $(0, \vartheta_0)$, and the kernel $\{k_\vartheta(r)\}$ is assumed to be an approximate identity[4] for $\vartheta \to 0+$, i.e., for some σ_1, σ_2 with $-\infty \leqq \sigma_1 < \sigma_2 \leqq \infty$:

(2.2)

(i) $k_\vartheta \in L^1 \cap L^1_{\sigma_1,\sigma_2}$ for each $\vartheta \in (0, \vartheta_0)$,

(ii) $\displaystyle\lim_{\vartheta \to 0+} \int_0^\infty k_\vartheta(r)\frac{dr}{r} = 1$,

(iii) $\|k_\vartheta\|_1 \leqq M$, the constant M being independent of $\vartheta \in (0, \vartheta_0)$,

(iv) $\displaystyle\lim_{\vartheta \to 0+} \left(\int_0^{1-\delta} + \int_{1+\delta}^\infty\right)[1 + r^\sigma]|k_\vartheta(r)|\frac{dr}{r} = 0$ for each (fixed) $\delta \in (0, 1)$ and $\sigma \in (\sigma_1, \sigma_2)$.

Then $I_\vartheta(f; r)$ approximates every $f \in X_{\sigma_1,\sigma_2}$ in the X_{σ_1,σ_2}-topology as $\vartheta \to 0+$. To prove this convergence theorem, we need the following result concerning continuity in X_{σ_1,σ_2}.

LEMMA 2.1. *Let $f \in X_{\sigma_1,\sigma_2}$. Then for each $\sigma \in (\sigma_1, \sigma_2)$,*

$$\|r^\sigma[f(r) - f(r/\rho)]\|_X \leqq (1 + \rho^\sigma)\|r^\sigma f(r)\|_X, \tag{2.3}$$

$$\lim_{\rho \to 1} \|r^\sigma[f(r) - f(r/\rho)]\|_X = 0. \tag{2.4}$$

[4] Approximate identities for $\vartheta \to \vartheta_0-$ are defined similarly.

Proof. Relation (2.3) follows by Minkowski's inequality. To prove (2.4), let $X = L^p$, $1 \leqq p < \infty$, and $\sigma \in (\sigma_1, \sigma_2)$ be fixed. Substituting $r = e^{-x}$, $\rho = e^{-t}$ (cf. (1.7)) and setting $h(x) = e^{-\sigma x} f(e^{-x})$, it follows that

$$\|r^\sigma f(r) - (r/\rho)^\sigma f(r/\rho)\|_p^p = \int_{-\infty}^{\infty} |h(x) - h(x-t)|^p \, dx.$$

Since $r^\sigma f(r) \in L^p$ implies $\int_{-\infty}^{\infty} |h(x)|^p \, dx < \infty$, the usual theorem concerning continuity in the mean assures that the right-hand side tends to zero as $t \to 0$. Therefore,

$$\lim_{\rho \to 1} \|r^\sigma f(r) - (r/\rho)^\sigma f(r/\rho)\|_p = 0.$$

This implies (2.4) since by Minkowski's inequality,

$$\|r^\sigma[f(r) - f(r/\rho)]\|_p \leqq \|r^\sigma f(r) - (r/\rho)^\sigma f(r/\rho)\|_p + |1 - \rho^\sigma| \, \|r^\sigma f(r)\|_p.$$

In case $X = C$, the proof follows more immediately (cf. (1.2)); note that $r^\sigma f(r) \in C$ if and only if $h(x) = e^{-\sigma x} f(e^{-x})$ is bounded and uniformly continuous on $(-\infty, \infty)$.

THEOREM 2.2. *Let $\{k_\vartheta(r)\}$ satisfy* (2.2). *Then for every $f \in X_{\sigma_1, \sigma_2}$*,

$$\|r^\sigma I_\vartheta(f; r)\|_X \leqq \|r^\sigma k_\vartheta(r)\|_1 \|r^\sigma f(r)\|_X, \qquad \sigma \in (\sigma_1, \sigma_2), \tag{2.5}$$

$$\lim_{\vartheta \to 0+} \|r^\sigma[I_\vartheta(f; r) - f(r)]\|_X = 0, \qquad \sigma \in (\sigma_1, \sigma_2). \tag{2.6}$$

Proof. Relation (2.5) follows by (1.9). To prove (2.6), by Minkowski's inequality,

$$\begin{aligned} \|r^\sigma[I_\vartheta(f; r) - f(r)]\|_X &\leqq \left\| \int_0^\infty r^\sigma \left[f\left(\frac{r}{\rho}\right) - f(r) \right] k_\vartheta(\rho) \frac{d\rho}{\rho} \right\|_X \\ &\quad + \left| 1 - \int_0^\infty k_\vartheta(\rho) \frac{d\rho}{\rho} \right| \|r^\sigma f(r)\|_X = J_1 + J_2, \end{aligned} \tag{2.7}$$

say. Obviously, $\lim_{\vartheta \to 0+} J_2 = 0$ by (2.2) (ii). Concerning J_1, by the generalized Minkowski inequality and (2.3),

$$\begin{aligned} J_1 &\leqq \int_0^\infty \left\| r^\sigma \left[f\left(\frac{r}{\rho}\right) - f(r) \right] \right\|_X |k_\vartheta(\rho)| \frac{d\rho}{\rho} \\ &\leqq \int_{1-\delta}^{1+\delta} \left\| r^\sigma \left[f\left(\frac{r}{\rho}\right) - f(r) \right] \right\|_X |k_\vartheta(\rho)| \frac{d\rho}{\rho} \\ &\quad + \|r^\sigma f(r)\|_X \left(\int_0^{1-\delta} + \int_{1+\delta}^\infty \right) (1 + \rho^\sigma) |k_\vartheta(\rho)| \frac{d\rho}{\rho} = J_1^1 + J_1^2, \end{aligned}$$

say. In view of (2.4), to each $\varepsilon > 0$ there exists δ with $0 < \delta < 1$ such that $\|r^\sigma[f(r/\rho) - f(r)]\|_X < \varepsilon$ for all $|1 - \rho| < \delta$. This implies $J_1^1 < \varepsilon M$ by (2.2) (iii). Now take δ fixed. Then $\lim_{\vartheta \to 0+} J_1^2 = 0$ by (2.2) (iv), and (2.6) is completely established.

In the following we always assume that the kernel $\{k_\vartheta(r)\}$ is an approximate identity in the sense of (2.2). Though not each of the hypotheses in (2.2) is used at each time explicitly, we make the assumption in our further approximation theoretical investigations in order to be sure that the process actually converges to the given function f.

3. Saturation in X_{σ_1,σ_2}. Let $f\in X_{\sigma_1,\sigma_2}$ and $I_\vartheta(f;r)$ be a singular integral as given by (2.1). The main purpose of this paper is to study saturation for this convolution integral in the X_{σ_1,σ_2}-topology. Thus the problem to be considered is the precise characterization of the class $F[X_{\sigma_1,\sigma_2};I_\vartheta]$ of functions $f\in X_{\sigma_1,\sigma_2}$ for which the order of approximation by $I_\vartheta(f;r)$ in the X_{σ_1,σ_2}-topology is exactly of best order $O(\vartheta^\gamma)$, i.e., for which

$$D(f;I_\vartheta)=\|r^\sigma[I_\vartheta(f;r)-f(r)]\|_X=O(\vartheta^\gamma),\qquad \vartheta\to 0+,$$

for each $\sigma\in(\sigma_1,\sigma_2)$, and the approximation is best possible in the sense that if $D(f;I_\vartheta)=o(\vartheta^\gamma)$ for each $\sigma\in(\sigma_1,\sigma_2)$, then necessarily $f(r)=0$ (a.e.). This concept is essentially that of saturation as introduced by Favard [10]; $F[X_{\sigma_1,\sigma_2};I_\vartheta]$ is called the Favard (or saturation) class which, of course, is assumed to contain at least one non-null function.

To study saturation phenomena of the approximation process $I_\vartheta(f;r)$, the convolution structure of $I_\vartheta(f;r)$ suggests the application of the integral transform method as outlined by Butzer [4]. Thus, following Butzer [3] in the Fourier transform case and Berens–Butzer [1] in the unilateral Laplace transform case, the kernel $\{k_\vartheta(r)\}$ is assumed to satisfy the following conditions:

(3.1) Given a kernel $\{k_\vartheta(r)\}\subset L^1_{\sigma_1,\sigma_2}$, let there exist $\psi(s)\not\equiv 0$, holomorphic on the strip A, and $\gamma>0$ such that for every $s\in A$,

$$\lim_{\vartheta\to 0+}\vartheta^{-\gamma}[k_\vartheta^\wedge(s)-1]=\psi(s).$$

(3.2) Given a kernel $\{k_\vartheta(r)\}\subset L^1_{\sigma_1,\sigma_2}$, let there exist $v_\vartheta\in BV_{\sigma_1,\sigma_2}$ for each $0<\vartheta<\vartheta_0$ such that, with ψ and γ as in (3.1), the representation

$$\vartheta^{-\gamma}[k_\vartheta^\wedge(s)-1]=\psi(s)v_\vartheta^\vee(s)$$

holds for all $s\in A$ and such that $\int_0^\infty r^\sigma|dv_\vartheta(r)|=O(1)$ as $\vartheta\to 0+$ for each $\sigma\in(\sigma_1,\sigma_2)$.

Conditions of the type (3.1), (3.2) are by now standard in the study of saturation of convolution integrals; the reader is referred to Butzer–Nessel [6, § 12.6] for detailed bibliographical comments.

THEOREM 3.1. *Let* $f\in X_{\sigma_1,\sigma_2}$ and $\{k_\vartheta(r)\}$ *satisfy* (3.1). *If there exists* $g\in X_{\sigma_1,\sigma_2}$ *such that*

$$\lim_{\vartheta\to 0+}\left\|r^\sigma\left[\frac{I_\vartheta(f;r)-f(r)}{\vartheta^\gamma}-g(r)\right]\right\|_X=0\tag{3.3}$$

for each $\sigma\in(\sigma_1,\sigma_2)$, *then* $\psi(s)f^\wedge(s)=g^\wedge(s)$ *for all* $s\in A$.

Proof. In case $X=L^1$, (1.5) and (1.10) imply (with $\mathrm{Re}(s)=\sigma$)

$$\left|\frac{k_\vartheta^\wedge(s)-1}{\vartheta^\gamma}f^\wedge(s)-g^\wedge(s)\right|\leqq\left\|r^\sigma\left[\frac{I_\vartheta(f;r)-f(r)}{\vartheta^\gamma}-g(r)\right]\right\|_1,\tag{3.4}$$

so that the assertion follows by (3.1) and (3.3). In case of arbitrary X-spaces, the estimate (1.3) assures that (3.3) is also valid in L^1-norm, completing the proof.

COROLLARY 3.2. *If* $f \in X_{\sigma_1,\sigma_2}$ *and* $\{k_\vartheta(r)\}$ *satisfies* (3.1), *then*

$$\|r^\sigma[I_\vartheta(f;r) - f(r)]\|_X = o(\vartheta^\gamma), \qquad \vartheta \to 0+,$$

for each $\sigma \in (\sigma_1, \sigma_2)$ *implies* $f(r) = 0$ *(a.e.) on* $(0, \infty)$. *Moreover, the null function is the only invariant element of the singular integral* $I_\vartheta(f;r)$.

Proof. Theorem 3.1 implies $\psi(s)f^\wedge(s) = 0$ on A. Since $\psi(s)$ is holomorphic and does not vanish identically on A, it follows that $f^\wedge(s) = 0$ on A, so that $f(r) = 0$ (a.e.) by the uniqueness theorem. Obviously, the null function is an invariant element of $I_\vartheta(f;r)$, i.e., $I_\vartheta(f;r) = f(r)$ (a.e.) for every $\vartheta \in (0, \vartheta_0)$. Conversely, if $f \in X_{\sigma_1,\sigma_2}$ is invariant, then $[k_\vartheta^\wedge(s) - 1]f^\wedge(s) = 0$ on A by the convolution theorem. By (3.1) it follows that $\psi(s)f^\wedge(s) = 0$ on A, and thus $f(r) = 0$ (a.e.) as above.

For the next result it is convenient to introduce the following classes of functions:

$$V[X_{\sigma_1,\sigma_2}; \psi(s)] = \begin{cases} \{f \in C_{\sigma_1,\sigma_2} | \psi(s)f^\wedge(s) = g^\wedge(s), g \in L^\infty_{\sigma_1,\sigma_2}\}, \\ \{f \in L^1_{\sigma_1,\sigma_2} | \psi(s)f^\wedge(s) = \mu^\vee(s), \mu \in BV_{\sigma_1,\sigma_2}\}, \\ \{f \in L^p_{\sigma_1,\sigma_2} | \psi(s)f^\wedge(s) = g^\wedge(s), g \in L^p_{\sigma_1,\sigma_2}\}, & 1 < p < \infty. \end{cases}$$

THEOREM 3.3. *Let* $f \in X_{\sigma_1,\sigma_2}$ *and* $\{k_\vartheta(r)\}$ *satisfy* (3.1). *Then the approximation*

$$\|r^\sigma[I_\vartheta(f;r) - f(r)]\|_X = O(\vartheta^\gamma), \qquad \vartheta \to 0+, \tag{3.5}$$

for each $\sigma \in (\sigma_1, \sigma_2)$ *implies* $f \in V[X_{\sigma_1,\sigma_2}; \psi(s)]$.

Proof. Setting $h_\vartheta(r) = \vartheta^{-\gamma}[I_\vartheta(f;r) - f(r)]$, it follows by the convolution theorem and (3.1) that in any case

$$\lim_{\vartheta\to 0+} h_\vartheta^\wedge(s) = \lim_{\vartheta \to 0+} \vartheta^{-\gamma}[k_\vartheta^\wedge(s) - 1]f^\wedge(s) = \psi(s)f^\wedge(s) \tag{3.6}$$

for every $s \in A$. Let $[c_1, c_2]$ be an arbitrary closed subinterval of (σ_1, σ_2), and set

$$n(r) = \begin{cases} r^{c_1}, & 0 < r \leqq 1, \\ r^{c_2}, & 1 < r < \infty. \end{cases} \tag{3.7}$$

We first treat the cases that X is either L^p, $1 < p < \infty$, or C, the latter case being subsumed in the following under the case $p = \infty$. Then the hypothesis (3.5) gives that, for sufficiently small ϑ, there exists a constant M such that

$$\|n(r)h_\vartheta(r)\|_p \leqq \|r^{c_1}h_\vartheta(r)\|_p + \|r^{c_2}h_\vartheta(r)\|_p \leqq M.$$

Hence by the weak* compactness theorem for L^p, $1 < p \leqq \infty$, there exist a null sequence $\{\vartheta_{c_1,c_2,j}\}$ and a function $g_{c_1,c_2}(r)$ with $n(r)g_{c_1,c_2}(r) \in L^p$ such that

$$\lim_{j\to\infty} \int_0^\infty \Phi(r)n(r)h_{\vartheta_{c_1,c_2,j}}(r)\frac{dr}{r} = \int_0^\infty \Phi(r)n(r)g_{c_1,c_2}(r)\frac{dr}{r} \tag{3.8}$$

for every $\Phi \in L^{p'}$. Certainly, $n(r)g_{c_1,c_2}(r) \in L^p$ implies $r^\sigma g_{c_1,c_2}(r) \in L^p$ for every $\sigma \in [c_1, c_2]$, and putting

$$\Phi(r) = \begin{cases} r^{s-c_1}, & 0 < r \leqq 1, \\ r^{s-c_2}, & 1 < r < \infty, \end{cases} \tag{3.9}$$

with $c_1 < \operatorname{Re}(s) < c_2$, (3.8) in particular gives by (3.6) that

$$g^{\wedge}_{c_1,c_2}(s) = \lim_{j\to\infty} h^{\wedge}_{\vartheta_{c_1,c_2,j}}(s) = \psi(s) f^{\wedge}(s) \tag{3.10}$$

for every s with $c_1 < \operatorname{Re}(s) < c_2$. The assertion would follow if one could show that g_{c_1,c_2} is indeed independent of the particular choice of c_1, c_2. To this end, let $[c_1', c_2']$, $[c_1'', c_2'']$ be two arbitrary closed subintervals of (σ_1, σ_2), and let d_1, d_2 be such that

$$[c_1', c_2'] \cup [c_1'', c_2''] \subset [d_1, d_2] \subset (\sigma_1, \sigma_2).$$

Then it follows by (3.10) and the uniqueness theorem that $g_{c_1',c_2'}(r) = g_{d_1,d_2}(r) = g_{c_1'',c_2''}(r)$ a.e. Hence there exists a function g such that $r^{\sigma} g(r) \in L^p$ for each $\sigma \in (\sigma_1, \sigma_2)$ and $\psi(s) f^{\wedge}(s) = g^{\wedge}(s)$ for every $s \in A$.

Let $X = L^1$. Setting (cf. (3.7))

$$\mu_{\vartheta}(r) = \int_1^r n(\rho) h_{\vartheta}(\rho) \frac{d\rho}{\rho},$$

the hypothesis (3.5) implies for sufficiently small ϑ that there exists a constant M such that

$$\int_0^{\infty} |d\mu_{\vartheta}(r)| = \|n(r) h_{\vartheta}(r)\|_1 \leqq \|r^{c_1} h_{\vartheta}(r)\|_1 + \|r^{c_2} h_{\vartheta}(r)\|_1 \leqq M.$$

If C_0 denotes the subset of all $\Phi \in C$ for which $\lim_{r\to 0+} \Phi(r) = \lim_{r\to\infty} \Phi(r) = 0$, the theorem of Helly–Bray (cf. [17, p. 31]) assures the existence of a null sequence $\{\vartheta_{c_1,c_2,j}\}$ and of $\mu_{c_1,c_2} \in BV_{1\mathrm{oc}}(0, \infty)$ with $\int_0^{\infty} n(r) |d\mu_{c_1,c_2}(r)| < \infty$ such that

$$\lim_{j\to\infty} \int_0^{\infty} \Phi(r) n(r) h_{\vartheta_{c_1,c_2,j}}(r) \frac{dr}{r} = \int_0^{\infty} \Phi(r) n(r)\, d\mu_{c_1,c_2}(r) \tag{3.11}$$

for every $\Phi \in C_0$. Obviously, $\int_0^{\infty} r^{\sigma} |d\mu_{c_1,c_2}(r)| < \infty$ for every $\sigma \in [c_1, c_2]$, and using Φ of (3.9), then (3.6) and (3.11) deliver $\psi(s) f^{\wedge}(s) = \mu^{\vee}_{c_1,c_2}(s)$ for every s with $c_1 < \operatorname{Re}(s) < c_2$. Now it follows as above that $\mu(r) = \mu_{c_1,c_2}(r)$ indeed belongs to BV_{σ_1,σ_2} and that $\psi(s) f^{\wedge}(s) = \mu^{\vee}(s)$ for all $s \in A$. Thus $f \in V[L^1_{\sigma_1,\sigma_2}; \psi(s)]$, and the proof is complete.

THEOREM 3.4. *If* $\{k_{\vartheta}(r)\}$ *satisfies* (3.2), *then* $f \in V[X_{\sigma_1,\sigma_2}; \psi(s)]$ *implies* (3.5) *for every* $\sigma \in (\sigma_1, \sigma_2)$.

Proof. Let $X = L^p$, $1 < p < \infty$, and $g \in L^p_{\sigma_1,\sigma_2}$ be such that $\psi(s) f^{\wedge}(s) = g^{\wedge}(s)$ for all $s \in A$. Then in view of (3.2),

$$\begin{aligned}(\vartheta^{-\gamma}[I_{\vartheta}(f; r) - f(r)])^{\wedge}(s) &= \vartheta^{-\gamma}[k^{\wedge}_{\vartheta}(s) - 1] f^{\wedge}(s) \\ &= v^{\vee}_{\vartheta}(s) \psi(s) f^{\wedge}(s) = g^{\wedge}(s) v^{\vee}_{\vartheta}(s).\end{aligned} \tag{3.12}$$

Defining the convolution $g * dv_{\vartheta}$ of $g \in L^p_{\sigma_1,\sigma_2}$ and $v_{\vartheta} \in BV_{\sigma_1,\sigma_2}$ by

$$(g * dv_{\vartheta})(r) = \int_0^{\infty} g(r/\rho)\, dv_{\vartheta}(\rho), \tag{3.13}$$

one has similarly to (1.8)–(1.10) that $g * dv_\vartheta \in L^p_{\sigma_1,\sigma_2}$ and

$$\|r^\sigma(g * dv_\vartheta)(r)\|_p \leqq \|r^\sigma g(r)\|_p \int_0^\infty r^\sigma |dv_\vartheta(r)|, \tag{3.14}$$

$$[g * dv_\vartheta]^\wedge(s) = g^\wedge(s) v_\vartheta^\vee(s), \qquad s \in A. \tag{3.15}$$

Hence it follows by (3.12) and the uniqueness theorem that for every $f \in V[L^p_{\sigma_1,\sigma_2}; \psi(s)]$, $1 < p < \infty$, there exists $g \in L^p_{\sigma_1,\sigma_2}$ such that the representation

$$\vartheta^{-\gamma}[I_\vartheta(f; r) - f(r)] = \int_0^\infty g(r/\rho)\, dv_\vartheta(\rho) \quad \text{a.e.} \tag{3.16}$$

holds. By (3.14) this implies the approximation (3.5) in the case of the present X-spaces.

If $X = C$ and $g \in L^\infty_{\sigma_1,\sigma_2}$ is such that $\psi(s) f^\wedge(s) = g^\wedge(s)$, then one has again (3.16) so that

$$\|r^\sigma[I_\vartheta(f; r) - f(r)]\|_C \leqq \vartheta^\gamma \|r^\sigma g(r)\|_\infty \int_0^\infty r^\sigma |dv_\vartheta(r)|.$$

Finally, let $X = L^1$ and $\mu \in BV_{\sigma_1,\sigma_2}$ be such that $\psi(s) f^\wedge(s) = \mu^\vee(s)$. Similarly to (3.12) one has

$$\left[\int_1^r \frac{I_\vartheta(f; \rho) - f(\rho)}{\vartheta^\gamma} \frac{d\rho}{\rho}\right]^\vee(s) = \frac{k_\vartheta^\wedge(s) - 1}{\vartheta^\gamma} f^\wedge(s) = \mu^\vee(s) v_\vartheta^\vee(s). \tag{3.17}$$

According to the convolution theorem for functions μ, $v_\vartheta \in BV_{\sigma_1,\sigma_2}$ there exists a uniquely determined function $\mu * dv_\vartheta \in BV_{\sigma_1,\sigma_2}$ such that

$$\int_0^\infty r^\sigma |d[\mu * dv_\vartheta](r)| \leqq \int_0^\infty r^\sigma |d\mu(r)| \cdot \int_0^\infty r^\sigma |dv_\vartheta(r)|, \tag{3.18}$$

$$[\mu * dv_\vartheta]^\vee(s) = \mu^\vee(s) v_\vartheta^\vee(s), \qquad s \in A \tag{3.19}$$

(compare with the bilateral Laplace analogue in [17, p. 257]). This implies by (3.17) and the uniqueness theorem for Mellin–Stieltjes transforms that

$$\int_1^r \frac{I_\vartheta(f; \rho) - f(\rho)}{\vartheta^\gamma} \frac{d\rho}{\rho} = (\mu * dv_\vartheta)(r).$$

Therefore by (3.18) and (3.2) for $\vartheta \to 0+$,

$$\vartheta^{-\gamma} \|r^\sigma[I_\vartheta(f; r) - f(r)]\|_1 = \int_0^\infty r^\sigma |d[\mu * dv_\vartheta](r)| = O(1).$$

This completes the proof of Theorem 3.4.

Combining the results so far obtained we arrive at the following saturation theorem for the approximation of functions $f \in X_{\sigma_1,\sigma_2}$ by convolution integrals of type (2.1) in the X_{σ_1,σ_2}-topology.

THEOREM 3.5. *Let $f \in X_{\sigma_1,\sigma_2}$ and $\{k_\vartheta(r)\}$ satisfy* (3.1), (3.2). *Then the singular integral $I_\vartheta(f; r)$ is saturated in X_{σ_1,σ_2} with order $O(\vartheta^\gamma)$, $\vartheta \to 0+$. A function f belongs to the Favard class $F[X_{\sigma_1,\sigma_2}; I_\vartheta]$ if and only if f belongs to $V[X_{\sigma_1,\sigma_2}; \psi(s)]$.*

Indeed, this is an immediate consequence of Corollary 3.2 and Theorems 3.3, 3.4, provided we can show that the Favard class contains at least one element which does not vanish identically. To this end, it follows by (3.2) and the convolution theorem (cf. (3.15)) that for every $f \in X_{\sigma_1,\sigma_2}$,

$$\begin{aligned}\psi(s)[f * dv_\vartheta]^\wedge(s) &= \psi(s)v_\vartheta^\vee(s)f^\wedge(s) = \vartheta^{-\gamma}[k_\vartheta^\wedge(s) - 1]f^\wedge(s) \\ &= (\vartheta^{-\gamma}[I_\vartheta(f;r) - f(r)])^\wedge(s)\end{aligned} \tag{3.20}$$

for every $s \in A$. This, first of all, gives $f * dv_\vartheta \in V[X_{\sigma_1,\sigma_2}; \psi(s)]$ for every $f \in X_{\sigma_1,\sigma_2}$ and $0 < \vartheta < \vartheta_0$. Now it is easy to see that there exists at least one $f_0 \in X_{\sigma_1,\sigma_2}$ such that $f_0 * dv_\vartheta$ is not the null function for all $0 < \vartheta < \vartheta_0$. For, otherwise, $f^\wedge(s)v_\vartheta^\vee(s) = 0$ by (3.15) and thus $I_\vartheta(f;r) = f(r)$ by (3.20) for every $f \in X_{\sigma_1,\sigma_2}$, a contradiction to Corollary 3.2 which states that the null function is the only invariant element of $I_\vartheta(f;r)$.

4. Dirichlet's problem for a wedge. For an arbitrary, fixed ϑ_0, $0 < \vartheta_0 < 2\pi$, let W denote the wedge in the plane as given (in its polar coordinate form) by $W = \{(r,\vartheta) | 0 < r < \infty, 0 < \vartheta < \vartheta_0\}$. Dirichlet's problem for W then calls for a solution $u(r,\vartheta)$ of Laplace's equation

$$\left[\left(r\frac{\partial}{\partial r}\right)^2 + \frac{\partial^2}{\partial \vartheta^2}\right]u(r,\vartheta) = 0, \qquad (r,\vartheta) \in W, \tag{4.1}$$

which attains prescribed values $f_1(r), f_2(r)$ at the boundaries $\vartheta = 0$, $\vartheta = \vartheta_0$:

$$\lim_{\vartheta \to 0+} u(r,\vartheta) = f_1(r), \qquad \lim_{\vartheta \to \vartheta_0 -} u(r,\vartheta) = f_2(r). \tag{4.2}$$

This problem, in its various interpretations, is a standard example for the use of Mellin transform methods in the solution of partial differential equations; see, for example, Colombo [7, p. 72], Lomen [13], Sneddon [14, p. 294], Zemanian [20, p. 121], also Uflyand [16, p. 119 ff]. In this paper we are interested in the following interpretation of a solution:

Given two functions $f_1, f_2 \in X_{\sigma_1,\sigma_2}$, we call for a function $u(r,\vartheta)$ on W such that with $E = r(\partial/\partial r)$, $D = \partial/\partial\vartheta$:

(4.3)

(i) $\partial^2 u(r,\vartheta)/\partial r^2$, $\partial^2 u(r,\vartheta)/\partial \vartheta^2$ exist for all $(r,\vartheta) \in W$,

(ii) $u(r,\vartheta)$ satisfies (4.1) for all $(r,\vartheta) \in W$,

(iii) $u(r,\vartheta) \in X_{\sigma_1,\sigma_2}$ for each $0 < \vartheta < \vartheta_0$, and for each $\sigma \in (\sigma_1,\sigma_2)$,

$$\lim_{\vartheta \to 0+} \|r^\sigma[u(r,\vartheta) - f_1(r)]\|_X = 0, \qquad \lim_{\vartheta \to \vartheta_0 -} \|r^\sigma[u(r,\vartheta) - f_2(r)]\|_X = 0,$$

(iv) $Eu \in AC_{\mathrm{loc}}(0,\infty)$ such that $E^2 u \in X_{\sigma_1,\sigma_2}$ for each $0 < \vartheta < \vartheta_0$,

(v) $D^{j+1}u \in X_{\sigma_1,\sigma_2}$ for each $0 < \vartheta < \vartheta_0$, $j = 0, 1$, and for each $\sigma \in (\sigma_1,\sigma_2)$,

$$\lim_{\eta \to 0} \|r^\sigma[\eta^{-1}\{(D^j u)(r,\vartheta + \eta) - (D^j u)(r,\vartheta)\} - (D^{j+1}u)(r,\vartheta)]\|_X = 0.$$

To solve this problem, let us assume that there exists a function $u(r,\vartheta)$ satisfying these conditions, and apply the Mellin transform to (4.1). Setting $[u(r,\vartheta)]^\wedge(s) = u^\wedge(s,\vartheta)$, it follows from (iii), (iv), and Lemma 1.1 that

$$[(E^2 u)(r,\vartheta)]^\wedge(s) = s^2 u^\wedge(s,\vartheta), \qquad s \in A, \quad 0 < \vartheta < \vartheta_0. \tag{4.4}$$

On the other hand, for $j = 0, 1$,

$$[(D^{j+1}u)(r,\vartheta)]^\wedge(s) = D^{j+1}u^\wedge(s,\vartheta), \qquad s\in A, \quad 0<\vartheta<\vartheta_0, \tag{4.5}$$

since by (v) (see also (1.3), (3.4)),

$$|\eta^{-1}[u^\wedge(s,\vartheta+\eta) - u^\wedge(s,\vartheta)] - [(Du)(r,\vartheta)]^\wedge(s)| = o(1), \qquad \eta\to 0,$$

in case $j = 0$, and similarly for $j = 1$. Hence, for each $j = 0, 1$ and $s \in A$, $(D^{j+1}u^\wedge)(s,\vartheta)$ exists for $\vartheta\in(0,\vartheta_0)$; in particular, $u^\wedge(s,\vartheta)$ and $(Du^\wedge)(s,\vartheta)$ are continuous functions of ϑ on $(0,\vartheta_0)$ for each $s\in A$. Since by (iii),

$$\lim_{\vartheta\to 0+} u^\wedge(s,\vartheta) = f_1^\wedge(s), \quad \lim_{\vartheta\to\vartheta_0-} u^\wedge(s,\vartheta) = f_2^\wedge(s) \tag{4.6}$$

for each $s\in A$, the Mellin transform of any solution of (4.3) is therefore necessarily a solution of the (ordinary) differential equations

$$\frac{\partial^2 u^\wedge(s,\vartheta)}{\partial\vartheta^2} + s^2 u^\wedge(s,\vartheta) = 0, \qquad s\in A, \tag{4.7}$$

satisfying the boundary conditions (4.6). By classical methods the solution of (4.6)–(4.7) is given by

$$u^\wedge(s,\vartheta) = f_1^\wedge(s)\frac{\sin(\vartheta_0-\vartheta)s}{\sin\vartheta_0 s} + f_2^\wedge(s)\frac{\sin\vartheta s}{\sin\vartheta_0 s}. \tag{4.8}$$

Hence, if (4.3) has a solution, then its Mellin transform is given by (4.8). In order to reconstruct the full solution, let $\sigma_0 = \pi/\vartheta_0$ and

$$\chi_{j,\vartheta}(r) = \frac{\sin\sigma_0\vartheta}{\vartheta_0}\,\frac{1}{r^{\sigma_0} + (-1)^j 2\cos\sigma_0\vartheta + r^{-\sigma_0}}, \qquad j = 1, 2. \tag{4.9}$$

Then $\chi_{j,\vartheta}\in L^1_{-\sigma_0,\sigma_0}\cap C_{-\sigma_0,\sigma_0}$ for $j = 1, 2$ and (cf. [9, p. 309 (12)])

$$\chi_{1,\vartheta}^\wedge(s) = \frac{\sin(\vartheta_0-\vartheta)s}{\sin\vartheta_0 s}, \quad \chi_{2,\vartheta}^\wedge(s) = \frac{\sin\vartheta s}{\sin\vartheta_0 s}. \tag{4.10}$$

Hence, if σ_1, σ_2 are such that $-\sigma_0 \leqq \sigma_1 < \sigma_2 \leqq \sigma_0$, then by the convolution and uniqueness theorems,

$$u(r,\vartheta) = (f_1 * \chi_{1,\vartheta})(r) + (f_2 * \chi_{2,\vartheta})(r) \tag{4.11}$$

for all $(r,\vartheta)\in W$ since both sides represent continuous functions. Apart from (iii) which is discussed below in detail, one checks directly that $u(r,\vartheta)$ as given by (4.11) *is* indeed a solution of (4.3). Thus the procedure shows that problem (4.3) has a *unique* solution for $(\sigma_1,\sigma_2)\subset(-\sigma_0,\sigma_0)$. Though the representation (4.11) of the solution is well known, the present approach, particularly the definition of a solution (really raised as an abstract Cauchy problem, cf. Butzer–Berens [5, p. 3 ff]), allows a *rigorous* treatment inspired by the study of Bochner–Chandrasekharan [2, p. 40 ff] on Fourier transforms (see also Butzer–Nessel [6, Chap. 7] and the literature cited there).

Concerning particular values of ϑ_0, let us mention the case $\vartheta_0 = \pi$, so that $\sigma_0 = 1$. Then W is the upper half-plane, and (4.11) reduces to the well-known singular integral of Cauchy–Poisson:

$$P(f; x; y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(x-u)}{x^2+u^2}\,du \tag{4.12}$$

upon setting

$$x = r\cos\vartheta, \quad y = r\sin\vartheta, \quad f(t) = \begin{cases} f_1(t) & \text{for} \quad t > 0 \\ f_2(-t) & \text{for} \quad t < 0. \end{cases}$$

Let us further mention (cf. § 1) that the above problem may also be raised in connection with the bilateral Laplace transform. Then we ask for a function $v(x, y)$ which solves the Laplace equation

$$\frac{\partial^2 v(x,y)}{\partial x^2} + \frac{\partial^2 v(x,y)}{\partial y^2} = 0 \tag{4.13}$$

in the strip $B = \{(x, y)|0 < x < \pi, -\infty < y < \infty\}$ and satisfies the boundary conditions

$$\lim_{x\to 0+} v(x, y) = g_1(y), \qquad \lim_{x\to\pi-} v(x, y) = g_2(y). \tag{4.14}$$

The solution then takes on the form (compare also Widder [18], [19]):

$$\begin{aligned} v(x, y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_1(y-t)\frac{\sin x}{\cosh t - \cos x}\,dt \\ &\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} g_2(y-t)\frac{\sin x}{\cosh t + \cos x}\,dt. \end{aligned} \tag{4.15}$$

Indeed, if $z = r\exp\{i\vartheta\}$ and $z' = x + iy$, then the wedge W and the strip B are related by the conformal mapping $z = \exp\{i(\vartheta_0/\pi)z'\}$, i.e., by $\vartheta = (\vartheta_0/\pi)x$ and $r = \exp\{-(\vartheta_0/\pi)y\}$, so that (4.11) is converted into (4.15) by setting $g_j(t) = f_j(\exp\{-(\vartheta_0/\pi)t\})$, $j = 1, 2$ (cf. (1.7)). Therefore the above treatment for (4.3) also covers the present version of the problem, and thus Dirichlet's problem for the strip B.

Let us now turn to condition (4.3) (iii). First we treat the *symmetric* problem (cf. Sneddon [14, p. 295 ff]), i.e., we assume $f_1(r) = f_2(r) \equiv f(r)$. Then the solution $u(r, \vartheta)$ as given by (4.11) is denoted by $S_\vartheta(f; r)$ and takes on the form

$$S_\vartheta(f; r) = (f * \chi_\vartheta)(r), \tag{4.16}$$

where the kernel $\{\chi_\vartheta(r)\}$ is given by $(\sigma_0 = \pi/\vartheta_0)$

$$\chi_\vartheta(r) = \chi_{1,\vartheta}(r) + \chi_{2,\vartheta}(r) = \frac{2\sin\sigma_0\vartheta}{\vartheta_0}\,\frac{r^{\sigma_0} + r^{-\sigma_0}}{r^{2\sigma_0} - 2\cos 2\sigma_0\vartheta + r^{-2\sigma_0}}. \tag{4.17}$$

$\{\chi_\vartheta(r)\}$ satisfies (2.2) for $\sigma_1 = -\sigma_0, \sigma_2 = \sigma_0$. Indeed, since $\chi_\vartheta(r) \geqq 0$, (4.10) implies

$$\|r^\sigma\chi_\vartheta(r)\|_1 = \int_0^\infty r^\sigma\chi_\vartheta(r)\frac{dr}{r} = \frac{\sin(\vartheta_0-\vartheta)\sigma + \sin\vartheta\sigma}{\sin\vartheta_0\sigma}$$

for every $\sigma \in (-\sigma_0, \sigma_0)$ and $0 < \vartheta < \vartheta_0$. In particular,

$$\int_0^\infty \chi_\vartheta(r) \frac{dr}{r} = 1 \tag{4.18}$$

for all $0 < \vartheta < \vartheta_0$, proving (ii) and (iii) of (2.2). Finally, given $0 < \delta < 1$, then for each $\sigma \in (-\sigma_0, \sigma_0)$,

$$|r^\sigma \chi_\vartheta(r)| \leqq \frac{2}{\vartheta_0} r^\sigma \frac{r^{\sigma_0} + r^{-\sigma_0}}{(r^{\sigma_0} - r^{-\sigma_0})^2} \leqq \frac{4}{\vartheta_0} \frac{r^{\sigma \pm \sigma_0}}{(1 - (1 \mp \delta)^{\pm 2\sigma_0})^2}$$

for $0 < r < 1 - \delta$ and $1 + \delta < r < \infty$, respectively. Since $\lim_{\vartheta \to 0+} \chi_\vartheta(r) = 0$ for $r \neq 1$, (2.2)(iv) follows by Lebesgue's dominated convergence theorem.

Concerning conditions (3.1), (3.2) for $\sigma_1 = -\sigma_0$, $\sigma_2 = \sigma_0$, in view of (4.10),

$$\lim_{\vartheta \to 0+} \frac{\chi_\vartheta^\wedge(s) - 1}{\vartheta} = s \tan(\vartheta_0 s/2) \equiv \lambda(s). \tag{4.19}$$

Thus (3.1) is satisfied with $\gamma = 1$ and $\psi(s) = \lambda(s)$. Concerning (3.2), we have

$$\frac{\chi_\vartheta^\wedge(s) - 1}{\vartheta \lambda(s)} = \frac{\cos(\vartheta - \vartheta_0)s}{s\vartheta \sin \vartheta_0 s} + \frac{\cos \vartheta s}{s\vartheta \sin \vartheta_0 s} + \frac{-\cos(\vartheta_0 s/2)}{s\vartheta \sin(\vartheta_0 s/2)}.$$

In view of [9, p. 316 (27), p. 315 (18)] the three terms on the right are the Mellin transform of

$$\begin{gathered}(1/2\pi\vartheta) \log[1 - 2r^{\sigma_0} \cos \sigma_0 \vartheta + r^{2\sigma_0}], \quad (1/2\pi\vartheta) \log[1 + 2r^{\sigma_0} \cos \sigma_0 \vartheta + r^{2\sigma_0}], \\ (-1/2\pi\vartheta) \log[1 - r^{2\sigma_0}]^2,\end{gathered} \tag{4.20}$$

respectively, for each s with $-\sigma_0 < \mathrm{Re}(s) < 0$. However, their sum

$$h_\vartheta(r) = \frac{1}{2\pi\vartheta} \log \frac{r^{2\sigma_0} - 2\cos 2\sigma_0 \vartheta + r^{-2\sigma_0}}{(r^{\sigma_0} - r^{-\sigma_0})^2}$$

is an element of $L^1_{-\sigma_0, \sigma_0}$; in fact, $h_\vartheta(r) \geqq 0$ and $h_\vartheta(r) = h_\vartheta(1/r)$. Therefore, since $\|r^\sigma h_\vartheta(r)\|_1 = h_\vartheta^\wedge(\sigma) = O(1)$ as $\vartheta \to 0+$ for each $\sigma \in (-\sigma_0, \sigma_0)$, condition (3.2) is satisfied by the kernel $\{\chi_\vartheta(r)\}$ with $v_\vartheta(r) = \int_1^r h_\vartheta(\rho)\rho^{-1}\, d\rho$.

Hence, if we pose the problem concerning in what manner the solution (4.11) of Dirichlet's problem (4.3) for the wedge W attains symmetric boundary values, the results of § 2 and § 3 give us the following theorem.

THEOREM 4.1. *For symmetric boundary values $f \in X_{-\sigma_0, \sigma_0}$, $\sigma_0 = \pi/\vartheta_0$, the solution* (4.16) *of Dirichlet's problem for the wedge W attains its boundary value at $\vartheta = 0$ in the $X_{-\sigma_0,\sigma_0}$-topology, i.e.,*

$$\lim_{\vartheta \to 0+} \|r^\sigma[S_\vartheta(f; r) - f(r)]\|_X = 0 \tag{4.21}$$

for every $f \in X_{-\sigma_0,\sigma_0}$ and $\sigma \in (-\sigma_0, \sigma_0)$. The approximation (4.21) *is saturated in $X_{-\sigma_0,\sigma_0}$ with order $O(\vartheta)$, $\vartheta \to 0+$, the Favard class being characterized as the class $V[X_{-\sigma_0,\sigma_0}; \lambda(s)]$.*

This result reveals that the solution $S_\vartheta(f; r)$ cannot tend too rapidly towards the boundary value f. In fact, if for each $\sigma \in (-\sigma_0, \sigma_0)$,

$$\|r^\sigma[S_\vartheta(f; r) - f(r)]\|_X = o(\vartheta), \qquad \vartheta \to 0+,$$

then necessarily $f(r) = 0$ (a.e.). Moreover, for $X = C$, for example,

$$\|r^{\sigma}[S_{\vartheta}(f; r) - f(r)]\|_C = O(\vartheta)$$

if and only if there exists $g \in L^{\infty}_{-\sigma_0,\sigma_0}$ such that $s \tan(\vartheta_0 s/2) f^{\wedge}(s) = g^{\wedge}(s)$.

Obviously, there are completely analogous results concerning the question as to how $S_{\vartheta}(f; r)$ attains symmetric boundary values at $\vartheta = \vartheta_0$. Thus $\{\chi_{\vartheta}(r)\}$ as given by (4.17) is also an approximate identity for $\vartheta \to \vartheta_0-$ and (cf. [5a])

$$\lim_{\vartheta \to \vartheta_0 -} \|r^{\sigma}[S_{\vartheta}(f; r) - f(r)]\|_X = 0 \tag{4.22}$$

for every $f \in X_{-\sigma_0,\sigma_0}$ and $\sigma \in (-\sigma_0, \sigma_0)$; the approximation (4.22) is saturated in $X_{-\sigma_0,\sigma_0}$ with order $O(\vartheta_0 - \vartheta)$ and Favard class characterized as $V[X_{-\sigma_0,\sigma_0}; \lambda(s)]$.

Next we consider the *one-sided* boundary value problem, i.e., we assume $f_1(r) = f(r)$, $f_2(r) = 0$ (cf. Zemanian [20, p. 121]). Then the solution $u(r, \vartheta)$ as given by (4.11) takes on the form $(f * \chi_{1,\vartheta})(r)$ and is denoted by $A_{\vartheta}(f; r)$; thus

$$A_{\vartheta}(f; r) = \frac{\sin \sigma_0 \vartheta}{\vartheta_0} \int_0^{\infty} \frac{f(r/\rho)}{\rho^{\sigma_0} - 2\cos\sigma_0\vartheta + \rho^{-\sigma_0}} \frac{d\rho}{\rho}. \tag{4.23}$$

Again it follows that the kernel $\{\chi_{1,\vartheta}(r)\}$ satisfies (2.2) for $\sigma_1 = -\sigma_0$, $\sigma_2 = \sigma_0$. Indeed, $\chi_{1,\vartheta}(r) \geqq 0$ and

$$\int_0^{\infty} \chi_{1,\vartheta}(r) \frac{dr}{r} = \frac{\vartheta_0 - \vartheta}{\vartheta_0}. \tag{4.24}$$

Concerning conditions (3.1), (3.2), in view of (4.10),

$$\lim_{\vartheta \to 0+} \frac{\chi_{1,\vartheta}^{\wedge}(s) - 1}{\vartheta} = -s \cot \vartheta_0 s \equiv \lambda_1(s). \tag{4.25}$$

Thus (3.1) is satisfied with $\gamma = 1$, $\psi(s) = \lambda_1(s)$, and $\sigma_1 = -\sigma_0$, $\sigma_2 = \sigma_0$. Moreover, for $-\sigma_0/2 < \operatorname{Re}(s) < \sigma_0/2$,

$$\frac{\chi_{1,\vartheta}^{\wedge}(s) - 1}{\vartheta\lambda_1(s)} = -\frac{\sin(\vartheta_0 - \vartheta)s}{\vartheta s \cos \vartheta_0 s} + \frac{\tan \vartheta_0 s}{\vartheta s}$$

is the Mellin transform of (cf. [9, p. 316 (27), (24)])

$$l_{\vartheta}(r) = \frac{1}{2\pi\vartheta} \log \frac{r^{\sigma_0/2} + r^{-\sigma_0/2} - 2\sin(\sigma_0(\vartheta_0 - \vartheta)/2)}{r^{\sigma_0/2} + r^{-\sigma_0/2} + 2\sin(\sigma_0(\vartheta_0 - \vartheta)/2)} + \frac{1}{\pi\vartheta} \log \left| \frac{1 + r^{\sigma_0/2}}{1 - r^{\sigma_0/2}} \right|.$$

It follows that (3.2) is satisfied with $\sigma_1 = -\sigma_0/2$, $\sigma_2 = \sigma_0/2$ so that the following theorem holds.

THEOREM 4.2. *For one-sided boundary values $f_1(r) \equiv f(r) \in X_{-\sigma_0,\sigma_0}$, $f_2(r) = 0$, the solution* (4.23) *of Dirichlet's problem for the wedge W attains its boundary value f at $\vartheta = 0$ in the $X_{-\sigma_0,\sigma_0}$-topology, i.e.,*

$$\lim_{\vartheta \to 0+} \|r^{\sigma}[A_{\vartheta}(f; r) - f(r)]\|_X = 0 \tag{4.26}$$

for every $f \in X_{-\sigma_0,\sigma_0}$ and $\sigma \in (-\sigma_0, \sigma_0)$, $\sigma_0 = \pi/\vartheta_0$. The approximation (4.26) *is saturated in $X_{-\sigma_0/2,\sigma_0/2}$ with order $O(\vartheta)$, the Favard class being characterized by $V[X_{-\sigma_0/2,\sigma_0/2}; \lambda_1(s)]$.*

Again one has similar results if one considers problem (4.3) for the (one-sided) boundary values $f_1(r) = 0, f_2(r) \equiv f(r)$. Then one is interested in the approximation of f by the singular integral

$$B_\vartheta(f; r) = (f * \chi_{2,\vartheta})(r) = \frac{\sin \sigma_0 \vartheta}{\vartheta_0} \int_0^\infty \frac{f(r/\rho)}{\rho^{\sigma_0} + 2\cos \sigma_0 \vartheta + \rho^{-\sigma_0}} \frac{d\rho}{\rho} \tag{4.27}$$

as $\vartheta \to \vartheta_0-$. It follows that $\{\chi_{2,\vartheta}(r)\}$ is a positive approximate identity for $\vartheta \to \vartheta_0-$ and

$$\lim_{\vartheta \to \vartheta_0 -} \|r^\sigma [B_\vartheta(f; r) - f(r)]\|_X = 0 \tag{4.28}$$

for every $f \in X_{-\sigma_0,\sigma_0}$ and $\sigma \in (-\sigma_0, \sigma_0)$, the approximation (4.28) being saturated in $X_{-\sigma_0/2,\sigma_0/2}$ with order $O(\vartheta_0 - \vartheta)$ and Favard class characterized by $V[X_{-\sigma_0/2,\sigma_0/2}; \lambda_1(s)]$.

Regarding arbitrary boundary values f_1, f_2, we first observe that the results of § 2 and § 3 are concerned with approximation processes of type (2.1) so that they do not directly cover the case of the general solution (4.11). However, certain results for the general case may be inferred from those given for symmetric (or one-sided) boundary values. Indeed, concerning the approximation for $\vartheta \to 0+$, one has for $\vartheta \in (0, \vartheta_0/2)$ that

$$\chi_{2,\vartheta}(r) \leqq \frac{\sin \sigma_0 \vartheta}{\vartheta_0(r^{\sigma_0} + r^{-\sigma_0})} = \chi_{2,\vartheta_0/2}(r) \sin \sigma_0 \vartheta, \tag{4.29}$$

and thus by (1.9), (4.27) for any $g \in X_{-\sigma_0,\sigma_0}$ and $\sigma \in (-\sigma_0, \sigma_0)$,

$$\|r^\sigma B_\vartheta(g; r)\|_X \leqq \|r^\sigma g(r)\|_X \|r^\sigma \chi_{2,\vartheta_0/2}(r)\|_1 \sin \sigma_0 \vartheta = O(\vartheta), \qquad \vartheta \to 0+. \tag{4.30}$$

Similarly, concerning approximation for $\vartheta \to \vartheta_0-$, one has for $\vartheta \in (\vartheta_0/2, \vartheta_0)$ that

$$\chi_{1,\vartheta}(r) \leqq \frac{\sin \sigma_0 \vartheta}{\vartheta_0(r^{\sigma_0} + r^{-\sigma_0})} = \chi_{1,\vartheta_0/2}(r) \sin \sigma_0(\vartheta_0 - \vartheta) \tag{4.31}$$

since $\pi - \sigma_0 \vartheta = \sigma_0(\vartheta_0 - \vartheta)$. Thus for any $g \in X_{-\sigma_0,\sigma_0}$ and $\sigma \in (-\sigma_0, \sigma_0)$ (cf. (4.23)),

$$\|r^\sigma A_\vartheta(g; r)\|_X \leqq \|r^\sigma g(r)\|_X \|r^\sigma \chi_{1,\vartheta_0/2}(r)\|_1 \sin \sigma_0(\vartheta_0 - \vartheta) = O(\vartheta_0 - \vartheta), \tag{4.32}$$

$\vartheta \to \vartheta_0-$. Therefore for the general solution $u(r, \vartheta)$ (cf. (4.11)) of (4.3) for arbitrary boundary values $f_1, f_2 \in X_{-\sigma_0,\sigma_0}$,

$$\begin{aligned} \|r^\sigma [u(r, \vartheta) - f_1(r)]\|_X &\leqq \|r^\sigma [S_\vartheta(f_1; r) - f_1(r)]\|_X + \|r^\sigma B_\vartheta(f_2 - f_1; r)\|_X \\ &\leqq \|r^\sigma [S_\vartheta(f_1; r) - f_1(r)]\|_X + O(\vartheta), \qquad \vartheta \to 0+ \end{aligned} \tag{4.33}$$

(cf. (4.16)), and similarly

$$\begin{aligned} \|r^\sigma [u(r, \vartheta) - f_2(r)]\|_X &\leqq \|r^\sigma [S_\vartheta(f_2; r) - f_2(r)\|_X + \|r^\sigma A_\vartheta(f_1 - f_2; r)\|_X \\ &\leqq \|r^\sigma [S_\vartheta(f_2; r) - f_2(r)]\|_X + O(\vartheta_0 - \vartheta), \quad \vartheta \to \vartheta_0-. \end{aligned} \tag{4.34}$$

Moreover, the converses are also valid, i.e., one has

$$\|r^\sigma [S_\vartheta(f_1; r) - f_1(r)]\|_X \leqq \|r^\sigma [u(r, \vartheta) - f_1(r)]\|_X + O(\vartheta), \tag{4.35}$$

$$\|r^\sigma [S_\vartheta(f_2; r) - f_2(r)]\|_X \leqq \|r^\sigma [u(r, \vartheta) - f_2(r)]\|_X + O(\vartheta_0 - \vartheta). \tag{4.36}$$

First of all, in view of (4.21)–(4.22) the estimates (4.33)–(4.34) show that the solution $u(r, \vartheta)$ attains arbitrary boundary values $f_1, f_2 \in X_{-\sigma_0,\sigma_0}$ in the sense of condition (4.3) (iii) (with $\sigma_1 = -\sigma_0, \sigma_2 = \sigma_0$). Moreover, combining (4.33)–(4.36) delivers the asymptotic relations

$$\|r^\sigma[u(r,\vartheta) - f_1(r)]\|_X = \|r^\sigma[S_\vartheta(f_1; r) - f_1(r)]\|_X + O(\vartheta), \qquad \vartheta \to 0+, \tag{4.37}$$

$$\|r^\sigma[u(r,\vartheta) - f_2(r)]\|_X = \|r^\sigma[S_\vartheta(f_2; r) - f_2(r)]\|_X + O(\vartheta_0 - \vartheta), \quad \vartheta \to \vartheta_0-, \tag{4.38}$$

for every $f_1, f_2 \in X_{-\sigma_0,\sigma_0}$ and $\sigma \in (-\sigma_0, \sigma_0)$. This in particular implies the following result.

THEOREM 4.3. *Let $f_1, f_2 \in X_{-\sigma_0,\sigma_0}$, $\sigma_0 = \pi/\vartheta_0$. Then for each $\sigma \in (-\sigma_0, \sigma_0)$ and $0 < \alpha \leqq 1$,*

$$\|r^\sigma[u(r,\vartheta) - f_1(r)]\|_X = O(\vartheta^\alpha) \quad \textit{if and only if} \quad \|r^\sigma[S_\vartheta(f_1; r) - f_1(r)]\|_X = O(\vartheta^\alpha), \tag{4.39}$$

$$\|r^\sigma[u(r,\vartheta) - f_2(r)]\|_X = O((\vartheta_0 - \vartheta)^\alpha) \quad \textit{if and only if} \quad \|r^\sigma[S_\vartheta(f_2; r) - f_2(r)]\|_X = O((\vartheta_0 - \vartheta)^\alpha) \tag{4.40}$$

for $\vartheta \to 0+$, $\vartheta \to \vartheta_0-$, respectively.

Thus a good deal of the approximation theory for the general solution $u(r, \vartheta)$ is reducible to that for the singular integral $S_\vartheta(f; r)$, corresponding to symmetric boundary values. Hence let us once more examine the one-sided case, particularly the saturation behavior of $A_\vartheta(f; r)$. Theorem 4.2 only studies saturation of $A_\vartheta(f; r)$ in $X_{-\sigma_0/2,\sigma_0/2}$ since condition (3.2) was only verified in these spaces. However, condition (3.1) is satisfied by the kernel $\{\chi_{1,\vartheta}(r)\}$ for the whole interval $(-\sigma_0, \sigma_0)$ (cf. (4.25)) so that Corollary 3.2, Theorem 4.1, and (4.39) indeed imply the following corollary.

COROLLARY 4.4. *The approximation* (4.26) *is saturated in $X_{-\sigma_0,\sigma_0}$ with order $O(\vartheta)$, the Favard class being characterized by $V[X_{-\sigma_0,\sigma_0}; \lambda(s)]$* (cf. (4.19)).

Obviously, a similar extension may be given concerning saturation of the singular integral $B_\vartheta(f; r)$.

However, concerning solutions $u(r, \vartheta)$ corresponding to arbitrary boundary values $f_1, f_2 \in X_{-\sigma_0,\sigma_0}$, we cannot at this stage supplement Theorem 4.3 by an assertion concerning saturation. Indeed, § 3 only treats saturation for singular integrals of type (2.1); examples were given by $S_\vartheta(f; r)$, $A_\vartheta(f; r)$, $B_\vartheta(f; r)$ which only involve *one* function f to be approximated. But in the general case, $u(r, \vartheta)$ of (4.11) involves two functions f_1, f_2, and one has to consider the approximations

$$\lim_{\vartheta \to 0+} \|r^\sigma[u(r,\vartheta) - f_1(r)]\|_X, \quad \lim_{\vartheta \to \vartheta_0-} \|r^\sigma[u(r,\vartheta) - f_2(r)]\|_X$$

simultaneously. This leads to the study of saturation for systems of coupled approximations, the principles of which will be outlined in [12]. On the other hand, one may consider, as a further example, antisymmetric boundary values $f_1(r) = -f_2(r)$ which may be discussed completely analogously to the symmetric case so that details may be omitted. The general case is then completely reducible to the symmetric and antisymmetric ones. Compare the corresponding treatment in [5a] in connection with Fourier transforms.

Acknowledgment. The authors would like to express their sincere gratitude to Professor Dr. P. L. Butzer who read the manuscript and gave many valuable suggestions.

REFERENCES

[1] H. Berens and P. L. Butzer, *On the best approximation for singular integrals by Laplace-transform methods*, On Approximation Theory, P. L. Butzer and J. Korevaar, eds., ISNM 5, Birkhäuser, 1964, pp. 24–42.

[2] S. Bochner and K. Chandrasekharan, *Fourier Transforms*, Ann. of Math. Studies no. 19, Princeton University Press, Princeton, 1949.

[3] P. L. Butzer, *Fourier transform methods in the theory of approximation*, Arch. Rational Mech. Anal., 5 (1960), pp. 390–415.

[4] ———, *Integral transform methods in the theory of approximation*, On Approximation Theory, P. L. Butzer and J. Korevaar, eds., ISNM 5, Birkhäuser, 1964, pp. 12–23.

[5] P. L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, Springer, Berlin, 1967.

[5a] P. L. Butzer, W. Kolbe and R. J. Nessel, *Approximation by functions harmonic in a strip*, Arch. Rational Mech. Anal., to appear.

[6] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, vol. I, One-Dimensional Theory, Birkhäuser, Basel, and Academic Press, New York, 1971.

[7] S. Colombo, *Les transformations de Mellin et de Hankel*, Centre National de la Recherche Scientifique, Paris, 1959.

[8] G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, Springer, Berlin, 1937.

[9] A. Erdélyi et al., *Tables of Integral Transforms. I*, McGraw-Hill, New York, 1954.

[10] J. Favard, *Sur l'approximation des fonctions d'une variable réelle*, Analyse Harmonique (Colloques Internat. Centre Nat. Rech. Sci. 15), Éditions Centre Nat. Rech. Sci., 1949, pp. 97–110.

[11] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd. ed., Cambridge University Press, London, 1952.

[12] W. Kolbe and R. J. Nessel, *Simultaneous saturation in connection with Dirichlet's problem for a wedge*, Applicable Analysis, An International Journal, to appear.

[13] D. Lomen, *Application of the Mellin transform to boundary-value problems*, Proc. Iowa Acad. Sci., 69 (1962), pp. 436–442.

[14] I. N. Sneddon, *Functional Analysis*, Encyclopedia of Physics, vol. II: Mathematical Methods II, S. Flügge, ed., Springer, Berlin, 1955, pp. 198–348.

[15] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, 2nd. ed., Oxford University Press, Oxford, 1948.

[16] Ya. S. Uflyand, *Integral Transforms in Problems of the Theory of Elasticity*, Izdat. Akad. Nauk SSSR, 1963. (In Russian.)

[17] D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.

[18] ———, *Functions harmonic on a strip*, Proc. Amer. Math. Soc., 12 (1961), pp. 67–72.

[19] ———, *Fourier cosine transforms whose real parts are non-negative in a strip*, Proc. Amer. Math. Soc., 16 (1965), pp. 1246–1252.

[20] A. H. Zemanian, *Generalized Integral Transformations*, Interscience, New York, 1968.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

MULTIPLE ASYMPTOTIC EXPANSIONS AND SINGULAR PROBLEMS*

KENNETH D. SHERE†

Abstract. Applications of multiple asymptotic expansions to singular differential equations have been investigated by means of four examples. The techniques of multiple asymptotic expansions are first applied to an equation with an essential singularity in the leading coefficient. The results are then compared to the results obtained by the techniques of H. Schmidt. Then two linear problems of singular perturbation are investigated. For a boundary value problem it is shown that the technique of multiple asymptotic expansions yields the same result as the two-variable expansion technique, and for an initial value problem it is shown that this technique improves upon the two-variable technique. Finally a nonlinear boundary value problem of singular perturbation is considered. It has been shown that, whenever the calculations are not too onerous, the technique of multiple asymptotic expansions yields new insights into the nature of the solution. All of the derivations are formal.

1. Introduction. The concept of asymptoticity is extended to double series of the form $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}x^{-n}e^{-\lambda mx}$ in [1]. There, the author demonstrates that the definition satisfies expected existence and algebraic properties and he applies this concept to elastic scattering from a Yukawa well. The definition is repeated below for convenience. In this note these series are applied formally to four differential equations.

The first equation is used to illustrate how the theory can be applied to some equations possessing an essential singularity at infinity and to demonstrate heuristically a connection between Schmidt's work [2] and multiple asymptotic series. Schmidt constructed asymptotic expansions of the form

$$\sum_{m=0}^{\infty} a_m(x)\exp(-\lambda mx),$$

where the $\{a_m(x)\}$ are of bounded variation. For the first example, it is shown that $a_m(x) \sim \sum_{n=0}^{\infty} a_{mn}x^{-n}$. As indicated in § 2, however, neither theory contains the other.

The last three equations and related side conditions are problems of singular perturbation. These examples are respectively a linear boundary value problem, a linear initial value problem and a nonlinear boundary value problem. The two-variable expansion technique involves the solution of a system of partial differential equations. The multiple asymptotic expansion technique applied to the same ordinary differential equation results in a new system of either ordinary differential or algebraic equations. This system is much easier to solve than the corresponding system of partial differential equations obtained from the two-variable expansion technique.

It is shown in these three examples that the same results can be obtained by either technique. In the example of a nonlinear boundary value problem it is shown how multiple asymptotic expansions can provide new insights both on the boundary layer and the nature of asymptotics. In particular, it may be possible to

* Received by the editors November 24, 1970, and in revised form May 24, 1971.
† U.S. Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland 20910.

eliminate spurious solutions by requiring the boundary conditions to be satisfied exactly and allowing the solution to have a jump discontinuity of sufficiently small order in its first derivative.

DEFINITION. Suppose that we are given the formal, not necessarily convergent, double sum

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}x^{-n}e^{-\lambda mx}, \tag{1}$$

where $\{a_{mn}: m, n = 0, 1, \cdots\}$ is an arbitrary sequence of complex numbers, $x \in R \equiv \{|x| > x_0 > 0, |\arg x| \leqq \alpha < \pi/4\}$ and $\lambda > 0$. Define

$$f_m(x) = \sum_{n=0}^{\infty} a_{mn}[1 - \exp(-f_{mn}x^2)]x^{-n}e^{-\lambda mx}, \tag{2}$$

where

$$f_{mn} \equiv \begin{cases} 0 & \text{for } a_{mn} = 0, \\ \min[1, 1/|a_{mn}|n!m!]/2 & \text{for } a_{mn} \neq 0. \end{cases} \tag{3}$$

The function $F(x)$ is asymptotic to (1) and is denoted by

$$F(x) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}x^{-n}e^{mx\lambda},$$

if

(a) $F_m \equiv \{f(x): f(x) \sim \sum_{n=0}^{\infty} a_{mn}x^{-n}e^{-\lambda mx} \quad \{x^{-n}e^{-\lambda mx}\}\}$ for each fixed $m = 0, 1, 2, \cdots$;
(b) $F_m^* \equiv \{f(x) \in F_m(x): f(x) - f_m(x) \sim 0 \quad \{x^{-n}e^{-\lambda \mu x}\}$ for each fixed $\mu = m, m+1, \cdots\}$;
(c) $F(x) \in F_0$;
(d) $F(x) - \sum_{\mu=0}^{m} f_\mu^* \in F_{m+1}$ for any $f_\mu^* \in F_\mu^*$.

The asymptotic symbols in (a) and (b) are taken in the Poincaré sense [3]. The series for $f_m(x)$, (1), converges uniformly in any compact subregion of R and absolutely in R (cf. [4]).

2. An equation with essential singularity. Consider the equation

$$e^{-x}u''(x) + x^h u'(x) + u(x) = 0. \tag{4}$$

The reduced equation of (4), $x^h u'(x) + u(x) = 0$, has an analytic solution at infinity whenever $h \geqq 1$; in particular, for $h \neq 1$, $u = k\exp(-x^{-(h-1)}/(h-1))$. It can be seen from the formal theory that the behavior of the reduced equation has a crucial effect on the form of the asymptotic solution. This is expected for a solution of the form (1) since the second term of (4), $x^h u'(x)$, dominates the leading term for large x in R.

Applying the techniques of Schmidt [2] to (4), we seek a solution of the form

$$y(x) \sim \sum_{m=0}^{\infty} g_m(x)e^{-mx}. \tag{5}$$

The asymptotic meaning of (5) was originally discussed by Schmidt. Upon substitution of (5) into (4) we see that $g_m(x)$ must satisfy

$$x^h g_m' - (mx^h - 1)g_m + g_{m-1}'' - 2(m-1)g_{m-1}' + (m-1)^2 g_{m-1} = 0 \tag{6}$$

with $g_{-1} = 0$. Hence, when $h = 1$,

$$(7)_0 \qquad g_0(x) = \alpha/x,$$

$$(7)_1 \qquad g_1(x) = -\frac{1}{x} e^x \int_\infty^x \frac{2\alpha\, e^{-t}}{t^3}\, dt.$$

Upon substitution of (1) into (4), with $\lambda = 1$, we obtain

$$\begin{aligned}
&\sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{\infty} [(n-1)a_{mn} + m a_{m,n+1}]x^{-n} + m a_{m0} x \right\} e^{-mx} \\
&= \sum_{m=1}^{\infty} \left\{ \sum_{n=2}^{\infty} [(n-2)(n-1)a_{m-1,n-2} + 2(m-1)(n-1)a_{m-1,n-1} \right. \\
&\qquad \left. + (m-1)^2 a_{m-1,n}]x^{-n} + (m-1)^2 a_{m-1,0} + (m-1)^2 a_{m-1,1} x^{-1} \right\} e^{-mx}.
\end{aligned} \tag{8}$$

From (8) we deduce that

$$\begin{aligned}
a_{0n} &= 0 \quad \text{for all } n \neq 1, \\
a_{m0} &= a_{m2} = a_{m3} = 0, && m = 0, 1, 2, \cdots, \\
a_{m1} &= 0, && m = 1, 2, \cdots, \\
a_{1n} &= (-1)^{n-1}(n-2)!\,a_{01}, && n = 4, 5, \cdots, \\
a_{25} &= a_{01}.
\end{aligned}$$

The constant a_{01} is arbitrary. Setting $a_{01} = -\alpha$, we have

$$u(x) \sim -\alpha\{x^{-1} + [2!x^{-4} - 3!x^{-5} + 4!x^{-6} - \cdots]e^{-x} + x^{-5}e^{-2x} + \cdots\}. \tag{9}$$

Comparing (9) with (5) and (7) we note, for example, that

$$g_0(x) \sim \sum_{n=0}^{\infty} a_{0n} x^{-n} \tag{10}$$

and

$$g_1(x) \sim \sum_{n=0}^{\infty} a_{1n} x^{-n}. \tag{11}$$

We observe that in general,

$$g_m(x) \sim \sum_{n=0}^{\infty} a_{mn} x^{-n}, \tag{12}$$

where the asymptotic signs of (10)–(12) are in the Poincaré sense, that is,

$$g_i(x) - \sum_{n=0}^{N} a_{in} x^{-n} = o(x^{-N}).$$

In other words, by substituting the asymptotic expansions for the coefficients of e^{-mx} found by Schmidt's theory we obtain a multiple asymptotic series.

If the form of the asymptotic series is suspected a priori, then the technique of multiple asymptotic sequences is easier to apply because it yields an infinite system of algebraic equations which must be solved in lieu of the infinite system of differential equations obtained by using Schmidt's technique.

When $h > 1$, $(7)_0$ becomes

$$g_0(x) = \alpha \frac{\exp(-x^{-(h-1)})}{h-1} = \alpha \sum_{n=0}^{\infty} (-1)^n \frac{x^{-n(h-1)}}{(h-1)^n n!}.$$

If h were not an integer, a modified multiple asymptotic expansion of the form $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn} x^{-nh} e^{-mx}$ would be used to obtain a formal solution. If $h < 1$, an entirely different form must be used. These more general expansions are discussed in [9].

Neither Schmidt's theory nor the theory of multiple asymptotic expansions contains the other. The coefficients $g_m(x)$ in Schmidt's theory must be of bounded variation but need not have convenient asymptotic representations. Likewise, multiple asymptotic expansions may be quite general and need not be representable in the form $\sum(\sum a_{mn} x^{-n}) e^{-\lambda mx}$ (cf. [9]).

3. Two singular perturbation problems. The initial value problem

$$\begin{aligned} &\varepsilon y'' + y' + y = 0, \\ &y(0) = 0, \qquad y'(0) = 1/\varepsilon \end{aligned} \tag{13}$$

has been used by Friedrichs [5] and Lick [6] to discuss boundary layer phenomena and two-variable expansions respectively. Erdélyi [7] used the boundary value problem

$$\begin{aligned} &\varepsilon y''(t) + y'(t) = h(t), \\ &y(0) = \alpha, \qquad y(1) = \beta \end{aligned} \tag{14}$$

to compare the two-variable expansion technique with the composite method of matching inner and outer expansions developed by Kaplun [8] and Lagerstrom. In this section we assume asymptotic solutions of the form

$$y(t, \varepsilon) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} y_{m,n}(t) \varepsilon^n e^{-tm/\varepsilon} \quad \text{as } \varepsilon \to 0 \tag{15}$$

in a sector of the origin containing $(0, \varepsilon_0)$ for some $\varepsilon_0 > 0$. In (15), the variable t is considered a parameter. The sector defining ε and the values of t are restricted by the requirement that t/ε and ε have positive real part. Otherwise, (15) would not be a multiple asymptotic series. By requiring the asymptotic form (15) to satisfy the side conditions we improve the results Lick obtained for (13) and obtain the same results as Erdélyi for (14).

Substituting (15) into (13) yields for each $m = 0, 1, 2, \cdots$,

$$m(m-1)y_{m0}\varepsilon^{-1} + \sum_{n=0}^{\infty} [m(m-1)y_{m,n+1} + (1-2m^2)y'_{mn} + y_{mn} + y''_{m,n-1}]\varepsilon^n = 0, \tag{16}$$

where $y_{m,-1} \equiv 0$ for all m. Consequently,

$$y_{mn} = 0 \quad \text{for all } m > 1 \tag{17}$$

and

$$(-1)^m y'_{mn} + y_{mn} = -y''_{m,n-1}, \qquad m = 0, 1 \text{ and for all } n. \tag{18}$$

Equation (18) can be solved recursively. The initial conditions are satisfied by $y_{00}(0) = -y_{10}(0) = 1$, $y_{0n}(0) = -y_{1n}(0)$ for $n = 1, 2, \cdots$ and $y'_{0n}(0) + y'_{1n}(0) - y_{1,n+1}(0) = 0$ for $n = 0, 1, 2, \cdots$. Summarizing, we have

$$\begin{aligned} y(t) \sim\ & e^{-t-t\varepsilon} + 2\,e^{-t}\varepsilon + (6 - 4t)\,e^{-t}\varepsilon^2 + \cdots \\ & - e^{-t/\varepsilon + t + t\varepsilon} - 2\varepsilon\, e^{-t/\varepsilon + t} - (6 - 4t)\varepsilon^2\, e^{t - t/\varepsilon} - \cdots . \end{aligned} \tag{19}$$

The exact solution of (13) is

$$y(t) = [\exp(\gamma_1 t) - \exp(\gamma_2 t)]/\sqrt{1 - 4\varepsilon}, \tag{20}$$

where

$$\gamma_{1,2} = -\frac{1}{2\varepsilon} \pm \frac{1}{2\varepsilon}\sqrt{1 - 4\varepsilon}. \tag{21}$$

Expansion of (20) in powers of ε yields

$$\begin{aligned} y(t) =\ & e^{-t-t\varepsilon} + 2\,e^{-t}\varepsilon + (6 - 4t)\,e^{-t}\varepsilon^2 + O(\varepsilon^3) \\ & - e^{-t/\varepsilon + t + t\varepsilon} - 2\varepsilon\, e^{-t/\varepsilon + t} - (6 - 4t)\varepsilon^2\, e^{t - t/\varepsilon} + O(\varepsilon^3\, e^{-t/\varepsilon}) \end{aligned} \tag{22}$$

as $\varepsilon \to 0$. Hence (19) is uniformly valid in $t \in [0, 1]$.

We now consider (14). Upon substitution of (15) we obtain for each $m = 0, 1, 2, \cdots$,

$$\sum_{n=0}^{\infty} [y''_{mn}\varepsilon^{n+1} + (1 - 2m)y'_{mn}\varepsilon^n + m(m - 1)y_{mn}\varepsilon^{n-1}] = h'(t)\delta_{0n}, \tag{23}$$

where δ_{ij} is the Kronecker delta. Solving (23) we have

$$\begin{aligned} y_{0n} &= h^{(n)}(t) + \alpha_n, \\ y_{1n} &= \beta_n, \\ y_{mn} &= 0, \qquad\qquad m = 2, 3, \cdots \text{ and for all } n, \end{aligned}$$

where α_n and β_n are constants to be determined by the boundary conditions. To satisfy the condition $y(0) = \alpha$ exactly we see that

$$\begin{aligned} &\beta_0 + \alpha_0 + h(0) = \alpha, \\ &(-1)^n h^{(n)}(0) + \alpha_n + \beta_n = 0. \end{aligned} \tag{24}$$

The boundary condition $y(1) = \beta$ cannot be satisfied exactly for all ε without overspecifying the constants α_n, β_n; however, these constants are specified exactly by requiring only that $y(1) = \beta + O(e^{-t/\varepsilon})$. This relaxation of the boundary condition is consistent with the results of Erdélyi. Hence,

$$h(1) + \alpha_0 = \beta, \qquad (-1)^n h^{(n)}(t) + \alpha_n = 0. \tag{25}$$

From (24) and (25) we see that

$$y(t) \sim \beta - h(1) + h(t) + \sum_{n=1}^{\infty} (-1)^n [h^{(n)}(t) - h^{(n)}(1)]\varepsilon^n$$
$$+ (\alpha - h(0) + h(1) - \beta)\, e^{-t/\varepsilon} + \sum_{n=1}^{\infty} (-1)^n [h^{(n)}(1) - h^{(n)}(0)]\varepsilon^n\, e^{-t/\varepsilon}.$$

Hence, by solving a system of ordinary differential equations rather than a system of partial differential equations, as was done in the two-variable expansion technique, we have obtained the same result as Erdélyi [7].

4. A nonlinear boundary value problem of singular perturbation. In this section we obtain an asymptotic solution to the boundary value problem

$$\varepsilon u''(t) + u^2(t) = 1, \qquad u(1) = u(-1) = 0. \tag{26}$$

This example has been used by Carrier [10, p. 181] to illustrate some of the problems in asymptotics. Carrier obtains an asymptotic solution which differs from the exact solution at the boundary points by $O(e^{-2/\sqrt{\varepsilon}})$. He then demonstrates that there are spurious asymptotic solutions which differ from his asymptotic solution at the endpoints by $O(e^{-1/\sqrt{\varepsilon}})$ in both value and slope. These spurious solutions have peaks in the interval $[-1, 1]$ as illustrated in [10, Fig. 6]. It is shown below that series techniques can lead to Carrier's results and can provide additional insights into the problem.

We make the Ansatz that

$$u(t) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn}(t)\varepsilon^{n/2}\, e^{-\lambda(t)m/\sqrt{\varepsilon}} \tag{27}$$

as $\varepsilon \to 0$. The asymptotic form (27) can be defined by setting $x = 1/\sqrt{\varepsilon}$ in Definition 1. A more detailed discussion of these "generalized" asymptotic sequences is given in [9]. The form of the expansion can be obtained either by using hindsight (cf. [10]) or by substituting formal sums into (26) and determining experimentally the types of terms necessary to obtain a recurrence relation.

Temporarily, the more general equation

$$\varepsilon u''(t) + 2b(1 - t^2)u(t) + u(t)^2 = 1 \tag{28}$$

is considered. Upon substitution of (27) into (28),

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \left[m^2\lambda'^2 u_{mn} + 2b(1 - t^2)u_{mn} + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} u_{\mu\nu}u_{m-\mu,n-\nu} \right] \cdot \varepsilon^{n/2} - m[\lambda'' u_{mn} + 2\lambda' u'_{mn}]\varepsilon^{(n+1)/2} + u''_{mn}\varepsilon^{(n+2)/2} \right\} \exp[-\lambda(t)m/\sqrt{\varepsilon}] = 1. \tag{29}$$

From (29), for $m = 0$,

$$2b(1 - t^2)u_{0n} + \sum_{\nu=0}^{n} u_{0,\nu}u_{0,n-\nu} + u''_{0,n-2} = \delta_{0n}, \tag{30}$$

where terms with negative subscripts are defined to be zero and δ_{0n} is the Kronecker δ. The solution of (30) is straightforward; for example,

$$u_{00} = -b(1 - t^2) \pm \sqrt{b^2(1 - t^2)^2 + 1} \tag{30_0}$$

and

$$u_{01} = \pm u''_{00}(t)/\sqrt{b^2(1 - t^2)^2 + 1}. \tag{30_1}$$

The sign of u_{00} is determined by setting $m = 1$ and $n = 0$. The negative sign must be chosen for $\lambda(t) > 0$. Furthermore,

$$\lambda(t) = \pm \int \sqrt{2}[b^2(1 - t^2)^2 + 1]\, dt + k, \tag{31}$$

where k is an arbitrary constant, to be determined. Since $\lambda(t)$ increases as a function of b^2 for constant t, the boundary layer ultimately diminishes as b^2 increases. This result for $\lambda(t)$ illustrates the effect of the middle term of (28), for $b \neq 0$, on the boundary layer—a result not obtained by the two-variable technique. The computations are cumbersome; we set $b = 0$ and restrict attention to (26). Hence

$$u_{00}(t) = -1, \qquad u_{0n}(t) = 0 \quad \text{for } n = 1, 2, \cdots \tag{30'}$$

and

$$\lambda(t) = \pm\sqrt{2}t + k. \tag{31'}$$

Some suitable choices of $\lambda(t)$ are:

$$\lambda_1(t) \equiv \sqrt{2}(1 - t) \quad \text{for } -1 \leqq t \leqq 1,$$

$$\lambda_2(t) \equiv \sqrt{2}(1 + t) \quad \text{for } -1 \leqq t \leqq 1,$$

$$\lambda_3(t) \equiv \begin{cases} \lambda_1(t) & \text{for } 0 \leqq t \leqq 1, \\ \lambda_2(t) & \text{for } -1 \leqq t \leqq 0. \end{cases}$$

Observe that $\lambda_3(t)$ has a continuous first derivative except at $t = 0$, where $\lambda_3'(t)$ has a jump discontinuity. Inspecting (29), we note that $\lambda'(t)$ occurs in two terms. In the first term we have $[\lambda'(t)]^2$, which exists and is continuous on $[-1, 1]$ if $[\lambda'(0)]^2 \equiv 2$, and in the fourth term we have $\lambda'(t)u'_{mn}(t)$. This term disappears if $u_{mn}(t)$ is a constant. Of course, we could have assumed a form with u_{mn} constant from the beginning. Conveniently, any of the above possible choices of $\lambda(t)$ leads to the same conclusion that u_{mn} is constant.

From (29), it is concluded that

$$u_{1n} = k_n \quad \text{for } n = 0, 1, 2, \cdots, \tag{32}$$

where $\{k_n\}$ is a set of constants to be determined. The determination of $u_{m0}\,(m = 2, 3, \cdots)$ is

$$(33)\qquad 2(m^2 - 1)u_{m0} = -\sum_{\mu=1}^{m-1} u_{\mu 0}u_{m-\mu,0}.$$

The general solution of (33) is

$$(34)\qquad u_{m0} = (-1/12)^m(m + 1)k_0^{m+1}.$$

Likewise,

$$2(m^2 - 1)u_{m1} = -\sum_{\mu=1}^{m-1}\sum_{\nu=0}^{1} u_{\mu\nu}u_{m-\mu,n-\nu}.$$

Choosing $\lambda(t) = \lambda_3(t)$ we can satisfy the boundary conditions (26) by setting $k_n = 0$ for $n > 0$ and determining k_0. Summarizing, we have

$$(35)\quad u(t) \sim -1 + k_0\, e^{-\lambda_3/\sqrt{\varepsilon}} \sum_{m=0}^{\infty} (m + 1)(-k_0/12)^m\, e^{-m\lambda_3/\sqrt{\varepsilon}} = -1 + y_3(t),$$

where

$$y_i(t) \equiv k_0\, e^{-\lambda_i/\sqrt{\varepsilon}}[1 + (k_0/12)\, e^{-\lambda_i/\sqrt{\varepsilon}}]^{-2}$$

for $i = 1, 2, 3$, and $k_0 \equiv 12(\sqrt{2} + \sqrt{3})^2$.

Using the two-variable technique, Carrier [10] obtains the result

$$(36)\qquad u(t) \sim -1 + y_1(t) + y_2(t),$$

which satisfies the boundary conditions only to $O[\exp(-2\sqrt{2}/\sqrt{\varepsilon})]$.

Both (35) and (36) are good approximations to the actual solution of (26), which can be computed as a Weierstrass elliptic function. Carrier's result (36) has a continuous second derivative. However, as Carrier indicates, there are spurious solutions which solve (26) to the same asymptotic accuracy as (36). For example,

$$u(t) \sim -1 + y_1(t) + y_2(t) + k_0\, e^{\sqrt{2}(x-x_0)/\sqrt{\varepsilon}}[1 + (k_0/12)\, e^{\sqrt{2}(x-x_0)/\sqrt{\varepsilon}}]^{-2}$$

for any $x_0 \in (-1, 1)$ (cf. [10, Fig. 6]). Whereas (35) is not differentiable at one point ($t = 0$), it has the advantages of satisfying the boundary conditions exactly and having no spurious solutions (known to the author). Furthermore, $y_3'(t)$ has a jump discontinuity of magnitude $O[\exp(-\sqrt{2/\varepsilon})]$ at $t = 0$.

Acknowledgment. The author thanks Mr. Julius W. Enig for reading part of this paper and Drs. Andrew H. Van Tuyl and Jay M. Solomon for useful discussions.

REFERENCES

[1] K. D. SHERE, *Introduction to multiple asymptotic expansions with an application to elastic scattering*, J. Mathematical Phys., 12 (1971), pp. 78–83.

[2] H. SCHMIDT, *Beiträge zu einer Theorie der allgemein asymptotischen Darstellungen*, Math. Ann., 113 (1937), pp. 629–656.

[3] H. POINCARÉ, *Sur les integrales irregulieres des equations linéaires*, Acta Math., 8 (1886), pp. 295–344.

[4] J. RITT, *On the derivatives of a function at a point*, Ann. of Math., 18 (1916), pp. 18–23.

[5] K. O. FRIEDRICHS, *Asymptotic phenomena in mathematical physics*, Bull. Amer. Math. Soc., 61 (1955), pp. 485–504.
[6] W. LICK, *Two-variable expansions and singular perturbation problems*, SIAM J. Appl. Math., 17 (1969), pp. 815–825.
[7] A. ERDÉLYI, *Two-variable expansions for singular perturbations*, J. Inst. Math. Appl., 4 (1968), pp. 113–119.
[8] S. KAPLUN, *Fluid Mechanics and Singular Perturbations*, P. A. Lagerstrom, L. N. Howard and C.-S. Liu, eds., Academic Press, New York, 1967.
[9] K. D. SHERE, *On multiple asymptotic expansions*, this Journal, 3 (1972), pp. 272–284.
[10] G. F. CARRIER, *Singular perturbation theory and geophysics*, SIAM Rev., 12 (1970), pp. 175–193.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

ON MULTIPLE ASYMPTOTIC EXPANSIONS*

KENNETH D. SHERE†

Abstract. In previous papers asymptotic meaning has been attached to the formal double sums $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}x^{-n}e^{-\lambda mx}$ ($\lambda > 0$) as $x \to \infty$ in a sector of the right half-plane. Formal applications of these sums to elastic scattering and to singular perturbation problems have also been given. In this paper the concept of asymptotic sequences and series are extended to define more general double sums. Multiplication and differentiation properties of these sequences and series are investigated. Application of these more general asymptotic expansions to systems of first order, linear ordinary differential equations is presented.

1. Introduction. A consistent theory of asymptotics was first developed in 1886 by Poincaré [9]. He defines a function $f(x)$ to have asymptotic expansion $\sum a_n/x^n$ as $x \to \infty$ in a region R of the complex plane, written

$$f(x) \sim \sum_{n=0}^{\infty} \frac{a_n}{x^n}, \tag{1.1}$$

if

$$f(x) = \sum_{k=0}^{n} a_k x^{-k} + R_n(x), \tag{1.2}$$

where

$$R_n(x) = o(x^{-n}). \tag{1.3}$$

The $\{a_n\}$ are constants with respect to x. Watson [16] remarks that Borel in 1896 made Poincaré's theory more precise by defining the sum in (1.1) as

$$S(x) = \int_0^{\infty} e^{-t}\phi\left(\frac{t}{x}\right) dt, \tag{1.4}$$

where

$$\phi(t) \equiv \sum_{n=0}^{\infty} \frac{a_n t^n}{n!}.$$

It is required that $\phi(t)$ converge.

A primary difficulty with Borel's theory is the required knowledge of the singularities of $\phi(t)$. In another effort to add precision to Poincaré's theory, Watson [16] in 1912 introduced the notion of "characteristics" which are somewhat analogous to the radius of convergence of a convergent series. In particular, he imposes the constraints:

$$|a_n| \leqq A\rho^n\Gamma(kn + 1) \qquad \text{for all } n; \tag{1.5}$$

$$|R_n x^{n+1}| \leqq B\sigma^n\Gamma(jn + 1) \quad \text{for all } n. \tag{1.6}$$

The characteristics A, B, ρ, σ, k and j are independent of x and n. Watson then

* Received by the editors November 24, 1970, and in revised form August 10, 1971.

† U.S. Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland 20910.

investigates the circumstances for which an asymptotic expansion with assigned characteristics uniquely determines an analytic function in R. When $k = j = 1$, the functions are "Borel summable."

Fourteen years after Watson's work, Carleman [1] introduced the concept of asymptotic sequences $\{\phi_n(x)\}$. Requiring $\phi_{n+1}(x) = o[\phi_n(x)]$ as $x \to \infty$, Carleman defined asymptotic expansions as

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \tag{1.7}$$

if

$$f(x) - \sum_{k=0}^{n} a_k \phi_k(x) = o[\phi_n(x)]. \tag{1.8}$$

Carleman's definition was generalized by Schmidt (1936) [12] who allowed a_n to be any function of x belonging to a prescribed class of functions A_0 with the properties:

(i) the constants are in A_0,
(ii) A_0 is closed under linear combinations of its elements,
(iii) $a_n(x)$ is bounded in R,
(iv) either $a_n(x) = 0$ or $a_n(x) \neq o(1)$.

Schmidt's theory is especially useful in the study of oscillatory functions.

The most general asymptotic expansion is due to Erdélyi [4], [5], and Erdélyi and Wyman [6]. Erdélyi defines

$$f(x) \sim \sum_{n=0}^{\infty} f_n(x) \quad \text{with respect to } \{\phi_n(x)\} \tag{1.9}$$

if

$$f(x) - \sum_{k=0}^{n} f_k(x) = o[\phi_n(x)]. \tag{1.10}$$

These generalized expansions are used in [5] and [6] to evaluate certain integrals not otherwise amenable to asymptotic treatment. This definition can also be used in singular perturbation theory (cf. [8]). These generalized expansions lose the usual uniqueness property and, as Riekstiņš [10] demonstrates, some expansions seem somewhat meaningless. In particular, Riekstiņš mentions that

$$\frac{\sin x}{x} \sim \sum_{k=0}^{\infty} \frac{k!\, e^{-[(k+1)/2k]x}}{(\ln x)^k} \tag{1.11}$$

with respect to $\{(\ln x)^{-k}\}$ as $x \to \infty$.

Riekstiņš also provides a further extension of Carleman's theory by allowing the "constants" $a_n(x, z)$ to depend on a parameter z in addition to the variable x. The $a_n(x, z)$ are taken as members of a prescribed class A_0 with properties (iii) and (iv). The uniqueness of these expansions depends on the class A_0. Riekstiņš then uses "neutrices" (cf. [3]) to evaluate some integrals involving modified Bessel functions.

As the definitions become more general, the techniques have become less accessible to the nonspecialist. For example, no criterion is provided for determining $f_n(x)$ in (1.9) and most nonspecialists are unfamiliar with neutrices. Both Erdélyi's and Riekstiņš' generalizations use "asymptotically negligible quantities" to improve upon the numerical accuracy of asymptotic expansions.

In [13] the author has presented a more systematic scheme for using asymptotically negligible quantities by attaching asymptotic meaning to the double series $\sum_{m=0}^{\infty} [\sum_{n=0}^{\infty} a_{mn}x^{-n}] \exp(-\lambda mx)$ as $x \to \infty$ in a sector $R \equiv \{|x| > x_0 > 0, |\arg x| < \pi/4; \lambda > 0\}$. He also applied these series to elastic scattering from a Yukawa potential. Series techniques have the distinct advantage of being easy to use in formal derivations. These double series expansions are called "multiple asymptotic expansions."

In § 2, basic definitions extend this concept to sums of the form $\sum_{m=0}^{\infty} [\sum_{n=0}^{\infty} a_{mn}\phi_{mn}(x)]$, where $\{\phi_{mn}(x): n = 0, 1, 2, \cdots; m = 0, 1, 2, \cdots\}$ is a multiple asymptotic sequence. These definitions involve some complicated constructions in order to determine a function with prescribed asymptotic properties. The first construction is used to obtain a function which is asymptotic to zero with respect to a given asymptotic sequence; this construction is a modification of a construction of van der Corput [2]. The second construction modifies Ritt's procedure [11] for determining a function with prescribed asymptotic series.

Multiple asymptotic expansions can be constructed with slightly less generality in terms of Schmidt's work by requiring the class A_0 to consist of all functions $a_n(x)$ which have an asymptotic expansion (in Carleman's or Poincaré's sense) with respect to $\{\psi_n\}$, where $\phi_1(x) \sim 0$ with respect to $\{\psi_n\}$. It is shown in a companion paper [14] that singular perturbation problems have a natural interpretation in terms of multiple asymptotic expansions. It is also shown in [14] that series techniques provide either the same or improved formal results when compared to the two-variable technique.

In § 3, multiple asymptotic sequences are classified according to their multiplication and differentiation properties. Also, several results relating to these properties are obtained. It is shown, for example, that formal multiplication and differentiation is valid under the same hypotheses required for simple asymptotic sequences $\{Q_n(x): n = 0, 1, 2, \cdots\}$.

A classification with respect to the dependence on the subscripts is presented in § 4.

These theoretical results are applied in § 5 to a linear, first order system of ordinary differential equations with a regular singularity,

$$Y'(x) = A(x)Y(x).$$

It is shown that whenever

$$A(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn}x^{-n}\psi^m(x), \qquad A_{00} = 0$$

and

$$\psi'(x) \sim \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} P_{mn}x^{-n}\psi^m(x),$$

formal series techniques may be used to obtain a valid asymptotic solution.

2. Basic definitions. Let S be the set of s-tuples over the nonnegative integers. Define $<$ and $\leqq$ on S to be the lexicographic order, that is, whenever $N = (n_1, \cdots, n_s)$ and $M = (m_1, \cdots, m_s)$ and p is the smallest integer such that $m_p \neq n_p$, then $M < N$ if $m_p < n_p$; if no such p exists, $M = N$. Also define λe_i to be the s-tuple with λ in the ith place and zero elsewhere.

DEFINITION 2.1. The set of functions $\{\phi_N(x): N \in S\}$ defined on an unbounded region R of the complex plane is called a *multiple asymptotic sequence* (or *asymptotic sequence*) if $\phi_N = o(\phi_M)$ as $|x| \to \infty$ in R whenever $M < N$. It is assumed that the E. Landau symbol o applies uniformly in $N \in S$.

We note that any subsequence of an asymptotic sequence is an asymptotic sequence.

For ease of presentation, we take $s = 2$ and we assume that whenever $\phi_M(x_0) = 0$ for some $x_0 \in R$, $\phi_N(x_0) = 0$ for all $N > M$. It is further assumed, without loss in generality, that $\phi_{0,0}(x) \equiv 1$ (cf. Schmidt [12]). In [13] the multiple asymptotic sequence defined by $\phi_{mn}(x) = x^{-n} \exp(-\lambda m x)$ $(n, m = 0, 1, 2, \cdots)$, $\lambda > 0$ and $R \equiv \{x : |x| > x_0 > 0, |\arg x| < \alpha < \pi/4\}$ is discussed in detail.

Suppose that we are given the formal, not necessarily convergent, sum

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}\phi_{mn}(x), \tag{2.1}$$

where $\{a_{mn}\}$ is an arbitrary sequence of complex constants and $\{\phi_{mn}(x)\}$ is a multiple asymptotic sequence. Modifying a construction of van der Corput [2] (cf. [4, p. 22]) we shall determine a nontrivial function $\psi(x) \sim 0$ with respect to $\{\phi_{m0}: m = 0, 1, 2, \cdots\}$ in the sense of Schmidt, that is, $\psi = o(\phi_{m0})$ for each m. The function $\psi(x)$ is then used in a modification of a construction of Ritt [11] to determine a function

$$f_m(x) \sim \sum_{n=0}^{\infty} a_{mn}\phi_{mn}(x). \tag{2.2}$$

The functions $\{f_m(x)\}$ are subsequently used to define (2.1) as an asymptotic series.

Construct the sequence $\{U_{m0}\}$ of induced neighborhoods (in R) of infinity such that

$$R = U_{00} \supset \overline{U}_{1,0} \supset \overline{U}_{2,0} \supset \cdots$$

and such that

$$|\phi_{m+1,0}(x)| \leqq \tfrac{1}{2}|\phi_{m,0}(x)|$$

whenever $x \in U_{m+1,0}$. The closure and complement of $U_{m,0}$ are $\overline{U}_{m,0}$ and $U^*_{m,0}$ respectively. Let

$$\psi(x) \equiv \phi_{m,0}(x), \qquad x \in U_{m,0} \cap U^*_{m+1,0}.$$

Modifying the techniques of Ritt [11] (cf. [13]), we define

$$f_m(x) \equiv \sum_{n=0}^{\infty} a_{mn}\left[1 - \exp\frac{-b_{mn}}{|\psi(x)|}\right]\phi_{mn}(x), \tag{2.3}$$

where

$$b_{mn} \equiv \begin{cases} 0 & \text{for } a_{mn} = 0, \\ (2|a_{mn}|m!n!)^{-1} & \text{for } a_{mn} \neq 0. \end{cases}$$

Using the inequality $|A[1 - \exp(-B)]| < 2$ whenever $\operatorname{Re} B > 0$ and $|B| \cdot |A| < 1$, we can show that the series for $f_m(x)$ converges absolutely for large enough x. By the uniformity of $o(\cdot)$, for some $x_0 \in R$,

$$|\phi_{m+1,0}(x)| < \tfrac{1}{2}|\phi_{m,0}(x)|$$

for all $x \in R$ such that $|x| > |x_0|$. If $\psi(x) = 0$, the series terminates and $f(x)$ is absolutely convergent. If $\psi(x) \neq 0$, then

$$(2.4) \qquad \begin{aligned} |f_m(x)| &\leqq 2 \sum_{n=0}^{\infty} |\phi_{mn}(x)|/m!n!|\psi(x)| \\ &\leqq 2e^{1/2}|\phi_{m0}(x)|/|\psi(x)|m!. \end{aligned}$$

Hence (2.3) is absolutely convergent for all $x \in R$, $|x| > |x_0|$. We note that in any compact subregion of $R \cap \{|x| > x_0\}$ for which $\psi(x)$ is nonzero, (2.3) is uniformly convergent. It is useful for property (4) below to observe that $f(x) \equiv \sum_{m=0}^{\infty} f_m(x)$ is an absolutely convergent series.

We now establish (2.2).

$$(2.5) \qquad \begin{aligned} \left|\frac{f_m - \sum_{n=0}^{N} a_{mn}\phi_{mn}}{\phi_{mN}}\right| &\leqq \frac{\sum_{n=0}^{N} |a_{mn}|\, e^{-b_{mn}/|\psi|}|\phi_{mn}|}{|\phi_{mN}|} \\ &\quad + \sum_{n=N+1}^{\infty} |a_{mn}|\left[1 - \exp\frac{-b_{mn}}{|\psi|}\right]\left(\frac{\phi_{mn}}{\phi_{mN}}\right) = o(1). \end{aligned}$$

The right-hand side of (2.5) $\to 0$ as $|x| \to \infty$ in R.

DEFINITION 2.2. A function $F(x)$ defined in R is said to be *asymptotic to* (2.1) *as* $|x| \to \infty$ *in* R, denoted

$$(2.6) \qquad F(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}\phi_{mn}(x),$$

if:

(a) $F_m(x) \equiv \{f(x): f(x) \sim \sum_{n=0}^{\infty} a_{mn}\phi_{mn}(x)\}$ for each fixed $m = 0, 1, 2, \cdots$;

(b) $F_m^* \equiv \{f \in F_m: f(x) - f_m(x) \sim 0 \quad \{\phi_{\mu n}: n = 0, 1, \cdots\}$ for each fixed $\mu = m, m+1, \cdots\}$;

(c) $F(x) \in F_0$;

(d) $F(x) - \sum_{\mu=0}^{m} f_\mu^* \in F_{m+1}$ for any $f_\mu^* \in F_\mu^*$.

The asymptotic symbols in (a) and (b) are taken in the sense of Schmidt, $f(x) - \sum_{n=0}^{N} a_{mn}\phi_{mn}(x) = o(\phi_{m,N+1})$. (a) and (b) are definitions, so only conditions (c) and (d) must be demonstrated in future results.

In a straightforward manner we can demonstrate the properties:

(1) termwise addition of multiple asymptotic series is valid;

(2) whenever a function $f(x)$ has an asymptotic series representation with respect to $\{\phi_{mn}\}$, this representation is unique;

(3) whenever (2.1) is convergent and $F(x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}\phi_{mn}$, then $F(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}\phi_{mn}$;

(4) an arbitrary series (2.1) is the asymptotic expansion of some function.

In order to apply the theory of multiple asymptotic expansions to differential equations, it is necessary to investigate the multiplication and differentiation properties of asymptotic sequences. This is done in the following section.

3. Multiplication and differentiation properties.

DEFINITION 3.1. An asymptotic sequence $\{\phi_N : N \in S\}$ is called a *multiplicative asymptotic sequence* if whenever $N, M \in S$,

$$\phi_N \phi_M \sim \sum_K c_{M,N,K}\phi_K, \tag{3.1}$$

where

$$\sum_K \equiv \sum_{k_1} \sum_{k_2} .$$

DEFINITION 3.2. An asymptotic sequence $\{\phi_N : N \in S\}$ is said to be an *exponential asymptotic sequence* if

$$\phi_N \phi_M = \phi_{N+M} \quad \text{for all } N, M \in S.$$

The properties of multiplicative asymptotic sequences were first investigated by Schmidt [12] who had restricted his attention to simple asymptotic sequences. The following two results are extensions of theorems proved by Schmidt.

LEMMA 3.1. *Let $\{\phi_N : N \in S\}$ be a multiplicative asymptotic sequence satisfying* (3.1); *then*

$$c_{M,N,K} = 0 \quad \textit{for } K < M + N.$$

Proof. Assume without loss of generality that $\phi_0 = 1$. Obviously

$$\phi_0 \phi_N \sim \phi_N$$

which implies

$$c_{0,N,K} = 0 \quad \text{for } K < N.$$

Assume that $c_{L,N,K} = 0$ for $K < N + L$ for all $L < M$. Then

$$\begin{aligned} \phi_N \phi_M &= (\phi_M/\phi_L)(\phi_L/\phi_N) = (\phi_M/\phi_L)O(\phi_{N+L}) \\ &= o(\phi_{N+L}) \quad \text{for all } L < M. \end{aligned}$$

Hence,

$$c_{M,N,K} = 0 \quad \text{for all } K < M + N.$$

THEOREM 3.2. *If $F(x) \sim \sum_M a_M \phi_M$ and $G(x) \sim \sum_N b_N \phi_N$ and $\{\phi_K : K \in S\}$ is a multiplicative asymptotic sequence satisfying* (3.1), *then*

$$H(x) \equiv F(x)G(x) \sim \sum_K h_K \phi_K,$$

where

$$h_K = \sum_M \sum_N a_M b_N c_{MNK}.$$

This theorem can be proved using techniques of the corresponding theorem (Theorem 2.2) in [13] and Lemma 3.1, which shows that h_K is a finite sum.

DEFINITION 3.3. An asymptotic sequence $\{\phi_N : N \in S\}$ is called a *differentiable asymptotic sequence* if

$$\phi'_N \sim \sum_K d_{N,K}\phi_K \quad \text{for each } N \in S.$$

The sequence $\{\phi_n \equiv e^{-nx^2} : n = 0, 1, 2, \cdots\}$ is an exponential asymptotic sequence as $x \to \infty$ in (x_0, ∞) which is not a differentiable asymptotic sequence. The sequence $\{\phi_{n,m} \equiv x^{-n}e^{-mx}\}$ is an exponential, differentiable asymptotic sequence. It is shown below that every exponential, differentiable asymptotic sequence of the form $\{x^{-n}\psi^m(x)\}$ behaves essentially like $\{x^{-n}e^{-mx}\}$ if $\psi(x)$ is analytic in R. That is, $\psi'(x)/\psi(x) \to \alpha \neq 0$ as $|x| \to \infty$ in R. The following theorem yields a result for differentiable asymptotic sequences which corresponds to the result of Lemma 3.1 for multiplicative asymptotic sequences.

THEOREM 3.3. *Let R be a region which contains a curve Γ originating in R and extending towards infinity and such that the distance $\rho(x)$ of $x \in \Gamma$ is bounded away from zero as $|x| \to \infty$ along Γ. Let $\{\phi_N : N \in S\}$ be a differentiable asymptotic sequence of functions analytic in R; then for each N, ϕ_N has a bounded logarithmic derivative in a subregion $R^* \subset R$ as $|x| \to \infty$; that is, $\phi'_N = O(\phi_N)$ as $|x| \to \infty, x \in R^*$.*

Proof. The proof is by induction. Since $\{\phi_N\}$ is differentiable,

$$\phi'_{e_1} \sim \sum_K d_{e_1,K}\phi_K$$

which implies that

$$\lim_{|x|\to\infty} \phi'_{e_1}(x) = d_{e_1,0} \quad \text{for } x \in R.$$

Since ϕ_{e_1} is analytic in R, for any $x_0 \in \Gamma$,

$$\phi'_{e_1}(x_0) = \frac{1}{2\pi i}\int_{|x-x_0|=\rho} \frac{\phi_{e_1}}{(x-x_0)^2}\,dx. \tag{3.2}$$

Because $\phi_{e_1}(x) \to 0$ as $|x| \to \infty$, $x \in R$, for any $\varepsilon > 0$ there is a $\xi > 0$ such that whenever $|x| > \xi$ and $x \in R$, $|\phi_{e_1}(x)| < \varepsilon$. Hence, for

$$x_0 \in \Gamma \cap \{x : |x| > \xi + \rho\},$$

one obtains from (3.2),

$$|\phi'_{e_1}(x_0)| \leqq \frac{1}{2\pi}\int \frac{\varepsilon}{|x-x_0|^2}|dx| = \frac{\varepsilon}{\rho}.$$

Since ρ is fixed and $\rho > 0$, and ε is arbitrarily small,

$$\lim_{|x|\to\infty} |\phi'_{e_1}(x)| = 0 \quad \text{for } x \in \Gamma.$$

Therefore $d_{e_1,0} = 0$.

Now assume that the theorem is true for all $N < M$. Let K be a fixed nonzero s-tuple and $K < M$. The sequence

$$\{1, \phi_M/\phi_K, \phi_{M+e_1}/\phi_K, \cdots\} \equiv \{1, \psi_{e_1}, \psi_{2e_1}, \cdots\}$$

is an asymptotic sequence of functions in a subregion $R^* \subset R$. Since ϕ_K has no limit point of zeros in R, we may draw very small circles about these zeros and construct Γ^* in R^* by changing Γ where necessary to avoid the circles. By the preceding paragraph,

$$\lim_{|x| \to \infty} \psi'_{e_1}(x) = 0 \quad \text{for } x \in R^*.$$

Since $\psi'_{e_1} = \phi'_M/\phi_K - (\phi_M/\phi_K)(\phi'_K/\phi_K) \to 0$ as $|x| \to \infty$ in R^*, $\phi_M = o(\phi_K)$ and $\phi'_K/\phi_K = O(1)$, then $\phi'_M = o(\phi_K)$ for every $K < M$. Hence,

$$\phi'_M = O(\phi_M)$$

or ϕ_M has a bounded logarithmic derivative.

COROLLARY 3.4. *Let the hypotheses of Theorem* 3.3 *be satisfied. If* $\{\phi_{mn}(x) \equiv \theta_n(x)\psi_m(x)\}$ *is a differentiable asymptotic sequence, then for each n,* d_{ne_2,ke_2} *is nonzero for some k. Furthermore,* $k \geqq n$.

Proof. If $d_{ne_2,ke_2} = 0$ for all k, then $\theta'_n(x) \sim 0$ with respect to $\{\theta_n(x): n = 0, 1, 2, \cdots\}$. Hence $\theta_n(x)$ is asymptotic to a constant with respect to $\{\theta_n\}$. This is a contradiction. Furthermore, $k \geqq n$ since θ'_n/θ_n has a finite limit.

LEMMA 3.5. *If, in addition to the hypotheses of Theorem* 3.3, $(x_0, \infty) \subset R$ *for some* $x_0 > 0$, *and* $\{\phi_N(x)\}$ *is real-valued on* (x_0, ∞), *then* $\phi_N(x)$ *has at most a finite number of zeros on* (x_0, ∞).

Proof. This lemma is proved by contradiction. Let M be the smallest element of S such that $d_{N,M} \neq 0$. Such an M exists by Theorem 3.3. If $\phi_N(x_j) = 0$ for $j = 0, 1, 2, \cdots$, then points p_j, q_j $(j = 0, 1, 2, \cdots)$ can be found such that $p_j, q_j \in (x_j, x_{j+1})$ and $\phi'_N/\phi_M(p_j) \leqq 0$, $\phi'_N/\phi_M(q_j) \geqq 0$. It can, therefore, be concluded that

$$\lim_{|x| \to \infty} \phi'_N/\phi_M = 0$$

when it exists. This contradicts that $d_{N,M} \neq 0$.

THEOREM 3.6. *Given the hypotheses of Lemma* 3.5 *and that M is the smallest element of S such that* $d_{N,M} \neq 0$, *then*

$$\lim_{|x| \to \infty} \phi'_N/\phi_M = d_{N,M} < 0 \quad \textit{for } x \in R.$$

Proof. For large $x \in (x_0, \infty)$, $\phi_N(x)$ is of constant sign for each N. Without loss of generality we specify that each $\phi_N(x) > 0$ for large enough x. Hence $\phi_N(x)$ decreases to zero as $x \to \infty$ in (x_0, ∞) and $\phi'_N(x)/\phi_M(x) < 0$ in some interval (ξ, ∞).

It is desirable to have a fairly general differentiation theorem of the type Schmidt found. If $\{\phi_N\}$ is a differentiable asymptotic sequence, $f(x)$ is regular in R and $f(x) \sim \sum f_N \phi_N$, then $f'(x) \sim \sum f_N \phi'_N \sim \sum g_N \phi_N$, where $g_N = \sum_{M=0}^{N} f_M\, d_{MN}$. Unfortunately, the author has been unable to prove such a theorem. This is due to the nondifferentiability of $f_m(x)$ in the definition of asymptotic series. However, for an important special case we obtain the following theorem.

THEOREM 3.7. *If* $\{\phi_{mn}(x) \equiv \theta^n(x)\psi^m(x)\}$ *is an exponential, differentiable asymptotic sequence in R satisfying* (3.2) *and the hypotheses of Theorem* 3.3, *then*

$$f(x) \sim \sum_N a_N \phi_N \quad \textit{implies } f'(x) \sim \sum_N b_N \phi_N,$$

where

$$b_N = \sum_{M=0}^{N} a_M \, d_{M,N}.$$

Proof. Define

$$f_m^* \equiv \sum_{n=0}^{\infty} a_{mn}\left[1 - \exp\frac{-b}{\psi}\right]\theta^n\psi^m,$$

where b_{mn} is defined in (2.3). It is seen in a straightforward manner that $f_m^* \in F_m^*$. We observe that $f \sim f_0^*\{\theta^n : n = 0, 1, \cdots\}$ and

$$\theta'(x) \sim \sum_{n=\nu}^{\infty} d_{e_2,ne_2}\theta^n \quad \{\theta^n\}.$$

Hence,

$$f' \sim f_0^{*\prime} \sim \sum_{n=0}^{\infty} b_{0n}\theta^n \quad \{\theta^n\},$$

where

$$b_{0n} = \begin{cases} \sum_{k=1}^{n-\nu+1} k a_{0k} d_{e_2,n-k+1} & \text{for } n \geqq \nu, \\ 0 & \text{for } n < \nu. \end{cases}$$

We next observe that

$$f - f_0^* \sim f_1^* \quad \{\theta^n\psi : n = 0, 1, 2, \cdots\}$$

since $f_1^* \in F_1^*$. By Schmidt's result [12, p. 646],

$$f' - f_0^{*\prime} \sim f_1^{*\prime} \quad \{\theta^n\psi : n = 0, 1, 2, \cdots\}.$$

Hence,

$$f' - g_0^* \sim f_1^{*\prime} + (f_0^{*\prime} - g_0^*), \tag{3.3}$$

where

$$g_0^* \equiv \sum_{n=0}^{\infty} b_{0n}\left[1 - \exp\frac{-b_{0n}}{\psi}\right]\theta^n.$$

The right-hand side of (3.3) gives the desired result for $m = 1$. The proof is completed by an inductive argument.

As indicated in the beginning of this section, we now demonstrate that $\{x^{-n}\psi^m(x)\}$ behaves essentially like $\{x^{-n}e^{-mx}\}$.

THEOREM 3.8. *If $\{\phi_{mn} \equiv x^{-n}\psi^m(x)\}$ is a differentiable asymptotic sequence as $|x| \to \infty$ in R, a region satisfying the hypotheses of Theorem 3.3, and*

$$\psi'(x) \sim \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} p_{mn}x^{-n}\psi^m(x),$$

then

$$\psi'(x)/\psi(x) \to p_{10} \neq 0 \quad \text{as } |x| \to \infty \text{ in } R.$$

Proof. Since ϕ_{mn} has a bounded logarithmic derivative, $p_{0,n} = 0$ for all n. It is now shown by contradiction that $p_{1,0} \neq 0$.

Let (q, r) be the smallest ordered pair for which $p_{q,r} \neq 0$ and suppose that $(q, r) > (1, 0)$. Then

$$\psi'/\psi^q \sim d_{qr}x^{-r} + d_{q,r+1}x^{-r-1} + \cdots \tag{3.4}$$

with respect to the asymptotic power series $\{x^n : n = 0, 1, 2, \cdots\}$. Integration of (3.4) for the case $q > 1$ yields

$$\frac{1}{1-q}\psi^{-q+1} \sim c + \frac{1}{1-r}p_{q,r}x^{-r+1} + \cdots.$$

However, $\psi \sim 0 \quad \{x^{-n}\}$ implies that ψ^{-1} does not have an asymptotic expansion with respect to $\{x^{-n}\}$. This contradicts the assumption that $q > 1$.

Now set $q = 1$ and assume $r > 0$. Integration of (3.3) yields

$$\psi(x) \sim x^{p_{1,1}} \exp \int_{\infty}^{x} f\,dt, \tag{3.5}$$

where

$$f(t) = p_{1,2}t^{-2} + p_{13}t^{-3} + \cdots$$

and $p_{1,i} = 0$ for $i < p$. From (3.4),

$$x^{-p_{1,1}}\psi(x) \sim \sum_{n=0}^{\infty} \delta_n x^{-n},$$

where δ_n is determined from the expansion of $\exp \int f(x)\,dx$. Hence $\delta_0 \neq 0$. This contradicts that $x^{-p_{1,1}}\psi(x) \sim 0$ with respect to $\{x^{-n}\}$. Hence $r = 0$ and $(q, r) = (1, 0)$. The proof is complete.

Before proceeding with applications of multiple asymptotic expansions to systems of differential equations we consider an additional classification.

4. Separable asymptotic sequences. Sometimes the behavior of $\{\phi_N\}$ with respect to its subscript is complicated as in the example

$$\{\phi_{(n,m)} \equiv x^{-n\exp(mx)} : n, m = 0, 1, 2, \cdots\}$$

as $|x| \to \infty$, $x \in (1, \infty)$. The author is unaware of physical applications which exhibit coupled behavior among the indices. We, therefore, introduce the following definition.

DEFINITION 4.1. An asymptotic sequence $\{\phi_N : N \in S\}$ is said to be *separable* if

$$\phi_{(n_1,\cdots,n_s)} = \psi_{n_1}^{(1)}(x) \cdots \psi_{n_s}^{(s)}(x),$$

where $\psi_{n_i}^{(i)}(x) \equiv \phi_{n_i e_i}$. The $\{\psi_{n_i}^{(i)}(x) : n_i = 0, 1, \cdots\}$, $i = 1, \cdots, s$, are called the *factors* of the asymptotic sequence $\{\phi_N\}$.

The factors of a multiple asymptotic sequence are simple asymptotic sequences. Obviously, when all of the factors of $\{\phi_N\}$ are multiplicative (differentiable), then $\{\phi_N\}$ is multiplicative (differentiable).

The asymptotic sequence

$$\{\phi_{n,m} \equiv x^{-n/2} e^{-mx^{1/2}} : n, m = 0, 1, 2, \cdots\}$$

is an example of a differentiable, separable asymptotic sequence whose factors are not differentiable asymptotic sequences.

A similar statement can be made about multiplicative, separable asymptotic sequences under the following more generalized definition.

DEFINITION 4.2. A sequence $\{\phi_N : N \in S\}$ is called a *generalized asymptotic sequence if* for each $N \in S$, $N \neq 0$, there exists an $M_N < N$ so that

$$\phi_N = o(\phi_{M_N}).$$

This definition makes the asymptotic sequences defined according to Schmidt's theory independent of the order in which the scalar functions are listed provided that $\phi_0 = 1$.

Consider the following example.

Example. Define

$$\phi_{n_1,n_2,n_3,n_4} \equiv \prod_{i=1}^{4} \psi_{n_i}^{(i)}(x) \equiv x^{-(n_1^2+n_2^2+n_3^2+n_4^2)}.$$

Observe that $\{\psi_{n_i}^{(i)}\}$ is a nonmultiplicative asymptotic sequence for each i. Lagrange's theorem says that the diophantine equation

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 = n$$

has at least one integral solution whenever n is a nonnegative integer [7]. Hence $\{\phi_N\}$ is a multiplicative generalized asymptotic sequence. This example demonstrates that a multiplicative, separable asymptotic sequence need not have multiplicative factors.

This example also shows that a generalized separable sequence need not be an exponential asymptotic sequence. It can be shown, however, that every exponential asymptotic sequence is a separable asymptotic sequence (define $\psi_{n_i}^{(i)} = \phi_{e_i}^{n_i}$).

In the interest of clarity and simplicity the generalized definition will not be used.

5. Systems of ordinary differential equations. In this section we apply the results of § 3 to systems of equations of the form

$$Y'(x) = A(x)Y(x), \tag{5.1}$$

where Y is a vector and $A(x)$ is a square matrix which has an asymptotic expansion with respect to $\{\phi_N\}$. We assume that ϕ_N is an exponential, differentiable asymptotic sequence. Without loss of generality we take $\phi_{mn}(x) = x^{-n}\psi^m(x)$. Indeed, if $\phi_{mn}(x) = \theta^n(x)\psi^m(x)$, the transformation $z^{-1} = \theta(x)$ takes (5.1) to the desired form.

Let

$$A(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{mn} x^{-n} \psi^m(x). \tag{5.2}$$

It is shown that whenever (5.1) has a regular singularity, that is, $A_{00} = 0$ and $A_{01} \neq 0$, it is reducible to the usual power series case. In particular, it is shown that there is an analytic matrix function $p(x)$ in R admitting the asymptotic expansion

$$p(x) \sim \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} p_{mn} x^{-n} \psi^{m}(x) \tag{5.3}$$

such that the transformation

$$Y(x) = p(x)W(x) \tag{5.4}$$

takes (5.1) into

$$W'(x) = B(x)W(x), \tag{5.5}$$

where

$$B(x) = p^{-1}(x)[-p'(x)I + A(x)p(x)] \tag{5.6}$$

is an asymptotic function admitting the asymptotic expansion

$$B(x) \sim \sum_{n=0}^{\infty} B_{0n} x^{-n} \quad \{x^{-n}\psi^{m}(x): n, m = 0, 1, \cdots\}. \tag{5.7}$$

By substituting (5.4) into (5.1), performing the necessary algebra and matching coefficients, it is seen that

$$\begin{aligned} \sum_{j=0}^{m} \sum_{i=0}^{n} p_{ji} B_{m-j,n-i} = {} & \sum_{j=0}^{m} \sum_{i=0}^{n} A_{m-j,n-i} p_{ji} + (n-1) p_{n-1,m} \\ & - \sum_{j=1}^{m} \sum_{i=0}^{n} (m+1-j) d_{ji} p_{m+1-j,n-1}. \end{aligned} \tag{5.8}$$

When $m = 0$, the last term on the right of (5.8) does not appear, and when $n = 0$, the middle term on the right does not appear. The coefficient d_{ji} corresponds to p_{ji} in Theorem 3.8.

For $m = 0$, (5.8) is

$$A_{0,1} p_{0,n-1} - [B_{0,1} - (n-1)I] p_{0,n-1} = B_{0,n-1} + H_{0,n-1}, \tag{5.9}$$

where $H_{0,n-1}$ is a known function of $p_{0,i}$, $B_{0,i}$ and $A_{0,i}$ for $i < n - 1$. Analyze (5.9) as in the regular singularity case for asymptotic power series [15, p. 21 ff.] to obtain some positive integer h such that $B_{0,n} = 0$ for all $n \geqq h$.

Whenever $m \neq 0$, (5.8) takes the form

$$B_{mn} + m d_{1,0} p_{mn} = J_{m,n-1}, \tag{5.10}$$

where $J_{m,n-1}$ is a known function of $B_{j,i}$, $p_{j,i}$ and $A_{j,i}$ for $(j, i) \leqq (m, n - 1)$. Since (5.10) has a unique solution for p_{mn} for any choice of $B_{m,n}$, choose $B_{mn} = 0$ and $p_{mn} = J_{m,n-1}/m d_{1,0}$.

Thus, (5.1) has been transformed to an equation with a well-known solution.

6. Concluding remarks. This paper is of an introductory nature. Much remains to be done both with the theory and application of multiple asymptotic series. For example, nondifferentiable, exponential asymptotic sequences of regular functions have not been considered. Also, we have not considered asymptotic sequences containing a parameter. It is implied in § 5 that more accurate results may be obtained by using multiple asymptotic expansions; an error analysis is needed to determine the effects analytically. Although some heuristic considerations are given in [14], no general theory has been developed yet. It is hinted in [14] that multiple asymptotic sequences may be used to solve linear ordinary differential equations of order at least two which have an essential singularity in the leading coefficient. Application to nonlinear equations is still an open question, although the remarks of Wasow [15, p. 215] are not encouraging. General results on this problem would also be useful. It may, for example, allow analytical treatment of chemically reacting fluid flow.

Acknowledgment. Part of this work appeared in the author's dissertation under the direction of Professor Ray G. Langebartel of the University of Illinois. Thanks are also extended to the editor for bringing to the author's attention references [6] and [10] and for providing the author with a translation of [10].

REFERENCES

[1] T. CARLEMAN, *Les fonctions quasi analytiques*, Collection de monographies sur la theorie des fonctions, Emile Borel, ed., Gauthier-Villars et C^{ie}, Paris, 1926.

[2] J. G. VAN DER CORPUT, *Asymptotic expansions. I*, J. Analyse Math., 4 (1954–56), pp. 341–417.

[3] ———, *Neutrices*, J. Soc. Indust. Appl. Math., 7 (1959), pp. 253–279.

[4] A. ERDÉLYI, *Asymptotic Expansions*, Dover, New York, 1956.

[5] ———, *General asymptotic expansions of Laplace integrals*, Arch. Rational Mech. Anal., 7 (1961), pp. 1–20.

[6] A. ERDÉLYI AND M. WYMAN, *The asymptotic evaluation of certain integrals*, Ibid., 14 (1963), pp. 217–260.

[7] G. H. HARDY AND E. M. WRIGHT, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1960.

[8] S. KAPLUN, *Fluid Mechanics and Singular Perturbations*, P. A. Lagerstrom, L. N. Howard and C.-S. Liu, eds., Academic Press, New York, 1967.

[9] H. POINCARÉ, *Sur les integrales irregulieres des equations linéaires*, Acta Math., 8 (1886), pp. 295–344.

[10] E. YA. RIEKSTIŅŠ, *On the use of neutrices for asymptotic representation of some integrals*, Latvian Math. Annual (Latviiskiy Mathematicheskiy Yezhegodnik), Riga, 1966, pp. 5–21.

[11] J. RITT, *On the derivatives of a function at a point*, Ann. of Math. (2), 18 (1916), pp. 18–23.

[12] H. SCHMIDT, *Beiträge zu einer Theorie der allgemein asymptotischen Darstellungen*, Math. Ann., 113 (1937), pp. 629–656.

[13] K. D. SHERE, *Introduction to multiple asymptotic expansions with an application to elastic scattering*, J. Mathematical Phys., 12 (1971), pp. 78–83.

[14] ———, *Multiple asymptotic expansions and singular problems*, this Journal, 3 (1972), pp. 263–271.

[15] W. WASOW, *Asymptotic Expansions for Ordinary Differential Equations*, Interscience, New York, 1965.

[16] G. N. WATSON, *A theory of asymptotic series*, Philos. Trans. Roy. Soc. London Ser. A, 211 (1911), pp. 279–313.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

UNIFORM ASYMPTOTIC EXPANSIONS OF INTEGRALS OF THE LIPSHITZ–HANKEL TYPE*

R. MUKI†

Abstract. In this note we consider integrals of the type

$$\int_0^\infty f(t)J_\nu(rt\sin\theta)\exp(-rt\cos\theta)\,dt$$

for which we establish asymptotic expansions as $r \to \infty$, that are uniformly valid for $0 \leqq \theta \leqq \pi/2$.

Solutions to problems in mathematical physics pertaining to a half-space[1] frequently involve integrals of the type

$$\int_0^\infty f(t)J_\nu(\rho t)\exp(-zt)\,dt. \tag{1}$$

Here (ρ, z) are the usual radial and axial cylindrical coordinates while J_ν is the Bessel function of the first kind of nonnegative order ν, whereas f is a real-valued function defined on $[0, \infty)$.

By introduction of spherical polar coordinates (r, θ) through

$$\rho = r\sin\theta, \qquad z = r\cos\theta, \qquad 0 \leqq r < \infty, \quad 0 \leqq \theta \leqq \pi/2, \tag{2}$$

the integral (1) is carried into

$$G_\nu(r, \theta) = \int_0^\infty f(t)J_\nu(rt\sin\theta)\exp(-rt\cos\theta)\,dt. \tag{3}$$

It is the purpose of the present note to obtain an asymptotic expansion of $G_\nu(r, \theta)$ as $r \to \infty$ that is uniformly valid for $0 \leqq \theta \leqq \pi/2$.

When the function f has the form

$$f(t) = J_\mu(at)/t^\lambda, \qquad 0 < t < \infty, \quad \mu + \nu + 1 > \lambda > -1, \tag{4}$$

where a is a positive constant, $G_\nu(r, \theta)$ reduces to a Lipschitz–Hankel integral, the analytical properties of which have been thoroughly investigated.[2] In applications, however, one often encounters cases in which f does not fall into the special class of functions characterized by (4). As one such example we may cite the elastostatic problem for a layered half-space considered in [5].

Our ultimate objective is to prove the following theorem.

THEOREM. *Let f be a real-valued function with the properties*:

(a) *f is N-times continuously differentiable on* $[0, \infty)$, *where N is a positive integer*;

(b) *the n-th derivative $f^{(n)}$, $n = 0, 1, 2, \cdots, N$, of f satisfies the inequality*

$$|f^{(n)}(t)| < Mt^{1/2-\alpha}, \qquad 1 \leqq t < \infty, \tag{5}$$

---

* Received by the editors March 30, 1971, and in revised form July 15, 1971.

† Mechanics and Structures Department, University of California, Los Angeles, California 90024.

[1] See, for instance, [3, Chap. 10].

[2] See [4, Chap. 13] and [2].

for some positive numbers M and α.
Further, the integral

$$\int_1^{\infty} f^{(n)}(t) \exp(iyt) \frac{dt}{\sqrt{t}}$$

exists for $a < y < \infty$, where a is a positive number, and there are positive numbers M_* and a_* such that

$$\left| \int_1^{t_0} f^{(n)}(t) \exp(iyt) \frac{dt}{\sqrt{t}} \right| < M_*, \qquad 1 \leqq t_0 < \infty, \quad a \leqq a_* < y < \infty. \tag{6}$$

Then, for each nonnegative ν, the integral (3) *obeys the estimate*

$$G_\nu(r, \theta) = \sum_{n=0}^{N-1} \psi_n(\theta, \nu) f^{(n)}(0) r^{-n-1} + o(r^{-N}) \quad \text{as } r \to \infty, \qquad 0 \leqq \theta \leqq \pi/2, \tag{7}$$

in which

$$\psi_n(\theta, \nu) = \frac{\Gamma(\nu + n + 1)}{n!\Gamma(\nu + 1)} \left(\frac{\sin\theta}{2} \right)^\nu F\left(\frac{\nu + n + 1}{2}, \frac{\nu - n}{2}; \nu + 1; \sin^2\theta \right). \tag{8}$$

The hypergeometric series in (8) *converges uniformly and absolutely for* $0 \leqq \theta \leqq \pi/2$. *In particular,*

$$\begin{aligned} G_\nu(r, \pi/2) &= \int_0^\infty f(t) J_\nu(rt)\, dt \\ &= \frac{\sqrt{\pi}}{2^\nu} \sum_{n=0}^{N-1} \frac{\Gamma(\nu + n + 1)}{n!\Gamma((\nu - n + 1)/2)\Gamma((\nu + n + 2)/2)} \frac{f^{(n)}(0)}{r^{n+1}} + o(r^{-N}) \end{aligned} \tag{9}$$

as $r \to \infty$.

Observe that the hypergeometric series in (8) degenerates into a polynomial in $\sin\theta$ when $\nu - n$ is an even negative integer. Note also that by virtue of the assumptions (a) and (b) the integral in (3) is convergent for all $r > a$ and $0 \leqq \theta \leqq \pi/2$.

We prove the theorem first for a degenerate function f which, in addition to hypotheses (a) and (b), satisfies the condition

$$f^{(n)}(0) = 0, \qquad n = 0, 1, 2, \cdots, N - 1. \tag{10}$$

To this end, define a sequence of functions $\{J_\nu^m(x; \gamma)\}$ ($0 \leqq x < \infty, 0 \leqq \gamma < \infty$, $m = 0, 1, 2, \cdots, N$) by means of

$$\begin{aligned} J_\nu^0(x; \gamma) &= J_\nu(x) \exp(-\gamma x), \\ J_\nu^m(x; \gamma) &= -\int_x^\infty J_\nu^{m-1}(t; \gamma)\, dt, \qquad m = 1, 2, \cdots, N. \end{aligned} \tag{11}$$

In view of known asymptotic properties of Bessel functions,[3] if

$$J_\nu^m(x; \gamma) = \operatorname{Re}\left\{ \sqrt{\frac{2}{\pi x}} \frac{1}{(\gamma - i)^m} \exp\left[-\gamma x - i\left(x + \frac{2\nu - 3}{4}\pi \right) \right] \right\} + \mathscr{J}_\nu^m(x; \gamma), \tag{12)^4}$$

$$0 < x < \infty, \quad 0 \leqq \gamma < \infty,$$

[3] See [4, p. 197].

[4] Here and throughout the remainder of the proof it is understood that $m = 0, 1, 2, \cdots, N$.

then there exist positive numbers x_* and M_1 such that

(13) $$|\mathscr{J}_\nu^m(x;\gamma)| < M_1 x^{-3/2} \exp(-\gamma x), \qquad x_* < x < \infty, \quad 0 \leqq \gamma < \infty.$$

Further, since

(14)[5] $$|J_\nu(x)| \leqq 1, \qquad 0 \leqq \nu < \infty, \quad 0 \leqq x < \infty,$$

we have

(15) $$|J_\nu^m(x;\gamma)| \leqq \frac{1}{\gamma^m} \exp(-\gamma x), \qquad 0 < x < \infty, \quad 0 < \gamma < \infty.$$

Finally, we gather from (11), (12), (13) and (14) that there is a positive number M_2 such that

(16) $$|J_\nu^m(x;\gamma)| < M_2, \qquad 0 \leqq x < \infty, \quad 0 \leqq \gamma < \infty.$$

Formal application of N-fold integration by parts to the right-hand member of (3) leads, with the aid of (10) and (11), to

(17) $$G_\nu(r,\theta) = \frac{(-1)^N}{(r\sin\theta)^N} \int_0^\infty f^{(N)}(t) J_\nu^{(N)}(rt\sin\theta;\cot\theta)\,dt, \qquad a_* < r < \infty, \quad 0 < \theta \leqq \pi/2.$$

Because of (12), (13), (15), (16) and conditions (a), (b), the integral in (17) as well as all those which appeared in the process of integration by parts exist. In addition, (3) and the power series representation of J_ν furnish

(18) $$G_0(r,0) = \frac{1}{r^N} \int_0^\infty f^{(N)}(t) \exp(-rt)\,dt, \qquad 0 < r < \infty,$$
$$G_\nu(r,0) = 0, \qquad 0 < r < \infty, \quad 0 < \nu < \infty.$$

At this point observe that hypotheses (a) and (b) imply the existence of a positive constant M_3 such that

(19) $$|f^{(N)}(t)| < \begin{cases} M_3, & 0 \leqq t \leqq 1, \\ M_3 t^{1/2-\alpha}, & 1 \leqq t < \infty. \end{cases}$$

Since $\alpha > 0$, we conclude from (15), (17), (18) and (19) that, for $1 < r < \infty$,

(20) $$|G_\nu(r,0)| < \frac{(1+\sqrt{\pi})M_3}{r^{N+1}},$$
$$|G_\nu(r,\theta)| < \frac{(1+\sqrt{\pi})2^{(N+1)/2} M_3}{r^{N+1}}, \qquad 0 < \theta \leqq \pi/4.$$

Our next task is to obtain an estimate of $G_\nu(r,\theta)$ as $r \to \infty$ for $\pi/4 \leqq \theta \leqq \pi/2$. To this end we note from (16) and (19) that, for any $\varepsilon > 0$,

(21) $$\left| \int_0^\infty f^{(N)}(t) J_\nu^N(rt\sin\theta;\cot\theta)\,dt \right| < \frac{\varepsilon}{3}, \qquad 0 < r < \infty, \quad \pi/4 \leqq \theta \leqq \pi/2,$$

[5] This result is quoted without proof in [1, p. 362]. A proof of (14) may be supplied with the aid of the Bessel integral [4, p. 19], an integral representation of a product of two Bessel functions [4, p. 150], and by an appeal to the continuous dependence of $J_\nu(x)$ on ν.

provided δ is the smaller of $\varepsilon/(3M_2M_3)$ and 1. Further, with the aid of (12), (13) and (19), we establish the estimate

$$\left|\int_\delta^1 f^{(N)}(t)J_\nu^N(rt\sin\theta;\cot\theta)\,dt\right| < \frac{4M_3}{\sqrt{r}}\left[\sqrt{\frac{2}{\pi}}+\frac{M_1}{x_*}\right], \quad \sqrt{2}x_*/\delta < r, \quad \pi/4 \leqq \theta \leqq \pi/2. \tag{22}$$

Finally, we claim that there exist two positive numbers r_* and M_0 such that

$$\left|\int_1^\infty f^{(N)}(t)J_\nu^N(rt\sin\theta;\cot\theta)\,dt\right| < \frac{M_0}{\sqrt{r}}, \qquad r_* < r, \quad \pi/4 \leqq \theta \leqq \pi/2. \tag{23}$$

To confirm (23), we note that

$$\int_1^\infty f^{(N)}(t)J_\nu^N(rt\sin\theta;\cot\theta)\,dt = I_1(r,\theta) + I_2(r,\theta), \quad a_* < r < \infty, \quad \pi/4 \leqq \theta \leqq \pi/2, \tag{24}$$

where

$$\begin{aligned} I_1(r,\theta) &= \int_1^\infty f^{(N)}(t)[J_\nu^N(x;\gamma) - \mathscr{J}_\nu^N(x;\gamma)]\,dt, \\ I_2(r,\theta) &= \int_1^\infty f^{(N)}(t)\mathscr{J}_\nu^N(x;\gamma)\,dt, \\ x &= rt\sin\theta, \qquad \gamma = \cot\theta. \end{aligned} \tag{25}$$

To establish an estimate for I_1, we observe first that

$$\left|\int_1^\infty f^{(N)}(t)\exp(-ty\cot\theta)\exp(iyt)\frac{dt}{\sqrt{t}}\right| < M_*, \quad a_* < y < \infty, \quad \pi/4 \leqq \theta \leqq \pi/2, \tag{26}$$

which is obtained with the aid of (6) and Abel's test of uniform convergence.[6] Then, from (12), (25), (26) follows the inequality

$$|I_1(r,\theta)| < 2M_*/\sqrt{\pi r}, \qquad \sqrt{2}a_* < r < \infty, \quad \pi/4 \leqq \theta \leqq \pi/2. \tag{27}$$

On the other hand, (25), (19) and (18) imply

$$|I_2(r,\theta)| < 2M_1M_3r^{-3/2}/\alpha, \qquad \sqrt{2}x_* < r < \infty, \quad \pi/4 \leqq \theta \leqq \pi/2. \tag{28}$$

Thus, one infers from (24), (25), (27) and (28), the desired inequality (23).

We now gather from (17), (21), (22) and (23) that, given $\varepsilon > 0$ there exists a positive number r_0 such that

$$|G_\nu(r,\theta)| < 2^{N/2}\varepsilon/r^N, \qquad r_0 < r; \quad \pi/4 \leqq \theta \leqq \pi/2. \tag{29}$$

Combining (29) and (20) one sees that the theorem is true if the first $N - 1$ derivatives of f vanish at the origin, as specified by (10).

[6] See [6, p. 481].

We are now in a position to prove the theorem as stated originally. Thus, let f be a function that meets conditions (a), (b) and is otherwise arbitrary. Further, define functions $\hat{f}$ and $\tilde{f}$ through

(30)[7]
$$\hat{f}(t) = J_N(t) \sum_{l=0}^{N-1} A_l t^{l-N}, \qquad 0 < t < \infty,$$
$$\hat{f}(0) = A_0(J_N^{(N)}(0)/N!),$$
$$\tilde{f}(t) = f(t) - \hat{f}(t), \qquad 0 < t < \infty,$$

where the constant coefficients A_l are determined recursively from

(31)[8]
$$A_0 = N!f(0)/J_N^{(N)}(0), \qquad A_1 = N!f^{(1)}(0)/J_N^{(N)}(0),$$
$$A_l = \frac{N!}{J_N^{(N)}(0)}\left[\frac{f^{(l)}(0)}{l!} - \sum_{s=1}^{[l/2]} \frac{J_N^{(N+2s)}(0)}{(N+2s)!}A_{l-2s}\right], \qquad l = 2, 3, \cdots, N-1.$$

It is clear from (30) and the asymptotic behavior of $J_N(t)$ as $t \to \infty$ that $\tilde{f}$ satisfies the conditions (a) and (b). Moreover, the first $N - 1$ derivatives of $\tilde{f}$ vanish at the origin, as is apparent from (30) and (31). Thus, one draws from the result just established that

(32)
$$\int_0^\infty \tilde{f}(t)J_\nu(rt \sin\theta) \exp(-rt\cos\theta)\, dt = o(r^{-N}) \quad \text{as } r \to \infty, \qquad 0 \leqq \theta \leqq \pi/2.$$

Next, we recall that for $l = 0, 1, 2, \cdots, N-1$,

(33)[9]
$$\int_0^\infty t^{l-N} J_N(t) J_\nu(rt\sin\theta)\exp(-rt\cos\theta)\,dt$$
$$= \frac{\sin^\nu\theta}{2^\nu\Gamma(\nu+1)} \sum_{k=0}^\infty \frac{(-1)^k\Gamma(l+\nu+2k+1)}{k!(N+k)!2^{N+2k}r^{l+2k+1}} \cdot F\left(\frac{\nu+l+2k+1}{2}, \frac{\nu-l-2k}{2}; \nu+1; \sin^2\theta\right),$$
$$2 \leqq r < \infty, \quad 0 \leqq \theta \leqq \pi/2,$$

the series in (33) being uniformly and absolutely convergent on its region of definition.

We now infer from (30), (31), (33), (8) and

(34)
$$J_N^{(N+2k)}(0) = \frac{(-1)^k(N+2k)!}{2^{N+2k}k!(N+k)!}, \qquad k = 0, 1, 2, \cdots,$$

[7] $J_N^{(s)} = d^sJ_N/dx^s$.

[8] $[x]$ denotes the largest integer not exceeding x.

[9] See [4, p. 399]. A proof of the uniform and absolute convergence of the series on the region
$$R = \{(r,\theta) | r\cos\theta > 1, r(1-\cos\theta) > 1\}$$
is given on pp. 399–401 in [4].

that

$$\int_0^\infty \hat{f}(t)J_\nu(rt\sin\theta)\exp(-rt\cos\theta)\,dt$$

(35)
$$\begin{aligned} &= \sum_{l=0}^{N-1}\sum_{k=0}^{\infty}\frac{(l+2k)!J_N^{(N+2k)}(0)}{(N+2k)!r^{l+2k+1}}\psi_{l+2k}(\theta,\nu) \\ &= \sum_{n=0}^{N-1} f^{(n)}(0)\psi_n(\theta,\nu)r^{-n-1} + \sum_{n=N}^{\infty} n!B_n\psi_n(\theta,\nu)r^{-n-1}, \\ &\qquad 2 \leqq r < \infty, \quad 0 \leqq \theta \leqq \pi/2, \end{aligned}$$

provided

(36)
$$B_n = \sum_{s=[(n+1-N)/2]}^{[n/2]} \frac{J_N^{(N+2s)}(0)}{(N+2s)!}A_{n-2s}, \qquad n = N, N+1, \cdots.$$

The rearrangement of the first series in (35) is permissible since this series is absolutely convergent. Thus, (30), (32), (35) imply (7). Further, (7), (8) and

(37)
$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (a+b-c<0), \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi},$$

yield the estimate (9). This completes the proof.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1964.

[2] G. Eason, B. Noble and I. N. Sneddon, *On certain integrals of Lipshitz–Hankel type involving products of Bessel functions*, Philos. Trans. Roy. Soc. London Ser. A, 247 (1955), pp. 529–551.

[3] I. N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York, 1951.

[4] G. N. Watson, *The Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1944.

[5] R. A. Westmann, *Layered systems subjected to asymmetric surface shears*, Proc. Roy. Soc. Edinburgh, 66 (1964), pp. 140–149.

[6] T. J. I'A. Bromwich, *The Theory of Infinite Series*, Macmillan, New York, 1949.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

ON THE APPLICABILITY OF LYAPUNOV'S THEOREM IN HILBERT SPACE*

A. PAZY†

Abstract. Let X be a Banach space, $T(t)$ a strongly continuous semigroup of bounded operators on X, $p \geqq 1$ and

$$\|x\|_p = \left(\int_0^\infty \|T(t)x\|^p\,dt\right)^{1/p}.$$

We determine, by means of a necessary and sufficient condition, those semigroups $T(t)$ for which $\|\cdot\|_p$ defines a norm on X which is equivalent to the original norm $\|\cdot\|$ on X.

1. Introduction. In [1], R. Datko extends a well-known theorem of A. M. Lyapunov concerning Hurwitzian matrices ($n \times n$ matrices with eigenvalues in the half-plane $\operatorname{Re} z < 0$) to strongly continuous semigroups of operators on a complex Hilbert space. Lyapunov's result can be stated as follows: Let A be a complex $n \times n$ matrix and A^* its adjoint. Then A has all its characteristic roots lying in the half-plane $\operatorname{Re} z < 0$ if and only if the solution B of the matrix equation $A^*B + BA = -I$ (the identity matrix) is a unique positive definite Hermitian matrix. The extension of this result is the following theorem.

THEOREM. *A necessary and sufficient condition that the semigroup $T(t)$ of class C_0 on a complex Hilbert space H satisfy the condition $\|T(t)\| \leqq M e^{-\mu t}$, where $M \geqq 1$ and $\mu > 0$, is the existence of an Hermitian endomorphism B on H with $B \geqq 0$, such that the relation $2\operatorname{Re}(BAx, x) = -\|x\|^2$ holds for every $x \in D(A)$ (the domain of the infinitesimal generator A of the semigroup $T(t)$).*

Note that the condition $\|T(t)\| \leqq M e^{-\mu t}$ implies that the spectrum of A lies in the half-plane $\operatorname{Re} z \leqq -\mu$.

The usefulness of Lyapunov's theorem in ordinary differential equations is that it allows for an explicit representation of a Lyapunov function as a positive definite quadratic form. Using this representation one may then, for example, study the effects of perturbations on asymptotically stable, linear constant coefficient systems of ordinary differential equations. Consider say the perturbed system

$$\frac{du}{dt} = Au + f(t,u), \qquad f(t,0) = 0. \tag{1}$$

Let A have its eigenvalues in the left half-plane $\operatorname{Re} z < 0$. By Lyapunov's theorem we then have a positive definite Hermitian matrix B such that

$$\frac{d}{dt}(Bu, u) = -\|u\|^2 + 2\operatorname{Re}(Bu, f(t,u)). \tag{2}$$

Hence, one may obtain conditions on the nonlinear perturbation $f(t, u)$ to insure that $d(Bu, u)/dt$ is negative definite (negative semidefinite) thus insuring asymptotic stability (stability) of the zero equilibrium. It is essential, however, that for (Bu, u)

* Received by the editors June 17, 1971.

† Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel.

to be a Lyapunov function, we have an estimate of the form $(Bu, u) \geqq b\|u\|^2$ for some $b > 0$. In other words the Hermitian form B has to define an equivalent norm on H. If H has finite dimension, this is always the case. However, if H has infinite dimension, it is not. The purpose of the present note is to characterize those semigroups $T(t)$ for which the Hermitian form B defines an equivalent norm on H and Lyapunov's theorem is applicable in the usual way to infinite-dimensional systems (see Zubov [2]). For example, consider the case when A is an unbounded closed linear operator on H and the solutions to (1) are viewed in some "mild" sense (see Browder [3], Kato [4]). It turns out that for semigroups of "hyperbolic" type, B defines an equivalent norm on H while it does not define an equivalent norm for semigroups of "parabolic" type. The precise conditions are given in Theorems 1 and 2 below.

2. Equivalent norms "generated" by a semigroup. Let H be a Hilbert space and let $T(t)$ be a strongly continuous semigroup of bounded operators on H. From the theorem quoted in the Introduction it follows that if $\|T(t)\| \leqq M e^{-\mu t}$, then there exists an Hermitian endomorphism B satisfying $2 \operatorname{Re}(BAx, x) = -\|x\|^2$ for every $x \in D(A)$. It is not difficult to show (see [1]) that

$$(Bx, x) = \int_0^\infty \|T(t)x\|^2 \, dt$$

for every $x \in X$. Therefore, B defines an equivalent norm on H if there exists a constant $b > 0$ such that

$$\int_0^\infty \|T(t)x\|^2 \, dt \geqq b\|x\|^2 \quad \text{for every } x \in X.$$

We shall consider the following, slightly more general, problem. Let X be a Banach space, $T(t)$ a strongly continuous semigroup of bounded operators on X, $p \geqq 1$, and

(3) $$\|x\|_p = \left(\int_0^\infty \|T(t)x\|^p \, dt\right)^{1/p}.$$

Our problem is to determine for what semigroups $T(t)$ is $\|\cdot\|_p$ a norm on X which is equivalent to the original norm $\|\cdot\|$ on X. The answer to this problem is given in the following two theorems.

THEOREM 1. *Let $p \geqq 1$ be fixed. $\|x\|_p$ is finite for every $x \in X$ if and only if there exist constants $M \geqq 1$ and $\mu > 0$ such that $\|T(t)\| \leqq M e^{-\mu t}$.*

THEOREM 2. *If $\|x\|_p$ is finite for every $x \in X$, it defines a norm $\|\cdot\|_p$ on X, equivalent to the original norm $\|\cdot\|$, if and only if there exist $t_0 > 0$ and $c > 0$ such that*

(4) $$\|T(t_0)x\| \geqq c\|x\| \quad \textit{for every } x \in X.$$

Remark. From Theorem 1 it follows, in particular, that if for some $p \geqq 1$, $\|x\|_p$ is finite for every $x \in X$, then this is true for every $p \geqq 1$.

Proof of Theorem 1. It is clear that if $\|T(t)\| \leqq M e^{-\mu t}$, then $\|x\|_p$ is finite for every $x \in X$ and every $p \geqq 1$.

Let $p \geqq 1$ be given and let $\|x\|_p$ be finite for every $x \in X$. Using the boundedness of any semigroup by $M e^{\omega t}$, for some real ω and $M \geqq 1$, and the finiteness of $\|x\|_p$

one obtains easily that $\lim_{t\to\infty} T(t)x = 0$ for every $x \in X$ (see, e.g., Lemma 3 of [1]). Hence, $\|T(t)x\|$ is bounded for every $x \in X$, and from the uniform boundedness theorem we have

$$\|T(t)\| \leqq M, \qquad M \geqq 1. \tag{5}$$

Let $S: X \to L^p(R^+; X)$ be defined by $Sx = T(t)x$. S is a linear operator defined on all of X, and it is readily seen to be closed. By the closed graph theorem S is bounded, and we obtain

$$\left(\int_0^\infty \|T(t)x\|^p\,dt\right)^{1/p} \leqq M_1\|x\|. \tag{6}$$

Now, let $0 < \rho < M^{-1}$ and define

$$t_x(\rho) = \sup\{t : \|T(s)x\| \geqq \rho\|x\| \quad \text{for every } s \in [0, t]\}.$$

Since $\|T(t)x\| \to 0$ as $t \to \infty$, $t_x(\rho)$ is finite for every $x \neq 0$. Using (6) we have

$$t_x(\rho)\rho^p\|x\|^p \leqq \int_0^{t_x(\rho)} \|T(t)x\|^p\,dt \leqq \int_0^\infty \|T(t)x\|^p\,dt \leqq M_1^p\|x\|^p,$$

and therefore, $t_x(\rho) \leqq (M_1/\rho)^p$. Thus, for $t > (M_1/\rho)^p$ we have

$$\|T(t)x\| \leqq \|T(t - t_x(\rho))\|\,\|T(t_x(\rho))x\| \leqq M\rho\|x\| = \rho'\|x\| \tag{7}$$

with $\rho' < 1$. Therefore, if $t > (M_1/\rho)^p$, we have $\|T(t)\| \leqq \rho' < 1$. This together with the semigroup property imply $\|T(t)\| \leqq M\,e^{-\mu t}$.

Proof of Theorem 2. From Theorem 1 it follows that $\|T(t)\| \leqq M\,e^{-\mu t}$ and that $\|x\|_p \leqq M_1\|x\|$. It is therefore sufficient to prove that (4) is equivalent to $\|x\|_p \geqq m\|x\|$ for some $m > 0$ and all $x \in X$. Suppose (4) does not hold. Then for every $t > 0$ and every $\varepsilon > 0$ there exists an element $x \in X$ such that $\|x\| = 1$ and $\|T(t)x\| < \varepsilon$. But

$$\int_0^\infty \|T(t)x\|^p\,dt = \int_0^\tau \|T(t)x\|^p\,dt + \int_0^\infty \|T(t+\tau)x\|^p\,dt \leqq \tau M_1^p\|x\|^p + M_1^p\|T(\tau)x\|^p.$$

Given $\varepsilon > 0$, we first choose τ such that τM_1^p is small, and then we choose x such that $\|x\| = 1$ and $\|T(\tau)x\|$ is small. Therefore, given any $\varepsilon > 0$, there exists an element $x \in X$ such that $\|x\| = 1$ and $\|x\|_p < \varepsilon$ and no estimate of the form $\|x\|_p \geqq m\|x\|$ is possible. The condition (4) is therefore necessary.

Let $\|T(t_0)x\| \geqq c\|x\|$ for every $x \in X$ and let $0 \leqq t \leqq t_0$. Then

$$c\|x\| \leqq \|T(t_0)x\| = \|T(t_0 - t)T(t)x\| \leqq M\|T(t)x\|$$

for every $x \in X$, and therefore,

$$\int_0^\infty \|T(t)x\|^p\,dt \geqq \int_0^{t_0} \|T(t)x\|^p\,dt \geqq \left(\frac{c}{M}\right)^p t_0\|x\|^p;$$

that is, $\|x\|_p \geqq m\|x\|$, and the condition (4) is sufficient.

COROLLARY 1. *Let $T(t)$ be a strongly continuous semigroup of bounded operators on a Banach space X, satisfying $\|T(t)\| \leqq M\,e^{-\mu t}$, $M \geqq 1$, $\mu > 0$. If for $t > 0$,*

$R(T(t))$ (the range of $T(t)$) is dense in X, then $\|\cdot\|_p$ defines an equivalent norm on X if and only if $T(t)$ can be extended to a group of bounded operators on X (that is, A is a generator of a group of bounded operators on X).

Proof. From Theorem 2 it follows that $\|\cdot\|_p$ is equivalent to $\|\cdot\|$ if and only if there exist $t_0 > 0$ and $c > 0$ such that $\|T(t_0)x\| \geqq c\|x\|$ for every $x \in X$. As in the proof of Theorem 2 this implies $\|T(t)x\| \geqq c'\|x\|$ for every $0 \leqq t \leqq t_0$. Since $R(T(t))$ is dense, $T(t)$ is invertible for every $0 \leqq t \leqq t_0$. Using the semigroup property we find that $T(t)$ is invertible for every $t \geqq 0$. Defining $T(-t) = (T(t))^{-1}$ we obtain a C_0 group of bounded operators. On the other hand if $T(t)$ is a C_0 group, $T(t)$ is invertible for every $t \geqq 0$ and we have $\|T(t)x\| \geqq (\|T(-t)\|)^{-1}\|x\|$ for every $x \in X$.

From the proof of Corollary 1 it follows that if A generates a group of operators such that $\|T(t)x\| \leqq M e^{-\mu t}\|x\|$, then $\|\cdot\|_p$ is always equivalent to $\|\cdot\|$. In particular, if A is a bounded operator and $\|T(t)\| \leqq M e^{-\mu t}$, $\|\cdot\|_p$ is always equivalent to $\|\cdot\|$. Thus, for bounded operators A the situation is identical to the situation in the finite-dimensional case.

We conclude with two simple examples which illustrate the use of the previous results.

Example 1. Let H be a Hilbert space, and let A be a self-adjoint (unbounded) operator. By Stone's theorem iA generates a group $U(t)$ of unitary operators on H. Let $\varepsilon > 0$ and consider the operator $iA - \varepsilon I$. This operator generates a group $T(t)$ of bounded operators given by $T(t) = e^{-\varepsilon t}U(t)$. By our previous results $\|\cdot\|_2$ is an equivalent norm on H. Thus for the "hyperbolic" case Lyapunov's method is applicable.

Example 2. Let X be a Banach space and let A be the infinitesimal generator of a holomorphic semigroup $T(t)$ satisfying $\|T(t)\| \leqq M e^{-\mu t}$. If A is unbounded, then $\|\cdot\|_p$ does not define an equivalent norm on X. The reason for this is the following: suppose $\|\cdot\|_p$ is an equivalent norm; then by Theorem 2 there exist $t_0 > 0$ and $c > 0$ such that $\|T(t_0)x\| \geqq c\|x\|$; therefore if $x \in D(A)$, we have $\|T(t_0)Ax\| \geqq c\|Ax\|$, but since $T(t)$ is holomorphic, $AT(t_0)$ is a bounded operator, and therefore,

$$\|Ax\| \leqq \frac{1}{c}\|AT(t_0)\|\,\|x\| \quad \text{for every } x \in D(A).$$

Since $D(A)$ is dense this holds for every $x \in X$, and A is bounded. Therefore, $\|\cdot\|_p$ does not define an equivalent norm in the "parabolic" case.

Acknowledgment. I wish to thank Professors B. Weiss and M. Slemrod for several stimulating conversations.

REFERENCES

[1] R. DATKO, *Extending a theorem of A. M. Liapunov to Hilbert space*, J. Math. Anal. Appl., 32 (1970), pp. 610–616.

[2] V. I. ZUBOV, *Methods of A. M. Liapunov and Their Applications*, P. Noordhoff, Gröningen, Netherlands, 1964.

[3] F. E. BROWDER, *Nonlinear equations of evolution*, Ann. of Math., 80 (1964), pp. 485–523.

[4] T. KATO, *Nonlinear evolution equations in Banach spaces*, Proc. Symposium on Applied Mathematics, 17 (1965).

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

REMARKS ON SINGULAR PERTURBATION OF CERTAIN NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS*

SEYMOUR V. PARTER†

Abstract. Under some relatively simple assumptions on $f(x, u, v)$, this report discusses the existence and asymptotic behavior of solutions of $\varepsilon u'' + f(x, u, u')u' = 0$, $u'(0) - au(0) = A$, $u'(1) + bu(1) = B$.

1. Introduction. Consider the nonlinear two-point boundary value problem

(1.1) $$\varepsilon u'' + f(x, u(x), u'(x))u' = 0, \qquad 0 \leqq x \leqq 1,$$

(1.2) $$u'(0) - au(0) = A \geqq 0, \qquad a > 0,$$

(1.3) $$u'(1) + bu(1) = B > 0, \qquad b > 0.$$

Let $\varepsilon > 0$ and assume:

H1: $f(x, u, u')$ is continuous in the region

$$R \equiv \{(x, u, u')|0 \leqq x \leqq 1, 0 \leqq u \leqq B/b, 0 \leqq u' \leqq A + aB/b\};$$

H2: $f(x, u, u') \geqq \beta > 0$ for all $(x, u, u') \in R$.

Recently D. S. Cohen [2] used the "shooting method" to study this problem under somewhat more restrictive hypotheses. Our approach is based on a priori estimates and the Schauder fixed-point theorem. The physical motivations for this problem as well as other interesting background facts are discussed in [2].

2. Results. For $\varepsilon > 0$ let H1′ and H2′ be the hypotheses H1 and H2 with R replaced by R', where

(2.1) $$R' \equiv \{(x, u, u')|0 \leqq x \leqq 1, -A/a \leqq u \leqq B/b, 0 \leqq u' \leqq A + aB/b\}.$$

Let W be the set of all functions $v(x) \in C^1[0, 1]$ which satisfy

(2.2) $$v'(0) - av(0) = A, \qquad v'(1) + bv(1) = B,$$

(2.3) $$(x, v(x), v'(x)) \in R' \quad \text{for all } x \in [0, 1].$$

Let $\varepsilon > 0$ be fixed, let $v(x) \in W$ and let $u(x) \in C^2[0, 1]$ be the *unique* solution of the *linear* boundary value problem

(2.4) $$\varepsilon u'' + f(x, v(x), v'(x))u' = 0, \qquad 0 \leqq x \leqq 1,$$

(2.5) $$u'(0) - au(0) = A, \qquad u'(1) + bu(1) = B.$$

LEMMA 1. *Assume that* H1′ *and* H2′ *hold. Then* $u(x) \in W$.

Proof. Since $u(x) \not\equiv$ const., the maximum principle [3] tells us that $|u'(x)| > 0$ for $0 \leqq x \leqq 1$. Suppose

$$u'(x) < 0, \qquad 0 \leqq x \leqq 1.$$

* Received by the editors June 1, 1971, and in revised form July 26, 1971.

† Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706. This work was supported by the U.S. Office of Naval Research under Contract N00014-67-A-0128-0004.

Then

$$u(0) > u(1). \tag{2.6}$$

On the other hand,

$$u(1) = \frac{B - u'(1)}{b} > 0, \qquad u(0) = \frac{u'(0) - A}{a} < 0$$

which contradicts (2.6). Thus

$$u'(x) > 0, \qquad 0 \leqq x \leqq 1. \tag{2.7}$$

Hence

$$-A/a \leqq -A/a + u'(0)/a = u(0) < u(1) = B/b - u'(1)/b \leqq B/b$$

and

$$-A/a \leqq u(x) \leqq B/b. \tag{2.8}$$

Finally, since

$$u'' = -\frac{1}{\varepsilon} f(x, v(x), v'(x))u'(x) \leqq 0,$$

$u'(x)$ assumed its maximum at $x = 0$. Thus,

$$0 \leqq u'(x) \leqq u'(0) = A + au(0) \leqq A + aB/b,$$

which completes the proof.

Let T denote the mapping described above, i.e.,

$$T: W \to W \tag{2.9}$$

and

$$T(v) = u. \tag{2.10}$$

LEMMA 2. *T is continuous in the $C^1[0, 1]$ topology.*

Proof. Let $v_1(x), v_2(x) \in W$ and let

$$T(v_1) = u_1, \quad T(v_2) = u_2, \quad w = T(v_1) - T(v_2). \tag{2.11}$$

Then $w(x)$ satisfies the equation

$$\begin{aligned} &\varepsilon w'' + f(x, v_1, v_1')w' = [f(x, v_2, v_2') - f(x, v_1, v_1')]u_2'(x), \qquad 0 \leqq x \leqq 1, \\ &w'(0) - aw(0) = 0, \qquad w'(1) + bw(1) = 0. \end{aligned} \tag{2.12}$$

The lemma now follows from standard estimates. That is, as $v_2 \to v_1$ and $v_2' \to v_1'$, w and $w' \to 0$.

We now remind the reader of the well-known Schauder fixed-point theorem (see [1, p. 97]).

THEOREM (Schauder). *If T is a continuous mapping of a closed convex set W in a Banach space X into a compact set $W_0 \subset W$, then T has a fixed point in W_0.*

THEOREM 1. *For every $\varepsilon > 0$ there exists (at least one) a solution $u(x, \varepsilon)$ of* (1.1), (1.2), (1.3) *and that solution $u(x, \varepsilon) \in W$.*

Proof. For fixed $\varepsilon > 0$ let

$$K = \frac{1}{\varepsilon}\left(A + \frac{aB}{b}\right) \max \{|f(x, u, u')|; (x, u, u') \in R'\}.$$

Let X be the Banach space $C^1[0, 1]$ and let W be the W defined above. Let W_0 be the set of all $w(x) \in W$ for which

$$|w''| \leqq K.$$

Then, using the Ascoli–Arzela lemma (see [1]) we see that W_0 is a compact subset of the closed convex set $W \subset X$. Thus we may apply the Schauder fixed-point theorem, and the theorem follows.

LEMMA 3. *There are a sequence $\varepsilon_n \to 0+$ and a constant $\bar{u}$ such that*

$$\max_{0 \leqq x \leqq 1} |u(x, \varepsilon_n) - \bar{u}| \to 0 \quad \textit{as } \varepsilon_n \to 0+. \tag{2.13}$$

Proof. The solutions $u(x, \varepsilon)$ are uniformly bounded and equicontinuous. Hence there are a sequence $\varepsilon_n \to 0+$ and a function $U(x)$ such that

$$\max_{0 \leqq x \leqq 1} |u(x, \varepsilon_n) - U(x)| \to 0 \quad \text{as } \varepsilon_n \to 0+. \tag{2.14}$$

However, we claim $U(x) \equiv$ const. Consider the function

$$\phi(x, \varepsilon_n) = e^{\beta x/\varepsilon_n}[u(x, \varepsilon_n) - u(1, \varepsilon_n)].$$

Then $\phi(x, \varepsilon_n)$ satisfies the equations

$$\begin{gathered} \varepsilon\phi'' + [f - 2\beta]\phi - \frac{1}{\varepsilon}[f\beta - \beta^2]\phi = 0, \\ \phi(1, \varepsilon_n) = 0, \qquad |\phi(0, \varepsilon_n)| \leqq 2B/b. \end{gathered} \tag{2.15}$$

Applying H2′ we see that

$$|\phi(x, \varepsilon_n)| \leqq 2B/b$$

which implies

$$|u(x, \varepsilon_n) - u(1, \varepsilon_n)| \leqq (2B/b)\, e^{-\beta x/\varepsilon_n}.$$

Thus, for all $x \in (0, 1)$,

$$u(x, \varepsilon_n) \to \lim u(1, \varepsilon_n) \quad \text{as } \varepsilon_n \to 0+. \tag{2.16}$$

But because of the uniform convergence in (2.14) we see that

$$U(x) \equiv U(1),$$

and the lemma is proved.

LEMMA 4. *Under the hypothesis above,*

$$\lim u(x, \varepsilon_n) = \bar{u} = B/b.$$

Proof. Let $x \in (0, 1)$. Then

$$\frac{u(x, \varepsilon_n) - u(1, \varepsilon_n)}{x - 1} = u'(1, \varepsilon) + \frac{1}{2}u''(\xi, \varepsilon_n)(x - 1).$$

Since $u''(\xi, \varepsilon_n)(x - 1) > 0$, we have

$$\frac{u(x, \varepsilon_n) - u(1, \varepsilon_n)}{x - 1} \geqq u'(1, \varepsilon_n) \geqq 0.$$

Then, using Lemma 3, we have

$$0 \geqq \limsup u'(1, \varepsilon_n) \geqq \liminf u'(1, \varepsilon_n) \geqq 0$$

and

$$u'(1, \varepsilon_n) \to 0.$$

But then

$$u(1, \varepsilon_n) = \frac{B - u'(1, \varepsilon_n)}{b} \to B/b.$$

THEOREM 2. *Let* $\{u(x, \varepsilon)\}$ *be solutions of* (1.1), (1.2), (1.3) *which lie in* W. *Then*

$$u(1, \varepsilon) \to B/b \quad \text{as } \varepsilon \to 0+ \tag{2.17}$$

and

$$\max_{0 \leqq x \leqq 1} |u(x, \varepsilon) - B/b| \to 0 \quad \text{as } \varepsilon \to 0+. \tag{2.18}$$

Proof. Suppose (2.17) is false. Then there is a sequence $\varepsilon_n \to 0+$ such that

$$u(1, \varepsilon_n) \to c_0 \neq B/b. \tag{2.19}$$

However, we may extract a subsequence $\varepsilon_{n'}$ which converges as in Lemma 3. Then applying Lemma 4,

$$u(1, \varepsilon_{n'}) \to B/b$$

which contradicts (2.19). Thus (2.17) is established. Then the argument of Lemma 3 using the comparison function $\phi(x, \varepsilon)$ leads to the conclusion that

$$u(x, \varepsilon) \to B/b \quad \text{for all } x \in (0, 1].$$

But, an equicontinuous and bounded family which converges on a dense set converges uniformly.

Remark. We cannot expect that $u'(x, \varepsilon)$ will converge to 0 uniformly on the entire interval $[0, 1]$. Indeed,

$$u'(0, \varepsilon) = A + au(0, \varepsilon) \to A + aB/b.$$

However, we easily obtain the following result.

THEOREM 3. *Let* $\delta > 0$. *Then*

$$\max\{|u'(x, \varepsilon)|, \delta \leqq x \leqq 1\} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. Observe that

$$u'' < 0, \qquad u' > 0.$$

Hence, if $\delta \leqq x \leqq 1$, then

$$|u'(x, \varepsilon)| \leqq u'(\delta, \varepsilon).$$

Thus it suffices to prove that

$$u'(\delta, \varepsilon) \to 0. \tag{2.20}$$

But we now proceed as in the proof of Lemma 4. Let $y \in (0, \delta)$. Then

$$\frac{u(y, \varepsilon) - u(\delta, \varepsilon)}{y - \delta} \geqq u'(\delta, \varepsilon) \geqq 0,$$

and we see that (2.20) holds.

Finally, let us return to our original problem. Suppose we do not have H1′ but only H1. Let

$$\tilde{f}(x, u, u') = \begin{cases} f(x, u, u'), & (x, u, u') \in R, \\ f(x, 0, u'), & (x, u, u') \in R' \quad \text{but} \quad u \leqq 0. \end{cases}$$

Let us replace $f(x, u, u')$ by $\tilde{f}(x, u, u')$. Then the solutions $\tilde{u}(x, \varepsilon)$ obtained in Theorem 1 are solutions of the original problem if $\tilde{u}(0, \varepsilon) \geqq 0$. However since $\tilde{u}(0, \varepsilon) \to B/b > 0$ we have: under the hypotheses H1 and H2 there is an $\varepsilon_0 > 0$ such that there exists a solution of (1.1), (1.2), (1.3) for all $\varepsilon \in (0, \varepsilon_0)$.

REFERENCES

[1] L. Bers, *Topology*, Lecture notes, Courant Institute of Mathematical Sciences, New York, 1956–1957.

[2] D. S. Cohen, *Singular perturbation of nonlinear two-point boundary value problems*, to appear.

[3] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

PERIODIC SOLUTIONS OF A CLASS OF WEAKLY NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS*

A. K. AZIZ† AND S. L. BRODSKY‡

Abstract. In a recent paper A. C. Lazer gave a technique for the determination of periodic solutions of weakly nonlinear ordinary differential equations of the form $dx/dt = \varepsilon f(t, x, \varepsilon)$, where x and f are real k-vectors and ε is a real parameter. In the present paper a method similar to that of Lazer is developed for the determination of periodic solutions of the weakly nonlinear hyperbolic system of the form $u_{xy} = \varepsilon f(x, y, u, u_x, u_y, \varepsilon)$. Both problems of obtaining periodic solutions in a strip and in the large are considered.

1. Introduction. In this paper we consider the hyperbolic system

$$u_{xy} = \varepsilon f(x, y, u, u_x, u_y, \varepsilon), \tag{1.1}$$

where $u = \operatorname{col}(u_1, \cdots, u_n)$, $f = \operatorname{col}(f_1, \cdots, f_n)$, and discuss a method for determining the solutions of this system which are periodic. We shall examine both the problems of obtaining periodic solutions in the strip and in the large.

The questions of existence and uniqueness of periodic solutions of (1.1) have been considered by various authors and in particular by L. Cesari [5], [6] and J. Hale [8], who extended methods they had used in the study of similar problems in ordinary differential equations. In [1] and [2] the problems of existence and uniqueness of periodic solutions of

$$u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y, u, u_x, u_y) \tag{1.2}$$

have been considered. The approach in [1], [2] is different from that used by other authors but depends on the presence of the damping factors a and b and thus is not applicable to the present problem.

The method we shall exploit here is an extension of a technique developed by A. Lazer [10] for the determination of periodic solutions of ordinary differential equations of the form

$$dx/dt = \varepsilon f(x, y, t, \varepsilon),$$
$$dy/dt = Ay + \varepsilon g(x, y, t, \varepsilon),$$

where x and f are real k-vectors, y and g are real $(n - k)$-vectors,

$$f(x, y, t + T, \varepsilon) \equiv f(x, y, t, \varepsilon),$$
$$g(x, y, t + T, \varepsilon) \equiv g(x, y, t, \varepsilon),$$

* Received by the editors July 15, 1971.

† Division of Mathematics, University of Maryland Baltimore County, Baltimore, Maryland 21228, and Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland. The work of this author was supported in part by the Atomic Energy Commission under Contract AEC-AT-(40-1)3433, and in part by the U.S. Naval Ordnance Laboratory, White Oak, Maryland.

‡ U.S. Naval Ordnance Laboratory, White Oak, Maryland. Now at Office of Naval Research, Washington, D.C. 20360.

ε is a real parameter, and A is an $(n - k) \times (n - k)$ real matrix, none of whose eigenvalues are integral multiples of $2\pi i/T$. We should mention that the Lazer process has its roots in the fundamental work of L. Cesari [7] and J. Hale [9], which was modified to allow one to compute approximate periodic solutions.

In this paper a crucial role is played by an equivalent integral operator formulation in a suitable function space of the differential problem considered. Once this formulation is accomplished the main results are obtained in a manner similar to that in [10]. The present method appears to be simple, computationally feasible and provides a unified approach to both problems of obtaining periodic solutions in a strip and in the large. In the literature these problems have been treated separately, requiring different techniques (see, e.g., [4], [5], [6]).

In § 2 we deal with the periodic solutions of (1.1) in a strip and give a detailed description of the method employed in proving our results. In § 3 we extend our results to periodic solutions of (1.1) in the large. In this case, since the arguments used are similar to those described in § 2, we merely outline the proofs. The results we obtain can be applied to the related equation

$$u_{tt} - u_{xx} = \varepsilon f(t, x, u, u_t, u_x, \varepsilon)$$

which has been investigated by O. Vejvoda [13], J. Hale [8] and P. Rabinowitz [11].

2. Periodic solutions in a strip. In this section we consider the problem of determining solutions of

$$u_{xy} = \varepsilon f(x, y, u, u_x, u_y, \varepsilon), \tag{2.1}$$

in the strip $S = \{(x, y); -\infty < x < \infty, 0 \leqq y \leqq y_0\}$, satisfying the conditions

$$u(x, 0, \varepsilon) = \psi(x, \varepsilon), \quad u(x + T, y, \varepsilon) = u(x, y, \varepsilon), \quad u_y(x, 0, 0) = \phi_0,$$

for $(x, y) \in S$ and $0 \leqq \varepsilon \leqq \varepsilon_0$, where $u = \mathrm{col}(u_1, \cdots, u_n)$, $f = \mathrm{col}(f_1, \cdots, f_n)$, $\psi = \mathrm{col}(\psi_1, \cdots, \psi_n)$, $\phi_0 = (\phi_0^1, \cdots, \phi_0^n)$ is a constant vector and ε is a small parameter. The function ψ is taken to be continuous on $(-\infty, \infty) \times [0, \varepsilon_0]$, continuously differentiable with respect to x, and periodic in x with period T. We consider here the case $\psi(x, \varepsilon) \equiv \bar{0}$. For more general ψ see Remark 1. We then assume that the following hypotheses hold.

H_1: $f(x, y, u, p, q, \varepsilon)$ is a real n-vector-valued function defined on $(-\infty, \infty) \times [0, y_0] \times E^n \times E^n \times E^n \times [0, \varepsilon_0)$, $y_0 > 0$, $\varepsilon_0 > 0$, which is continuous and periodic in x with period T.

H_2: $f(x, y, u, p, q, \varepsilon)$ satisfies a Lipschitz condition in u and is such that $\partial f/\partial p = (\partial f_i/\partial p_j)$, $\partial f/\partial q = (\partial f_i/\partial q_j)$ are continuous.

H_3: $\phi_0 \in E^n$ is such that $\int_0^T f(s, 0, 0, 0, \phi_0, 0)\, ds = \bar{0}$.

H_4: the matrix $D = (1/T)\int_0^T (\partial f(s, 0, 0, 0, \phi_0, 0)/\partial q)\, ds$ is nonsingular.

In addition to the solution u which satisfies the equation and the conditions above, the method described below will generate the $u_y(x, 0, \varepsilon)$ which gives rise to that solution.

The notation used by Lazer [10] is adopted here. Thus, $|z|$ will denote the usual Euclidean norm for $z \in E^n$. For a real $n \times n$ matrix $A = (\alpha_{ij})$, $|A|$ will denote $(\sum \alpha_{ij}^2)^{1/2}$. For an n-vector-valued function z defined on some set for which $|z|$ is bounded, $\|z\|$ will denote the supremum of $|z|$ taken over its domain of definition.

Finally we let

$$M(u, p, q, \varepsilon) = \frac{1}{T}\int_0^T f(x, y, u, p, q, \varepsilon)\, ds.$$

Now we state the main theorem of this section.

THEOREM 2.1. *Suppose*:

(i) *hypotheses* H_1–H_4 *hold with* $\psi(x, \varepsilon) \equiv 0$;

(ii) $p_0(x, y, \varepsilon) \equiv \bar{0}, \qquad q_0(x, y, \varepsilon) \equiv \phi_0 \equiv \phi_0(y, \varepsilon),$

$$p_{n+1}(x, y, \varepsilon) = \varepsilon \int_0^y f(x, \eta, u_n(x, \eta, \varepsilon), p_n(x, \eta, \varepsilon), q_n(x, \eta, \varepsilon), \varepsilon)\, d\eta,$$

$$q_{n+1}(x, y, \varepsilon) = \varepsilon \int_0^x [f(\xi, y, u_n(\xi, y, \varepsilon), p_n(\xi, y, \varepsilon), q_n(\xi, y, \varepsilon), \varepsilon) - M(u_n, p_n, q_n, \varepsilon)]\, d\xi + \phi_n(y, \varepsilon),$$

$$\phi_{n+1}(y, \varepsilon) = \phi_n(y, \varepsilon) - D^{-1}M(u_n, p_n, q_{n+1}, \varepsilon),$$

$$u_n(x, y, \varepsilon) = \int_0^y q_n(x, \eta, \varepsilon)\, d\eta.$$

Then there exists a $y' > 0$, $\varepsilon' > 0$ *such that the sequences* $\langle p_n \rangle$, $\langle q_n \rangle$, $\langle u_n \rangle$ *and* $\langle \phi_n \rangle$ *converge uniformly on* $(-\infty, \infty) \times [0, y'] \times [0, \varepsilon']$ *to functions p, q, u and* ϕ *with the following properties*:

$$u_{xy} = \varepsilon f(x, y, u, p, q, \varepsilon),$$

$$u(x + T, y, \varepsilon) = u(x, y, \varepsilon), \quad u(x, 0, \varepsilon) = \bar{0}, \quad u_x(x, y, \varepsilon) = p(x, y, \varepsilon),$$

$$u_y(x, y, \varepsilon) = q(x, y, \varepsilon), \quad u_y(0, y, \varepsilon) = \phi(y, \varepsilon), \quad u_y(x, 0, 0) = \phi_0 = \phi(0, 0).$$

Furthermore, there is an $r > 0$ *such that if* $\varepsilon \leqq \varepsilon'$ *is small enough to insure that* $|p(x, y, \varepsilon)| \leqq r/2$, $|q(x, y, \varepsilon) - \phi_0| \leqq r$ *and* $|\phi(y, \varepsilon) - \phi_0| \leqq r/2$ *for* $(x, y) \in (-\infty, \infty) \times [0, y']$, *then u, p, q and* ϕ *are unique.*

Proof. For $0 < y' \leqq y_0$, $0 < \varepsilon' \leqq \varepsilon_0$, let $C(y', \varepsilon')$ be the set of ordered triples (p, q, ϕ), where:

(i) $p = p(x, y, \varepsilon)$ is a continuous, n-vector-valued function defined on $(-\infty, \infty) \times [0, y'] \times [0, \varepsilon']$ which satisfies $p(x + T, y, \varepsilon) = p(x, y, \varepsilon)$ and $p(x, 0, \varepsilon) = \bar{0}$;

(ii) $q = q(x, y, \varepsilon)$ is a continuous, n-vector-valued function defined on $(-\infty, \infty) \times [0, y'] \times [0, \varepsilon']$ which satisfies $q(x + T, y, \varepsilon) = q(x, y, \varepsilon)$ and $q(x, 0, 0) = \phi_0$;

(iii) $\phi = \phi(y, \varepsilon)$ is a continuous, n-vector-valued function defined on $[0, y'] \times [0, \varepsilon']$ with $\phi(0, 0) = \phi_0$.

For $r > 0$, $S(r, y', \varepsilon')$ will denote the subset of elements (p, q, ϕ) of $C(y', \varepsilon')$ such that $\|p\| \leqq r/2$, $\|q - \phi_0\| \leqq r$, $\|\phi - \phi_0\| \leqq r/2$. If one introduces the metric d on $C(y', \varepsilon')$ defined by

$$d[(p_1, q_1, \phi_1), (p_2, q_2, \phi_2)] = \|p_1 - p_2\| + \tfrac{1}{2}\|q_1 - q_2\| + \|\phi_1 - \phi_2\|$$

for $(p_i, q_i, \phi_i) \in C(y', \varepsilon')$, $i = 1, 2$, then the space $(C(y', \varepsilon'), d)$ is a complete metric space with $(S(r, y', \varepsilon'), d)$ a closed, and thus complete subspace.

We now define a mapping F on $C(y', \varepsilon')$ which takes an element $(p, q, \phi) \in C(y', \varepsilon')$ into the element $F(p, q, \phi) = (p^*, q^*, \phi^*)$, defined by

$$(2.2) \quad p^*(x, y, \varepsilon) = \varepsilon \int_0^y f(x, \eta, u(x, \eta, \varepsilon), p(x, \eta, \varepsilon), q(x, \eta, \varepsilon), \varepsilon)\, d\eta,$$

$$(2.3) \quad q^*(x, y, \varepsilon) = \varepsilon \int_0^x [f(\xi, y, u(\xi, y, \varepsilon), p(\xi, y, \varepsilon), q(\xi, y, \varepsilon), \varepsilon) - M(u, p, q, \varepsilon)]\, d\xi + \phi(y, \varepsilon),$$

$$(2.4) \quad \phi^*(y, \varepsilon) = \phi(y, \varepsilon) - D^{-1} M(u, p, q^*, \varepsilon),$$

where

$$u(x, y, \varepsilon) = \int_0^y q(x, \eta, \varepsilon)\, d\eta.$$

It follows easily, using the properties of the elements of $C(y', \varepsilon')$ and the fact that a primitive of a periodic function with mean zero is periodic, that F maps $C(y', \varepsilon')$ into itself.

We wish to show that, for $r > 0$, $y' > 0$, $\varepsilon' > 0$ sufficiently small,

$$(2.5a) \qquad F[S(r, y', \varepsilon')] \subseteq S(r, y', \varepsilon'),$$

and if in addition $(p_1, q_1, \phi_1), (p_2, q_2, \phi_2) \in S(r, y', \varepsilon')$,

$$(2.5b) \qquad d[F(p_1, q_1, \phi_1), F(p_2, q_2, \phi_2)] \leqq \tfrac{2}{3} d[(p_1, q_1, \phi_1), (p_2, q_2, \phi_2)].$$

This in turn implies by the contraction mapping principle the existence of a unique element $(p, q, \phi) \in S(r, y', \varepsilon')$ which is a fixed point of the mapping F. It then follows from (2.2), (2.3), (2.4) that

$$u(x, y, \varepsilon) = \int_0^y q(x, \eta, \varepsilon)\, d\eta$$

is a solution of the problem being considered. Conversely, if there is a solution of the problem with $(p, q, \phi) \in S(r, y', \varepsilon')$, then it must be a fixed point of the map, F. In addition, since for $r > 0$, $y' > 0$, $\varepsilon' > 0$, the triple (p_0, q_0, ϕ_0), as defined in the statement of the theorem, is in $S(r, y', \varepsilon')$, and since we can write $(p_{n+1}, q_{n+1}, \phi_{n+1}) = F(p_n, q_n, \phi_n)$, our theorem will follow from (2.5a) and (2.5b).

With the choice of space $S(r, y', \varepsilon')$ and operator F which we have made, the proof of (2.5a) and (2.5b) proceeds in a manner analogous to that of Lazer. We first conclude from our assumptions on f that there are scalar functions $\omega_i(r, y', \varepsilon')$, $i = 1, 2, 3$, which are continuous and nondecreasing in r, y', ε' with $\omega_i(0, 0, 0) = 0$, such that

$$(2.6) \quad \left| \frac{\partial f}{\partial q}(x, 0, 0, 0, \phi_0, 0)(q_2 - q_1) - [f(x, y, u_1, p_1, q_1, \varepsilon) - f(x, y, u_2, p_2, q_2, \varepsilon)] \right| \leqq L\|u_1 - u_2\| + \omega_1(r, y', \varepsilon')|p_1 - p_2| + \omega_2(r, y', \varepsilon')|q_1 - q_2|,$$

$$|f(x, y, u_1, p_1, q_1, \varepsilon) - f(x, y, u_2, p_2, q_2, \varepsilon)| \leqq L\|u_1 - u_2\| + \omega_1(r, y', \varepsilon')|p_1 - p_2| + \omega_3(r, y', \varepsilon')|q_1 - q_2|, \tag{2.7}$$

where the inequalities hold for all $(u_1, p_1, q_1), (u_2, p_2, q_2) \in S(r, y', \varepsilon')$, $0 \leqq y \leqq y' \leqq y_0$, $0 \leqq \varepsilon \leqq \varepsilon' \leqq \varepsilon_0$. The existence of $\omega_2(r, y', \varepsilon')$ and $\omega_3(r, y', \varepsilon')$ follows from the periodicity of f and $\partial f_i/\partial q_j$, their continuity, the Schwarz inequality and the identity

$$f(x, y, u, p, q_1, \varepsilon) - f(x, y, u, p, q_2, \varepsilon) - \frac{\partial f}{\partial q}(x, 0, 0, 0, \phi_0, 0)(q_1 - q_2)$$
$$= \int_0^1 \left[\frac{\partial f}{\partial q}(x, y, u, p, q_1 + s(q_2 - q_1), \varepsilon) - \frac{\partial f}{\partial q}(x, 0, 0, 0, \phi_0, 0)\right] ds(q_1 - q_2).$$

The existence of $\omega_1(r, y', \varepsilon')$ follows similarly, where this time we use the periodicity and continuity of $\partial f_i/\partial p_j$. The constant L denotes the Lipschitz constant on f (see hypothesis H_2).

Let

$$H(r, y', \varepsilon') = \max |f(x, y, u, p, q, \varepsilon)|$$

for $(x, y, \varepsilon) \in (-\infty, \infty) \times [0, y'] \times [0, \varepsilon']$ and $|u| \leqq y'(r + |\phi_0|)$, $|p| \leqq r/2$, $|q - \phi_0| \leqq r$, and set $I(r, y', \varepsilon') = \max |\int_0^T f(x, y, u, p, \phi_0, \varepsilon)\, dx|$ for $(y, \varepsilon) \in [0, y'] \times (0, \varepsilon')$, and $|u| \leqq y'(r + |\phi_0|)$. From hypothesis H_3 we have that $I(r, y', \varepsilon')$ tends to zero as $\varepsilon' \to 0$, $y' \to 0$ and $r \to 0$. Furthermore, by the periodicity and continuity of $\partial f/\partial q$, we can conclude the existence of constants b_1 and b_2 such that for all $z \in E^n$, $|D^{-1}z| \leqq b_1|z|$, and $|D^{-1}(\partial f(x, 0, 0, 0, \phi_0, 0)/\partial q)z| \leqq b_2|z|$. We now choose $r > 0$, $\varepsilon' > 0$, $y' > 0$ so small that

$$\begin{aligned} &\max (2T, y')\varepsilon' H(r, y', \varepsilon') \leqq \min (r/2, r/(6b_2)), \\ &b_1\omega_2(r, y', \varepsilon') \leqq 1/6, \\ &\max \{2(y'L + \omega_3(r, y', \varepsilon')), \omega_1(r, y', \varepsilon')\}\alpha \leqq 2/3, \end{aligned} \tag{2.8}$$

where

$$\alpha = \tfrac{4}{3}\varepsilon' T + \varepsilon' y' + 2\varepsilon' T b_2 + b_1.$$

Now we fix r, y', ε' for the remainder of the proof.

Let $(p, q, \phi) \in S(r, y', \varepsilon')$. Then from (2.2) and (2.8) and the periodicity of p^* we have

$$\|p^*\| \leqq \varepsilon y' H(r, y', \varepsilon') \leqq r/2. \tag{2.9}$$

By (2.3),

$$q^*(x, y, \varepsilon) - \phi_0 = (\phi(y, \varepsilon) - \phi_0) + \varepsilon \int_0^x [f(\xi, y, u, (\xi, y, \varepsilon), p(\xi, y, \varepsilon), q(\xi, y, \varepsilon), \varepsilon) - M(u, p, q, \varepsilon)]\, d\xi,$$

so using the periodicity of q^* and (2.8), we obtain

$$\|q^* - \phi_0\| \leqq \|\phi - \phi_0\| + 2\varepsilon' T H(r, y', \varepsilon') \leqq r/2 + r/2 = r \tag{2.10}$$

and

$$\|q^* - \phi\| \leqq 2\varepsilon' T H(r, y', \varepsilon') \leqq r/2. \tag{2.11}$$

Finally from (2.4) we have

$$\begin{aligned}\phi^*(y, \varepsilon) - \phi_0 = (\phi(y, \varepsilon) - \phi_0) &- \frac{D^{-1}}{T}\int_0^T \frac{\partial f}{\partial q}(s, 0, 0, 0, \phi_0, 0)(\phi(y, \varepsilon) - \phi_0)\, ds \\ &- \frac{D^{-1}}{T}\int_0^T \frac{\partial f}{\partial q}(s, 0, 0, 0, \phi_0, 0)(q^*(s, y, \varepsilon) - \phi(y, \varepsilon))\, ds \\ &+ \frac{D^{-1}}{T}\int_0^T \left[\frac{\partial f}{\partial q}(s, 0, 0, 0, \phi_0, 0)(q^*(s, y, \varepsilon) - \phi_0)\right. \\ &\left. - \{f(s, y, u, p, q^*, \varepsilon) - f(s, y, u, p, \phi_0, \varepsilon)\}\right] ds \\ &- \frac{D^{-1}}{T}\int_0^T f(s, y, u, p, \phi_0, \varepsilon)\, ds.\end{aligned}$$

The first two terms on the right of the above equation cancel, leaving by virtue of (2.6), (2.8), (2.10) and (2.11),

$$\begin{aligned}\|\phi^* - \phi_0\| &\leqq b_2\|q^* - \phi\| + b_1\omega_2(r, y', \varepsilon')\|q^* - \phi_0\| + \frac{b_1}{T}I(r, y', \varepsilon') \\ &\leqq 2b_2\varepsilon' T H(r, y', \varepsilon') + b_1 r\omega_2(r, y', \varepsilon') + \frac{b_1}{T}I(r, y', \varepsilon') \\ &\leqq \frac{r}{6} + \frac{r}{6} + \frac{r}{6} = \frac{r}{2}.\end{aligned} \tag{2.12}$$

Thus we have shown $\|p^*\| \leqq r/2$, $\|q^* - \phi_0\| \leqq r$, $\|\phi^* - \phi_0\| \leqq r/2$. The above inequalities imply that $F[S(r, y', \varepsilon')] \subseteq S(r, y', \varepsilon')$.

To prove (2.5b), let (p_1, q_1, ϕ_1), $(p_2, q_2, \phi_2) \in S(r, y', \varepsilon')$. Then from the definition of u_i, from the periodicity of u_i, p_i^*, $i = 1, 2$, and (2.2), (2.7), we obtain

$$\begin{aligned}\|u_1 - u_2\| &\leqq y'\|q_1 - q_2\|, \\ \|p_1^* - p_2^*\| &\leqq \varepsilon' y'[(y'L + \omega_3(r, y', \varepsilon'))\|q_1 - q_2\| + \omega_1(r, y', \varepsilon')\|p_1 - p_2\|].\end{aligned} \tag{2.13}$$

From (2.3) and (2.7) and the fact that q_1^* and q_2^* are periodic it follows that

$$\begin{aligned}\|q_1^* - q_2^*\| \leqq \|\phi_1 - \phi_2\| + 2\varepsilon' T[(y'L + \omega_3(r, y', \varepsilon'))\|q_1 - q_2\| \\ + \omega_1(r, y', \varepsilon')\|p_1 - p_2\|]\end{aligned} \tag{2.14}$$

and that

$$\begin{aligned}\|(q_1^* - q_2^*) - (\phi_1 - \phi_2)\| \leqq 2\varepsilon' T[(y'L + \omega_3(r, y', \varepsilon'))\|q_1 - q_2\| \\ + \omega_1(r, y', \varepsilon')\|p_1 - p_2\|].\end{aligned} \tag{2.15}$$

Finally, from (2.4) we have

$$\phi_1^*(y,\varepsilon) - \phi_2^*(y,\varepsilon) = (\phi_1(y,\varepsilon) - \phi_2(y,\varepsilon))$$
$$-\frac{D^{-1}}{T}\int_0^T \frac{\partial f}{\partial q}(s,0,0,0,\phi_0,0)(\phi_1(y,\varepsilon) - \phi_2(y,\varepsilon))\,ds$$
$$-\frac{D^{-1}}{T}\int_0^T \frac{\partial f}{\partial q}(s,0,0,0,\phi_0,0)[q_1^*(s,y,\varepsilon) - q_2^*(s,y,\varepsilon) - (\phi_1(y,\varepsilon) - \phi_2(y,\varepsilon))]\,ds$$
$$+\frac{D^{-1}}{T}\int_0^T \left[\frac{\partial f}{\partial q}(s,0,0,0,\phi_0,0)(q_1^*(s,y,\varepsilon) - q_2^*(s,y,\varepsilon)) - (f(s,y,u_1,p_1,q_1,\varepsilon) - f(s,y,u_2,p_2,q_2,\varepsilon))\right] ds.$$

Noting that the first two terms on the right cancel, we see that

$$\|\phi_1^* - \phi_2^*\| \leqq b_2\|(q_1^* - q_2^*) - (\phi_1 - \phi_2)\| + b_1\omega_2(r,y',\varepsilon')\|q_1^* - q_2^*\| + b_1L\|u_1 - u_2\| + b_1\omega_1(r,y',\varepsilon)\|p_1 - p_2\|, \tag{2.16}$$

where we have made use of (2.6). By (2.14) and (2.15) and the above inequality, we have

$$\begin{aligned}\|\phi_1^* - \phi_2^*\| \leqq{}& [2\varepsilon' b_2 T\omega_1(r,y',\varepsilon') + 2\varepsilon' T b_1\omega_1(r,y',\varepsilon')\omega_2(r,y',\varepsilon') + b_1\omega_1(r,y',\varepsilon')]\|p_1 - p_2\|\\ &+[b_1y'L + 2\varepsilon'T(y'L + \omega_3(r,y',\varepsilon'))(b_2 + b_1\omega_2(r,y',\varepsilon'))]\|q_1 - q_2\|\\ &+b_1\omega_2(r,y',\varepsilon')\|\phi_1 - \phi_2\|.\end{aligned} \tag{2.17}$$

Thus, combining (2.13), (2.14) and (2.16), we get

$$\begin{aligned}d[F(p_1,q_1,\phi_1),F(p_2,q_2,\phi_2)] &= \tfrac{1}{2}\|q_1^* - q_2^*\| + \|p_1^* - p_2^*\| + \|\phi_1^* - \phi_2^*\|\\ &\leqq [b_1y'L + (y'L + \omega_3(r,y',\varepsilon'))\\ &\quad\cdot(\varepsilon'y' + \varepsilon'T + 2\varepsilon'Tb_2 + 2\varepsilon'Tb_1\omega_2(r,y',\varepsilon'))]\|q_1 - q_2\|\\ &\quad+[\varepsilon'T + \varepsilon'y' + 2\varepsilon'Tb_2\\ &\quad+2\varepsilon'b_1T\omega_2(r,y',\varepsilon') + b_1]\|p_1 - p_2\|\omega_1(r,y',\varepsilon')\\ &\quad+[\tfrac{1}{2} + b_1\omega_2(r,y',\varepsilon')]\|\phi_1 - \phi_2\|.\end{aligned}$$

Observing that $b_1\omega_3(r,y',\varepsilon') \geqq 0$ and recalling from (2.8) that $b_1\omega_2(r,y',\varepsilon') \leqq \frac{1}{6}$, we have

$$d[F(p_1,q_1,\phi_1),F(p_2,q_2,\phi_2)] < \alpha(y'L + \omega_3(r,y',\varepsilon'))\|q_1 - q_2\| + \alpha\omega_1(r,y',\varepsilon')\|p_1 - p_2\| + \tfrac{2}{3}\|\phi_1 - \phi_2\|,$$

where

$$\alpha = \tfrac{4}{3}\varepsilon'T + \varepsilon'y' + 2\varepsilon'Tb_2 + b_1.$$

From (2.8) we then conclude

$$d[F(p_1, q_1, \phi_1), F(p_2, q_2, \phi_2)] \leqq \tfrac{2}{3}[\tfrac{1}{2}\|q_1 - q_2\| + \|p_1 - p_2\| + \|\phi_1 - \phi_2\|].$$

Hence, (2.5b) holds, and the proof of Theorem 2.1 is complete.

Remark 1. We note that if $\psi(x, \varepsilon) \neq 0$, the change of variable $v(x, y, \varepsilon) = u(x, y, \varepsilon) - \psi(x, \varepsilon)$ leads to the problem

$$v_{xy} = f(x, y, v + \psi, v_x + \psi_x, v_y, \varepsilon) = \varepsilon H(x, y, v, v_x, v_y, \varepsilon),$$
$$v(x, 0, \varepsilon) = \bar{0}, \qquad v_y(x, 0, 0) = \phi_0,$$
$$v(x + T, y, \varepsilon) = v(x, y, \varepsilon).$$

The conditions H_1, H_2 imposed on f can be shown to carry over to H, and so if H also satisfies H_3, H_4, the solution for nonzero ψ can be obtained by using Theorem 1 and the transformation above.

It should also be observed that by a slight modification of the proof we can replace (2.5b) by

$$\text{(2.5b}') \qquad d[F(p_1, q_1, \phi_1), F(p_2, q_2, \phi_2)] \leqq kd[(p_1, q_1, \phi_1), (p_2, q_2, \phi_2)]$$

for (p_1, q_1, ϕ_1), $(p_2, q_2, \phi_2) \in S(r, y', \varepsilon')$, where k is any number with $0 < k < 1$. Of course, for different k we shall have to choose r, y', ε' differently. In general, however, since the initial approximation $(p_0, q_0, \phi_0) \in S(r, y', \varepsilon')$ and $(p_{n+1}, q_{n+1}, \phi_{n+1}) = F(p_n, q_n, \phi_n)$ we can use the estimate (2.5b′) to obtain the error bound

$$d[(p_n, q_n, \phi_n), (p, q, \phi)] \leqq \frac{k^n}{1 - k} d[(p_1, q_1, \phi_1), (p_0, q_0, \phi_0)],$$

where (p_n, q_n, ϕ_n) is the nth iterate and (p, q, ϕ) is the fixed point which gives rise to the solution. From the above inequality we have

$$\|p_n - p\|, \|\phi_n - \phi\| \leqq \frac{k^n}{1 - k}\{\tfrac{1}{2}\|q_1 - \phi_0\| + \|p_1\| + \|\phi_1 - \phi_0\|\},$$
$$\|q_n - q\| \leqq \frac{2k^n}{1 - k}\{\tfrac{1}{2}\|q_1 - \phi_0\| + \|p_1\| + \|\phi_1 - \phi_0\|\},$$

which in turn gives us the estimate

$$\|u_n - u\| \leqq 2y' \frac{k^n}{1 - k}\{\tfrac{1}{2}\|q_1 - \phi_0\| + \|p_1\| + \|\phi_1 - \phi_0\|\}.$$

Remark 2. The b_1 which appears in Theorem 2.1 is merely $\max_{i,j} |D_{ij}^{-1}|$ and hence can be determined. We can then get the same result as in Theorem 2.1 if instead of continuous differentiability with respect to p we assume that f satisfies a Lipschitz condition with respect to p and that $b_1 L < \frac{1}{6}$, where L is the Lipschitz constant (the choice of $\frac{1}{6}$ here corresponds to the choice $k = \frac{2}{3}$ in (2.5b′)).

Remark 3. The condition $u_y(x, 0, 0) = \phi_0$ means that we are considering problems (1.1) with solutions of the form $u(x, y, \varepsilon) = h(y, \varepsilon) + g(x, y, \varepsilon)$. Here $h(y, \varepsilon)$ is an n-vector-valued function continuous on $(-\infty, \infty) \times [0, \varepsilon']$ and continuously differentiable with respect to y, with $h(0, \varepsilon) = \bar{0}$, $h_y(0, 0) = \phi_0$. $g(x, y, \varepsilon)$ is an n-vector-valued function, continuous on $(-\infty, \infty) \times (-\infty, \infty) \times [0, \varepsilon']$ and continuously differentiable on $(-\infty, \infty) \times (-\infty, \infty)$ with $g(x + T, y, \varepsilon) = g(x, y, \varepsilon)$, $g(x, 0, \varepsilon) = \bar{0}$ and $g_y(x, y, \varepsilon) = \bar{0}$.

$$u(x, y, \varepsilon) = \psi(x, \varepsilon) + h(y, \varepsilon) + g(x, y, \varepsilon),$$

where ψ is given and h and g are as above and are determined so as to yield a solution of the type desired.

If $f(x, y, u, \varepsilon)$ is continuous on $(-\infty, \infty) \times [0, y_0] \times E^n \times [0, \varepsilon_0]$, continuously differentiable with respect to u, and satisfies $f(x + T, y, u, \varepsilon) \equiv f(x, y, u, \varepsilon)$, we can use a similar technique to find periodic solutions to

$$\begin{aligned} &u_{xy} = \varepsilon f(x, y, u, \varepsilon), \\ &u(x, 0, \varepsilon) = \psi(x, \varepsilon), \qquad \psi(x + T, \varepsilon) \equiv \psi(x, \varepsilon), \\ &u(x + T, y, \varepsilon) \equiv u(x, y, \varepsilon). \end{aligned}$$

As before we consider the problem with $\psi(x, \varepsilon) = \bar{0}$.

THEOREM 2.2. *Suppose*

$$\int_0^T f(x, 0, 0, 0)\, ds = \bar{0}$$

and

$$D \equiv \int_0^T \frac{\partial f}{\partial u}(x, 0, 0, 0)\, ds$$

is nonsingular. Then, if

$$u_0(x, y, \varepsilon) \equiv \bar{0}, \qquad \phi_0(y, \varepsilon) \equiv \bar{0}$$

and

$$u_{n+1}(x, y, \varepsilon) = \varepsilon \int_0^y \int_0^x [f(s, t, u_n(s, t, \varepsilon), \varepsilon) - M(u_n, \varepsilon)]\, ds\, dt + \phi_n(y, \varepsilon),$$

$$\phi_{n+1}(x, y, \varepsilon) = \phi_n(y, \varepsilon) - D^{-1} M(u_{n+1}, \varepsilon),$$

where

$$M(u, \varepsilon) = \frac{1}{T} \int_0^T f(s, y, u, (s, y, \varepsilon), \varepsilon)\, ds,$$

there are $y' > 0$, $\varepsilon' > 0$ such that the sequences $\langle u_n \rangle$, $\langle \phi_n \rangle$ converge uniformly on $(-\infty, \infty) \times [0, y'] \times [0, \varepsilon']$ to functions u, such that

$$\begin{aligned} &u_{xy} = \varepsilon f(x, y, u, \varepsilon), \\ &u(x + T, y, \varepsilon) \equiv u(x, y, \varepsilon), \qquad u(x, 0, \varepsilon) \equiv \bar{0}, \\ &u(0, y, \varepsilon) = \phi(y, \varepsilon), \qquad \phi(0, 0) = \bar{0}. \end{aligned}$$

Furthermore, there is an $r > 0$ such that if $\varepsilon < \varepsilon'$ is small enough so as to insure that $|u(x, y, \varepsilon)| \leqq r/2$, $|\phi(y, \varepsilon)| \leqq r/4$, then u and ϕ are unique.

As the proof of this theorem is so similar to that of Theorem 2.1, it is omitted.

3. Periodic solutions in the large. We now take up the problem of finding solutions of (1.1) which are periodic in the large. We seek, in particular, solutions of the problem

$$u_{xy} = \varepsilon f(x, y, u, u_x, u_y, \varepsilon), \tag{3.1}$$

for $-\infty < x, y < \infty$, satisfying the conditions

$$u(x, y, 0) = \theta_1(x) + \theta_2(y), \qquad u(x + T, y, \varepsilon) = u(x, y, \varepsilon) = u(x, y + T, \varepsilon)$$

for $-\infty < x, y < \infty$ and $0 \leqq \varepsilon \leqq \varepsilon_0$, where $u = \operatorname{col}(u_1, \cdots, u_n), f = \operatorname{col}(f_1, \cdots, f_n)$, $\theta_1 = (\theta_1^1, \cdots, \theta_1^n)$, $\theta_2 = \operatorname{col}(\theta_2^1, \cdots, \theta_2^n)$, and ε is a small parameter. $\theta_1(x) = \theta_1(x + T)$, $\theta_2(y) = \theta_2(y + T)$ are to be continuously differentiable. We shall consider the case $\theta_1(x) = \theta_2(y) = 0$, the nonzero case being handled by a transformation similar to that employed in § 2. We assume that the following hypotheses hold.

L_1: $f(x, y, u, p, q, \varepsilon)$ is a real n-vector-valued function defined on $E \times E \times E^n \times E^n \times E^n \times [0, \varepsilon_0]$, $\varepsilon_0 > 0$, which is continuous, continuously differentiable with respect to u, p, q and periodic in x and y with period T.

L_2:
$$\int_0^T f(s, 0, 0, 0, 0, 0)\, ds = \bar{0},$$
$$\int_0^T f(0, t, 0, 0, 0, 0)\, dt = \bar{0},$$
$$\int_0^T \int_0^T f(s, t, 0, 0, 0, 0)\, ds\, dt = \bar{0}.$$

L_3: The matrices

$$D_1 \equiv \frac{1}{T}\int_0^T \frac{\partial f}{\partial p}(0, t, 0, 0, 0, 0)\, dt,$$
$$D_2 \equiv \frac{1}{T}\int_0^T \frac{\partial f}{\partial q}(s, 0, 0, 0, 0, 0)\, ds$$

and

$$D_3 \equiv \frac{1}{T^2}\int_0^T \int_0^T \frac{\partial f}{\partial u}(s, t, 0, 0, 0, 0)\, ds\, dt$$

are nonsingular.

Introducing the notation

$$M_1(u, p, q, \varepsilon) = \frac{1}{T}\int_0^T f(x, t, u, p, q, \varepsilon)\, dt,$$
$$M_2(u, p, q, \varepsilon) = \frac{1}{T}\int_0^T f(s, y, u, p, q, \varepsilon)\, ds,$$
$$M_3(u, p, q, \varepsilon) = \frac{1}{T^2}\int_0^T \int_0^T f(s, t, u, p, q, \varepsilon)\, ds\, dt,$$

we then have the following theorem.

THEOREM 3.1. *Suppose*:

(i) *hypotheses* L_1–L_3 *hold with* $\theta_1(x) = \theta_2(y) = \bar{0}$;

(ii) $p_0(x, y, \varepsilon) = \bar{0} = \phi_0(x, \varepsilon)$, $q_0(x, y, \varepsilon) = \bar{0} = \psi_0(y, \varepsilon)$, $q_0(\varepsilon) = \bar{0}$,

$u_0(x, y, \varepsilon) = \bar{0}$,

$$p_{n+1}(x, y, \varepsilon) = \varepsilon \int_0^y [f(x, t, u_n, p_n, q_n, \varepsilon) - M_1(u_n, p_n, q_n, \varepsilon)]\, dt + \phi_n(x, \varepsilon),$$

$$q_{n+1}(x, y, \varepsilon) = \varepsilon \int_0^x [f(s, y, u_n, p_n, q_n, \varepsilon) - M_2(u_n, p_n, q_n, \varepsilon)]\, ds + \psi_n(y, \varepsilon),$$

$$\phi_{n+1}(x, \varepsilon) = \phi_n(x, \varepsilon) - D_1^{-1} M_1(u_n, p_{n+1}, q_n, \varepsilon),$$

$$\psi_{n+1}(y, \varepsilon) = \psi_n(y, \varepsilon) - D_2^{-1} M_2(u_n, p_n, q_{n+1}, \varepsilon),$$

$$a_{n+1}(\varepsilon) = a_n(\varepsilon) - D_3^{-1} M_3(u_{n+1}, p_n, q_n, \varepsilon),$$

$$u_{n+1}(x, y, \varepsilon) = \int_0^x \left[p_{n+1}(x, y, \varepsilon) - \frac{1}{T} \int_0^T p_{n+1}(\xi, y, \varepsilon)\, d\xi \right] ds$$
$$+ \int_0^y \left[q_{n+1}(0, t, \varepsilon) - \frac{1}{T} \int_0^T q_{n+1}(0, \eta, \varepsilon)\, d\eta \right] dt + a_n(\varepsilon).$$

Then there is an $\varepsilon' > 0$ *such that the sequences* $\langle p_n \rangle$, $\langle q_n \rangle$, $\langle u_n \rangle$, $\langle \phi_n \rangle$, $\langle \psi_n \rangle$, $\langle a_n \rangle$ *converge uniformly on their respective domains to functions* p, q, u, ϕ, ψ, a *such that*

$$u_{xy} = \varepsilon f(x, y, u, p, q, \varepsilon),$$

$$u(x + T, y, \varepsilon) \equiv u(x, y, \varepsilon) \equiv u(x, y + T, \varepsilon),$$

$$u_x(x, y, \varepsilon) = p(x, y, \varepsilon), \qquad u_y(x, y, \varepsilon) \equiv q(x, y, \varepsilon),$$

$$u_x(x, 0, \varepsilon) = \phi(x, \varepsilon), \qquad u_y(0, x, \varepsilon) = \psi(x, \varepsilon),$$

$$u(0, 0, \varepsilon) = a(\varepsilon).$$

Furthermore, there is an $r > 0$ *such that if* $\varepsilon \leqq \varepsilon'$ *is small enough to insure that* $|p(x, y, \varepsilon)| \leqq r/4$, $|q(x, y, \varepsilon)| \leqq r/4$, $|\phi(x, \varepsilon)| \leqq r/6$, $|\psi(y, \varepsilon)| \leqq r/6$, $|a(\varepsilon)| \leqq r/6$ *for* $-\infty < x, y < \infty$, *then* u, p, q, ϕ, ψ, *and* a *are unique.*

Proof. For $0 \leqq \varepsilon' \leqq \varepsilon_0$, $r > 0$, define $S(r, \varepsilon')$ to be the set of elements (p, q, ϕ, ψ, a) where:

(i) $p = p(x, y, \varepsilon)$ is an n-vector-valued function defined and continuous on $(-\infty, \infty) \times (-\infty, \infty) \times [0, \varepsilon']$ with $p(x + T, y, \varepsilon) \equiv p(x, y, \varepsilon) \equiv p(x, y + T, \varepsilon)$, $p(x, y, 0) = \bar{0}$ and $\|p\| \leqq r/4$;

(ii) $q = q(x, y, \varepsilon)$ is an n-vector-valued function defined and continuous on $(-\infty, \infty) \times (-\infty, \infty) \times [0, \varepsilon']$ with $q(x + T, y, \varepsilon) \equiv q(x, y, \varepsilon) \equiv q(x, y + T, \varepsilon)$, $q(x, y, 0) = \bar{0}$, and $\|q\| \leqq r/6$;

(iii) $\phi = \phi(x, \varepsilon)$ is an n-vector-valued function defined and continuous on $(-\infty, \infty) \times [0, \varepsilon']$ with $\phi(x + T, \varepsilon) \equiv \phi(x, \varepsilon)$, $\phi(x, 0) = \bar{0}$, and $\|\phi\| \leqq r/6$;

(iv) $\psi = \psi(y, \varepsilon)$ is an n-vector-valued function defined and continuous on $(-\infty, \infty) \times [0, \varepsilon']$ with $\psi(y + T, \varepsilon) \equiv \psi(y, \varepsilon)$, $\psi(y, 0) = \bar{0}$, and $\|\psi\| \leqq r/6$;

(v) $a = a(\varepsilon)$ is an n-vector-valued function defined and continuous on $[0, \varepsilon']$ with $a(0) = 0$, and $\|a\| \leqq r/6$.

For $(p_1, q_1, \phi_1, \psi_1, a_1), (p_2, q_2, \phi_2, \psi_2, a_2) \in S(r, \varepsilon')$ we define

$$d[(p_1, q_1, \phi_1, \psi_1, a_1), (p_2, q_2, \phi_2, \psi_2, a_2)] = \tfrac{1}{2}\|p_1 - p_2\| + \tfrac{1}{2}\|q_1 - q_2\| + \|\phi_1 - \phi_2\| + \|\psi_1 - \psi_2\| + \|a_1 - a_2\|.$$

Then $(S(r, \varepsilon'), d)$ is a complete metric space.

Now we define for $(p, q, \phi, \psi, a) \in S(r, \varepsilon')$ the mapping $G(p, q, \phi, \psi, a) = (p^*, q^*, \phi^*, \psi^*, a^*)$ by

$$p^*(x, y, \varepsilon) = \varepsilon \int_0^y [f(x, t, u(x, t, \varepsilon), p(x, t, \varepsilon), q(x, t, \varepsilon), \varepsilon) - M_1(u, p, q, \varepsilon)]\, dt + \phi(x, \varepsilon),$$

$$q^*(x, y, \varepsilon) = \varepsilon \int_0^x [f(s, y, u(s, y, \varepsilon), p(s, y, \varepsilon), q(s, y, \varepsilon), \varepsilon) - M_2(u, p, q, \varepsilon)]\, ds + \psi(y, \varepsilon),$$

$$\phi^*(x, \varepsilon) = \phi(x, \varepsilon) - D_1^{-1} M_1(u, p^*, q, \varepsilon),$$

$$\psi^*(y, \varepsilon) = \psi(y, \varepsilon) - D_2^{-1} M_2(u, p, q^*, \varepsilon),$$

$$a^*(\varepsilon) = a(\varepsilon) - D_3^{-1} M_3(u^*, p, q, \varepsilon),$$

where

$$u^*(x, y, \varepsilon) = \int_0^x \left[p^*(s, y, \varepsilon) - \frac{1}{T}\int_0^T p^*(\xi, y, \varepsilon)\, d\xi \right] ds + \int_0^y \left[q^*(0, t, \varepsilon) - \frac{1}{T}\int_0^T q^*(0, \eta, \varepsilon)\, d\eta \right] dt + a(\varepsilon).$$

Now for $0 < k < 1$, one can show in a manner similar to that of Theorem 2.1 that for suitable $r > 0$ and ε', G is well-defined, $G[S(r, \varepsilon')] \subseteq S(r, \varepsilon')$, and for

$$(p_1, q_1, \phi_1, \psi_1, a_1), (p_2, q_2, \phi_2, \psi_2, a_2) \in S(r, \varepsilon'),$$

$$d[G(p_1, q_1, \phi_1, \psi_1, a_1), G(p_2, q_2, \phi_2, \psi_2, a_2)] \leqq kd[(p_1, q_1, \phi_1, \psi_1, a_1), (p_2, q_2, \phi_2, \psi_2, a_2)],$$

and the theorem will follow by the contraction mapping principle.

Furthermore, since $(p_0, q_0, \phi_0, \psi_0, a_0) \in S(r, \varepsilon')$ for any $r > 0$, $\varepsilon' > 0$, we obtain again

$$\|p_n - p\|, \|q_n - q\| \leqq \frac{2k^n}{1 - k} d[(p_1, q_1, \phi_1, \psi_1, a_1), (p_0, q_0, \phi_0, \psi_0, a_0)],$$

$$\|\phi_n - \phi\|, \|\psi_n - \psi\|, \|a_n - a\|$$
$$\leqq \frac{k^n}{1-k} d[(p_1, q_1, \phi_1, \psi_1, a_1), (p_0, q_0, \phi_0, \psi_0, a_0)],$$
$$\|u_n - u\| \leqq \frac{k^{n-1}}{1-k}(8Tk + 1)\, d[(p_1, q_1, \phi_1, \psi_1, a_1), (p_0, q_0, \phi_0, \psi_0, a_0)],$$

or more precisely,

$$\|u_n - u\| \leqq \frac{k^{n-1}}{1-k}(8Tk + 1)[\tfrac{1}{2}\|p_1\| + \tfrac{1}{2}\|q_1\| + \|\phi_1\| + \|\psi_1\| + \|a_1\|].$$

We conclude this paper with examples.

(We take $\theta_1(x) = \theta_2(y) = \bar{0}$ throughout.)

Example 1. Consider the system (3.1) with

$$f(x, y, u, p, q, \varepsilon) = \psi_1(x, y)u + \psi_2(y)p + \psi_3(x)q + \varepsilon g(x, y, u, p, q, \varepsilon),$$

where $\psi_1(x, y)$, $\psi_2(y)$, $\psi_3(x)$ are continuous and periodic in both x and y with the same period T and g' satisfies the assumptions in L_1 on f. Suppose, further,

$$\int_0^T \int_0^T \psi_1(x, y)\, dx\, dy, \quad \int_0^T \psi_2(y)\, dy, \quad \int_0^T \psi_3(x)\, dx$$

are nonzero. It is easily shown that under these assumptions the hypotheses of Theorem 3.1 are satisfied, and we conclude the existence of a unique T-periodic solution to (3.1) with f given as above from Theorem 3.1.

Example 2.

$$f(x, y, u, p, q, \varepsilon) = u \cos u + \psi_1(y)p + \psi_2(x)q + \varepsilon g(x, y, u, p, q, \varepsilon),$$

where $\psi_1(y)$, $\psi_2(x)$ are 2π-periodic and are continuous and g is 2π-periodic and satisfies the hypothesis assumed on f in Theorem 3.1. Further,

$$\int_0^{2\pi} \psi_1(y)\, dy, \quad \int_0^{2\pi} \psi_2(x)\, dx$$

are nonzero. It is again directly verifiable that the hypotheses of Theorem 3.1 are satisfied, and we conclude that there is a unique 2π-periodic solution to (3.1) with f as above.

REFERENCES

[1] A. K. Aziz, *Periodic solutions of hyperbolic partial differential equations*, Proc. Amer. Math. Soc., 17 (1966), pp. 557–566.

[2] A. K. Aziz and A. M. Meyers, *Periodic solutions of hyperbolic partial differential equations in a strip*, Trans. Amer. Math. Soc., 146 (1969), pp. 167–178.

[3] A. K. Aziz and M. Horak, *Periodic solutions of hyperbolic partial differential equations in the large*, this Journal, to appear.

[4] L. Cesari, *Periodic solutions of hyperbolic partial differential equations*, Internat. Sympos. on Nonlinear Differential Equations and Nonlinear Mechanics (Colorado Springs, 1961), Academic Press, New York, 1963, pp. 33–57; also Proc. Internat. Sympos. Nonlinear Vibrations, vol. II (Kiev, 1961), Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1963, pp. 440–457.

[5] ———, *A criterion for the existence in a strip of periodic solutions of hyperbolic partial differential equations*, Rend. Cir. Mat. Palermo (2), 14 (1965), pp. 1–24.

[6] L. CESARI, *Existence in the large of periodic solutions of hyperbolic partial differential equations*, Arch. Rational Mech. Anal., 20 (1965), pp. 170–190.

[7] ———, *Sulla stabilità delle soluzioni dei sistemi di equazioni differenziali lineari a coefficienti periodici*, Atti. Accad. Italia Mem. Cl. Sci. Fis. Mat. Nat. (6), 11 (1950), pp. 333–395.

[8] J. HALE, *Periodic solutions of a class of hyperbolic equations containing a small parameter*, Arch. Rational Mech. Anal., 23 (1966), pp. 380–398.

[9] ———, *Oscillations in Nonlinear Systems*, McGraw-Hill, New York, 1963.

[10] A. C. LAZER, *On the computation of periodic solutions of weakly nonlinear differential equations*, SIAM J. Appl. Math., 15 (1967), pp. 1158–1170.

[11] P. H. RABINOWITZ, *Periodic solutions of nonlinear hyperbolic partial differential equations*, Comm. Pure Appl. Math., 20 (1967), pp. 145–205.

[12] ———, *Periodic solutions of nonlinear hyperbolic partial differential equations. II*, Ibid., 22 (1969), pp. 15–39.

[13] O. VEJVODA, *Nonlinear boundary problems for differential equations*, Differential Equations and Their Applications, Czechoslovakian Academy of Science, Prague, 1963, pp. 199–215.

[14] ———, *Periodic solutions of a linear and weakly nonlinear wave equation in one dimension*, Czechoslovak Math. J., 14 (1964), pp. 341–382.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

To my teacher D. V. Widder

INTEGRAL EQUATIONS ON CERTAIN COMPACT HOMOGENEOUS SPACES*

I. I. HIRSCHMAN, JR.†

Abstract. Let $\mathbf{T}_n = \mathbf{R}_n/\mathbf{Z}_n$, where $\mathbf{R}_n$ is n-dimensional Euclidean space and $\mathbf{Z}_n$ is the n-dimensional lattice group, be the n-dimensional torus. We denote elements of $\mathbf{T}_n$ by $\boldsymbol{\theta}, \boldsymbol{\varphi}$, etc. Let $d\boldsymbol{\varphi}$ be Haar measure on $\mathbf{T}_n$ so normalized that $\mathbf{T}_n$ has measure 1. Let $V(\boldsymbol{\theta}) \in L^\infty(\mathbf{T}_n)$, and let $L(\boldsymbol{\theta}) \in L^1(\mathbf{T}_n)$ be such that $L\hat{}(\mathbf{k}) \geqq 0, \mathbf{k} \in \mathbf{Z}_n$, where

$$L\hat{}(\mathbf{k}) = \int_{\mathbf{T}_n} L(\boldsymbol{\theta})\, e^{-i\mathbf{k}\cdot\boldsymbol{\theta}}\, d\boldsymbol{\theta}, \qquad \mathbf{k}\cdot\boldsymbol{\theta} = k_1\theta_1 + \cdots + k_n\theta_n.$$

Following earlier work of Kac who used probabilistic methods, Rosenblatt and Widom used classical methods to study the asymptotic distribution of the eigenvalues of the nonnegative integral operator

$$Sf\cdot(\boldsymbol{\theta}) = \int_{\mathbf{T}_n} \overline{V(\boldsymbol{\theta})} L(\boldsymbol{\theta} - \boldsymbol{\varphi}) V(\boldsymbol{\varphi}) f(\boldsymbol{\varphi})\, d\boldsymbol{\varphi}, \qquad f \in L^2(\mathbf{T}_n).$$

Let G be a compact Hausdorff group with elements x, y etc., and let K be a closed subgroup of G. In the present paper we show that results like those of Kac, Rosenblatt and Widom hold with $\mathbf{T}_n$ replaced by the compact homogeneous space G/K provided the following condition is satisfied. There exists a closed subgroup N of G contained in the normalizer of K such that if $L^1[G, K, N]$ consists of all functions $f(x)$ in $L^1[G]$ such that $f(k_1xk_2) = f(x)$ in $L^1[G]$ for all $k_1, k_2 \in K$ and such that $f(nxn^{-1}) = f(x)$ in $L^1[G]$ for all $n \in N$, then $L^1[G, K, N]$ is a commutative subalgebra of $L^1[G]$.

1. Introduction. In two extremely interesting studies of the connections between certain stochastic processes and the integral operators associated with them, Kac obtained (among many other results) an asymptotic formula for the eigenvalues of the operators

$$Tf\cdot(\mathbf{x}) = \int_{\mathbf{R}_n} V(\mathbf{x})|\mathbf{x} - \mathbf{y}|^{-\alpha} V(\mathbf{y}) f(\mathbf{y})\, d\mathbf{y}, \qquad 0 < \alpha < n.$$

See [4] and [5]. Here $\mathbf{R}_n$ is n-dimensional Euclidean space and $V(\mathbf{x})$ is a bounded nonnegative function on $\mathbf{R}_n$ with bounded support. For $\alpha = n - 2$ Kac's results are closely related to classical estimates due to Weyl for the eigenvalues of operators like the Laplace operator; see [2, Chap. VI]. The fact that Kac's results also held for other values of α suggested that such formulas might be valid in a rather general context. Subsequently Rosenblatt [6] and Widom [8] carried out studies of more general operators like T, using classical methods. The central result of Widom's elegant theory is as follows. Let $\mathbf{T}_n = \mathbf{R}_n/\mathbf{Z}_n$ be the n-dimensional torus, let $\boldsymbol{\theta}$ and $\boldsymbol{\varphi}$ be elements in $\mathbf{T}_n$, and let $d\boldsymbol{\varphi}$ be Haar measure on $\mathbf{T}_n$, normalized so that the measure of $\mathbf{T}_n$ is 1. For $V \in L^\infty[\mathbf{T}_n]$ set

$$M_V f\cdot(\boldsymbol{\theta}) = V(\boldsymbol{\theta}) f(\boldsymbol{\theta}).$$

Clearly M_V is a bounded linear transformation on $L^2[\mathbf{T}_n]$. Let $L\hat{}(\mathbf{k})$ be any nonnegative function on $\mathbf{Z}_n$ such that $L\hat{}(\mathbf{k}) \to 0$ as $\mathbf{k} \to \infty$. We *do not assume* that

* Received by the editors March 29, 1971.
† Department of Mathematics, Washington University, St. Louis, Missouri 63130.

$L^{\wedge}(\mathbf{k})$ is the Fourier transform of a function in $L^1[\mathbf{T}_n]$. The formula

$$R_{L^{\wedge}} f \cdot (\boldsymbol{\theta}) = \sum_{\mathbf{Z}_n} f^{\wedge}(\mathbf{k}) L^{\wedge}(\mathbf{k})\, e^{i\mathbf{k}\cdot\boldsymbol{\theta}}, \qquad f \in L^2[\mathbf{T}_n],$$

where $\mathbf{k} = (k_1, \cdots, k_n)$, $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n)$, $\mathbf{k} \cdot \boldsymbol{\theta} = k_1\theta_1 + \cdots + k_n\theta_n$, defines $R_{L^{\wedge}}$ as a nonnegative completely continuous operator on $L^2[\mathbf{T}_n]$. If

$$S = M_V^* R_{L^{\wedge}} M_V, \tag{1.1}$$

then S is a nonnegative completely continuous operator on $L^2[\mathbf{T}_n]$. Widom showed that if $L^{\wedge}(\mathbf{k})$ behaves somewhat regularly as $k \to \infty$, then (essentially)

$$N^+[\varepsilon, S] \sim |\{(\boldsymbol{\theta}, \mathbf{k}) : |V(\boldsymbol{\theta})|^2 L^{\wedge}(\mathbf{k}) > \varepsilon\}|_{\mathbf{T}_n \times \mathbf{Z}_n}, \tag{1.2}$$

where $N^+[\varepsilon, S]$ is the number of eigenvalues of S greater than ε, and $|\{\cdot\}|_{\mathbf{T}_n \times \mathbf{Z}_n}$ is the measure of the set $\{\cdot\}$ in the product measure space $\mathbf{T}_n \times \mathbf{Z}_n$. *If* it should happen that

$$L^{\wedge}(\mathbf{k}) = \int_{\mathbf{T}_n} L(\boldsymbol{\theta})\, e^{-i\mathbf{k}\cdot\boldsymbol{\theta}}\, d\boldsymbol{\theta}$$

for $L(\boldsymbol{\theta}) \in L^1[\mathbf{T}_n]$, then

$$Sf \cdot (\boldsymbol{\theta}) = \int_{\mathbf{T}_n} \overline{V(\boldsymbol{\theta})} L(\boldsymbol{\theta} - \boldsymbol{\varphi}) V(\boldsymbol{\varphi}) f(\boldsymbol{\varphi})\, d\boldsymbol{\varphi}. \tag{1.3}$$

Let G be a compact Hausdorff group with elements x, y, etc. The identity of G is e, and we denote by dx, dy, etc. Haar measure on G, normalized so that the measure of G itself is 1. Let K be a closed subgroup of G. Our objective in this paper is to show that, in principle, a result like Widom's holds for the homogeneous space G/K provided the following condition is satisfied: there exists a closed subgroup N of G contained in the normalizer of K in G, such that if $L^1[G, K, N]$ is the set of functions $f(x)$ in $L^1[G]$ which satisfy $f(kxk') = f(x)$ in $L^1[G]$ for all $k, k' \in K$ and $f(n^{-1}xn) = f(x)$ in $L^1[G]$ for all $n \in N$, then $L^1[G, K, N]$ is a *commutative* subalgebra of $L^1[G]$. If N_1 and N_2 are closed subgroups of the normalizer of K in G such that $L^1[G, K, N_1]$ and $L^1[G, K, N_2]$ are commutative, and if these algebras are distinct, then each gives rise to an analogue of Widom's formula. The qualification "in principle" is due to the fact that, to carry out our program in full, it is necessary to know a great deal about the irreducible unitary representations of G, information which is, more often than not, unavailable. Actually we shall study operators acting on K-right invariant functions on G, rather than operators acting on functions on G/K. However, it is all the same thing. Let $L^2[G/K]$ ($L^\infty[G/K]$) consist of all functions $f(x)$ in $L^2[G]$ ($L^\infty[G]$) with the property that $f(xk) = f(x)$ in $L^2[G]$ ($L^\infty[G]$) for all $k \in K$. Let S be the integral operator on the Hilbert space $L^2[G/K]$ defined by

$$Sf \cdot (x) = \int_G \overline{V(x)} L(y^{-1}x) V(y) f(y)\, dy, \qquad f \in L^2[G/K],$$

where:

(a) $V(x) \in L^\infty[G/K]$;

(b) $L(x) \in L^1[G, K, N]$ and the map $f \to f * L$ of the Hilbert space $L^2[G/K]$ into itself is nonnegative.

We shall show that if certain regularity conditions on the Fourier transform of L are satisfied, then a suitable analogue of Widom's formula (1.2) is valid. Moreover this result remains valid if S is defined by a formula analogous to (1.1) rather than, as above, by a formula analogous to (1.3).

2. Harmonic analysis. The assumption that $L^1[G, K, N]$ is commutative enters into the demonstration of the analogue of Widom's formula only very indirectly. This section is devoted to extracting the results we need from this assumption. Before embarking on this, it is convenient first to record some elementary facts concerning $L^1[G, K, N]$.

Let K and N be arbitrary closed subgroups of G. It is easy to see that the set of all functions in $L^1[G]$ for which

$$f(kxk') = f(x) \quad \text{in } L^1[G] \quad \text{for all } k, k' \in K \tag{2.1}$$

is a subalgebra of $L^1[G]$. Similarly the set of all functions $f \in L^1[G]$ such that

$$f(n^{-1}xn) = f(x) \quad \text{in } L^1[G] \quad \text{for all } n \in N \tag{2.1'}$$

is a subalgebra of $L^1[G]$.

With K and N still *arbitrary* closed subgroups of G, let $L^1[G, K, N]$ be the set of all functions in $L^1[G]$ with the invariance properties (2.1) and (2.1′). Clearly $L^1[G, K, N]$ is a subalgebra of $L^1[G]$.

LEMMA 2.1. *If K_1 is the smallest closed subgroup of G which contains K and is normalized by N, then*

$$L^1[G, K, N] = L^1[G, K_1, N].$$

Proof. Let $p = n^{-1}kn$ where $n \in N$, $k \in K$. Then

$$f(px) = f(n^{-1}knx) = f(n^{-1}knxn^{-1}n)$$
$$= f(knxn^{-1}) = f(nxn^{-1}) = f(x).$$

Similarly,

$$f(xp) = f(x).$$

It follows that if K_0 is the subgroup of G generated by all elements of the form $n^{-1}kn$ where $n \in N$, $k \in K$, then

$$f(qxq') = f(x) \quad \text{in } L^1[G] \quad \text{for all } q, q' \in K_0.$$

Let $q \in K_0$,

$$q = n_1^{-1}k_1n_1 \cdots n_\nu^{-1}k_\nu n_\nu;$$

then $n^{-1}qn^{-1} \in K_0$ since

$$n^{-1}qn = (n_1n)^{-1}k_1(n_1n) \cdots (n_\nu n)^{-1}k_\nu(n_\nu n).$$

It is evident that K_0 is the smallest subgroup of G containing K and normalized by N. A routine argument shows that $K_1 = \overline{K}_0$ is the smallest closed subgroup containing K and normalized by N, and that if $f \in L^1[G, K, N]$, then

$$f(sxs') = f(x) \quad \text{in } L^1[G] \quad \text{for all } s, s' \in K_1.$$

It follows that we can, and *henceforth do*, assume that N normalizes K.

LEMMA 2.2. *Let K and N be closed subgroups of G (such that N normalizes K). If for some $p \in G$,*

$$f(px) = f(x) \quad \text{in } L^1[G] \quad \text{for all } f \in L^1[G, K, N],$$

or

$$f(xp) = f(x) \quad \text{in } L^1[G] \quad \text{for all } f \in L^1[G, K, N],$$

then $p \in K$.

The proof will be given later in this section. Lemma 2.2 shows that if N normalizes K, then the notation $L^1[G, K, N]$ is "honest" so far as K is concerned. It is not always honest in N. Indeed, we shall have occasion to use the following evident fact. If N contains N_1 and is contained in the subgroup generated by N_1 and K and C, where C is the center of G, then

$$L^1[G, K, N] = L^1[G, K, N_1]. \tag{2.2}$$

SCHOLIUM 2.3. *Let $L^1[G, K, N]$ be commutative. If $K' \supset K$, $N' \supset N$, then $L^1[G, K', N']$ is commutative.*

This is because $L^1[G, K', N']$ is a subalgebra of $L^1[G, K, N]$. (In accordance with our convention we will, in applying this result, always choose K' so that it is normalized by N'.)

For each $\alpha \in \mathbf{A}$, an index set, let $x \to U(\alpha, x)$ be a continuous irreducible unitary representation of G on the Hilbert space $\mathbf{H}(a)$ with the inner product $\langle \cdot | \cdot \rangle_\alpha$. Let $d(\alpha)$ be the (necessarily finite) dimension of $\mathbf{H}(\alpha)$. We assume that if $\alpha_1, \alpha_2 \in \mathbf{A}$ and $\alpha_1 \neq \alpha_2$, the corresponding representations are not unitarily equivalent, and we assume that every continuous irreducible unitary representation of G is unitarily equivalent to one of the representations of $\mathbf{A}$.

For α fixed let $\xi(1), \cdots, \xi(d(\alpha))$ be an orthonormal basis in $\mathbf{H}(\alpha)$ and let

$$g(\alpha, i, j, x) = \langle U(\alpha, x)\xi(j) | \xi(i) \rangle_\alpha, \qquad i, j = 1, \cdots, d(\alpha).$$

For each $x \in G$, $[g(\alpha, i, j, x)]$ is a $d(\alpha) \times d(\alpha)$ unitary matrix. The $g(\alpha, i, j, x)$, $i, j = 1, \cdots, d(\alpha)$, are the representative functions of the representation α. The representative functions depend on the particular orthonormal basis used in $\mathbf{H}(\alpha)$, although this is not indicated in our notation. However, the $d(\alpha) \times d(\alpha)$ linear space of functions on G spanned by the representative functions $g(\alpha, i, j, x)$, $i, j = 1, \cdots, d(\alpha)$, is independent of the choice of basis. We recall the orthogonality relations

$$\int_G g(\alpha_1, i, j, x)\overline{g(\alpha_2, r, s, x)}\, dx = 0 \quad \text{if } \alpha_1 \neq \alpha_2, \tag{2.3'}$$

and

$$\int_G g(\alpha, i, j, x)\overline{g(\alpha, r, s, x)}\, dx = \delta(i, r)\delta(j, s)\, d(\alpha)^{-1}. \tag{2.3''}$$

For $f \in L^1[G]$ and $\alpha \in \mathbf{A}$ we set

$$[f\hat{\ }(\alpha)] = [f\hat{\ }(\alpha, i, j)] = \int_G f(x)\overline{[g(\alpha, i, j, x)]}\, dx. \tag{2.4}$$

This $d(\alpha) \times d(\alpha)$ matrix is called the αth Fourier coefficient of $f(x)$. We note that if

$f_1, f_2 \in L^1[G]$ and if

$$f(x) = f_1 * f_2 \cdot (x) = \int_G f_1(xy^{-1}) f_2(y)\, dy,$$

then

$$[f\hat{\ }(\alpha)] = [f_1\hat{\ }(\alpha)]\,[f_2\hat{\ }(\alpha)]. \tag{2.5}$$

For $f \in L^1[G]$ the (formal) Fourier series for $f(x)$ is

$$\begin{aligned} &\sum_{\alpha\in\mathbf{A}} d(\alpha) \sum_{i,j=1}^{d(\alpha)} f\hat{\ }(\alpha, i, j) g(\alpha, i, j, x) \\ &\qquad = \sum_{\alpha\in\mathbf{A}} d(\alpha)\, \mathrm{tr}\{[f\hat{\ }(\alpha)][g(\alpha, x)]^t\}. \end{aligned} \tag{2.6}$$

f is uniquely determined by $\{[f\hat{\ }(\alpha)]\}_{\alpha\in\mathbf{A}}$, the set of its Fourier coefficients, and if $f \in L^2[G]$, the series (2.6) converges unconditionally to f in $L^2[G]$.

Fix $\alpha \in \mathbf{A}$ and let, as before, $\xi(1), \cdots, \xi(d(\alpha))$ be an orthonormal basis for $\mathbf{H}(\alpha)$. Let

$$J\left[\sum_{i=1}^{d(\alpha)} a(i)\xi(i)\right] = \sum_{i=1}^{d(\alpha)} \overline{a(i)}\xi(i).$$

Then $x \to \overline{U(\alpha, x)} \equiv JU(\alpha, x)J$ is easily seen to be a continuous irreducible unitary representation of G on $\mathbf{H}(\alpha)$. It follows that there exists an element $\bar{\alpha}$ in A such that $\overline{U(\alpha, x)}$ and $U(\bar{\alpha}, x)$ are unitarily equivalent. Note that $d(\alpha) = d(\bar{\alpha})$ and that

$$\langle \overline{U(\alpha, x)}\xi(j) | \xi(i)\rangle = \overline{g(\alpha, i, j, x)}.$$

The mapping $k \to U(\alpha, k)$, the restriction of $U(\alpha, x)$ to K, is a unitary (though not necessarily irreducible) representation of K on $\mathbf{H}(\alpha)$. Clearly $\mathbf{H}(\alpha)$ can be written as the direct sum of two subspaces, invariant under $U(\alpha, k), k \in K$,

$$\mathbf{H}(\alpha) = \mathbf{H}_K(\alpha) + \mathbf{H}'(\alpha),$$

where $U(\alpha, k), k \in K$, leaves each vector in $\mathbf{H}_K(\alpha)$ fixed and acts nontrivially on every vector in $\mathbf{H}'(\alpha)$ (except the null vector). Let $d_K(\alpha) = \dim \mathbf{H}_K(\alpha)$. We may assume without loss of generality that the basis $\xi(1), \cdots, \xi(d(\alpha))$ has been so chosen that

$$\xi(1), \cdots, \xi(d_K(\alpha)) \in \mathbf{H}_K(\alpha). \tag{2.7}$$

Let

$$\mathbf{A}_K = \{\alpha \in \mathbf{A} : d_K(\alpha) \neq 0\}.$$

Note that $\alpha \in \mathbf{A}_K$ if and only if $\bar{\alpha} \in \mathbf{A}_K$ and that $d_K(\alpha) = d_K(\bar{\alpha})$.

We denote by $L^1[G/K](L^1[K\backslash G])$ the set of all functions f in $L^1[G]$ such that $f(x) = f(xk)$ in $L^1[G]$ for all $k \in K$ ($f(x) = f(kx)$ in $L^1[G]$ for all $k \in K$). We define $L^1[K\backslash G/K]$ to be $L^1[K\backslash G] \cap L^1[G/K]$.

The following result is well known and easily established.

LEMMA 2.4. *Let* (2.7) *hold. If* $f \in L^1[G/K]$, *then*

$$\begin{aligned} f\hat{\ }(\alpha, i, j) &= 0, \qquad \alpha \in \mathbf{A}\setminus\mathbf{A}_K, \\ f\hat{\ }(\alpha, i, j) &= 0, \qquad j > d_K(\alpha), \quad \alpha \in \mathbf{A}_K. \end{aligned} \tag{2.8}$$

Conversely if $f \in L^1[G]$ *and* (2.8) *is satisfied, then* $f \in L^1[G/K]$. *If* $f \in L^1[K \backslash G]$, *then*

$$(2.9) \qquad \begin{aligned} f\hat{\ }(\alpha, i, j) &= 0, \qquad \alpha \in \mathbf{A} \diagdown \mathbf{A}_K, \\ f\hat{\ }(\alpha, i, j) &= 0, \qquad i > d_K(\alpha), \quad \alpha \in \mathbf{A}_K. \end{aligned}$$

Conversely if $f \in L^1[G]$ *and if* (2.9) *is satisfied, then* $f \in L^1[K \backslash G]$. *If* $f \in L^1[K \backslash G/K]$,

$$(2.10) \qquad \begin{aligned} f\hat{\ }(\alpha, i, j) &= 0, \qquad \alpha \in \mathbf{A} \diagdown \mathbf{A}_K, \\ f\hat{\ }(\alpha, i, j) &= 0, \qquad j > d_K(\alpha) \quad or \quad i > d_K(\alpha), \quad \alpha \in \mathbf{A}_K. \end{aligned}$$

Conversely if $f \in L^1[G]$ *and if* (2.10) *is satisfied, then* $f \in L^1[K \backslash G/K]$.

We recall that as always N is a closed subgroup of G which normalizes K.

LEMMA 2.5. *If* $\gamma \in \mathbf{A}_K$, $n \in N$, *then*

$$U(\gamma, n)\mathbf{H}_K(\gamma) \subset \mathbf{H}_K(\gamma).$$

Proof. Let $\xi \in \mathbf{H}_K(\gamma)$. For $U(\gamma, n)\xi$ to belong to $\mathbf{H}_K(\gamma)$ it is necessary and sufficient that

$$U(\gamma, k)U(\gamma, n)\xi = U(\gamma, n)\xi \quad \text{for all } k \in K.$$

We have $U(\gamma, k)U(\gamma, n) = U(\gamma, kn)$. Since N normalizes K, $kn = nk'$ for some $k' \in K$. Thus

$$\begin{aligned} U(\gamma, k)U(\gamma, n)\xi &= U(\gamma, nk')\xi = U(\gamma, n)U(\gamma, k')\xi \\ &= U(\gamma, n)\xi, \end{aligned}$$

and our proof is complete.

It follows that $U_N(\gamma, n)$, the restriction of $U(\gamma, \cdot)$ to N, is a unitary representation of N on $\mathbf{H}_K(\gamma)$. Consequently $\mathbf{H}_K(\gamma)$ can be written as a finite direct sum of mutually orthogonal subspaces

$$\mathbf{H}_K(\gamma) = \sum_c \oplus \mathbf{H}_K(\gamma, c)$$

on each of which $U_N(\gamma, \cdot)$ acts irreducibly. (The number of summands in the above decomposition plays no significant role and we do not give it a name.) If we set

$$U_N(\gamma, n)\xi = U_N(\gamma, c, n)\xi, \qquad \xi \in H_K(\gamma, c),$$

then $n \to U_N(\gamma, c, n)$ is an irreducible unitary representation of N on $\mathbf{H}_K(\gamma, c)$. We put

$$d_K(\gamma, c) = \dim \mathbf{H}_K(\gamma, c).$$

Let $I(\gamma) = \{1, 2, \cdots, d_K(\gamma)\}$ and let

$$I(\gamma) = \bigcup_c I(\gamma, c)$$

be a decomposition of $I(\gamma)$ into disjoint sets of consecutive integers, such that the first set contains $d_K(\gamma, 1)$ integers, the second $d_K(\gamma, 2)$ integers, etc.

Given a $d_K(\gamma) \times d_K(\gamma)$ matrix we partition it into blocks as follows—the (c, c')th block corresponds to the indices $i \in I(\gamma, c)$, $j \in I(\gamma, c')$. It is easy to see that by choosing $\xi(1), \cdots, \xi(d_K(\gamma))$ correctly the matrix

$$[g(\gamma, i, j, n)]_{i,j=1}^{d_K(\gamma)}$$

can be made to have the following form. The entries in the (c, c')th block are 0 unless $c = c'$, and the (c, c)th and (c', c')th blocks are identical whenever $U_N(\gamma, c, n)$ and $U_N(\gamma, c', n)$ are unitarily equivalent. From this point on we *assume* that for every $\gamma \in \mathbf{A}_K$, the basis of $\mathbf{H}_K(\gamma)$ has been chosen in this way.

THEOREM 2.6. *$f \in L^1[G]$ belongs to $L^1[G, K, N]$ if and only if for each $\gamma \in \mathbf{A}$, $f\,\hat{}\,(\gamma, i, j) = 0$ whenever either i or j exceeds $d_K(\gamma)$, and the matrix*

$$[f\,\hat{}\,(\gamma, i, j)]_{i,j=1}^{d_K(\gamma)} \tag{2.11}$$

has the following form; the (c, c')-th block is 0 unless $U_N(\gamma, c, \cdot)$ and $U_N(\gamma, c', \cdot)$ are unitarily equivalent in which case it is a diagonal matrix with identical entries.

Proof. For $1 \leqq i \leqq d_K(\gamma)$ let $c(i)$ and $\varkappa(i)$ be such that i is the $\varkappa(i)$th number in $I(\gamma, c(i))$. Define $\delta^\circ(i, j)$ as 1 if $U_N(\gamma, c(i), \cdot)$ and $U_N(\gamma, c(j), \cdot)$ are unitarily equivalent and if $\varkappa(i) = \varkappa(j)$; otherwise $\delta^\circ(i, j)$ is 0. It is not hard to verify that if $1 \leqq i, j \leqq d_K(\gamma)$ and if $r \in I(\gamma, c(i))$, $s \in I(\gamma, c(j))$, then

$$\int_N g(\gamma, i, r, n)\overline{g(\gamma, j, s, n)}\, dn = \delta^\circ(i, j)\delta^\circ(r, s)\, d_K(\gamma, c(i))^{-1}.$$

(If $r \notin I(\gamma, c(i))$ or $s \notin I(\gamma, c(j))$, then the integral is 0 since $g(\gamma, i, r, n)$ or $g(\gamma, j, s, n)$ is identically 0.) Using this formula together with

$$g(\gamma, i, j, nxn^{-1}) = \sum_{r,s} g(\gamma, i, r, n) g(\gamma, r, s, x)\overline{g(\gamma, j, s, n)},$$

we find that

$$\int_N g(\gamma, i, j, nxn^{-1})\, dn = \delta^\circ(i, j)\, d_K(\gamma, c(i))^{-1} \sum_{\substack{r \in I(\gamma, c(i)) \\ s \in I(\gamma, c(j))}} g(\gamma, r, s, x)\delta^\circ(r, s). \tag{2.12}$$

Let $f \in L^1[G, K, N]$. We assert that if

$$\chi_\gamma(x) = \sum_{i=1}^{d(\gamma)} g(\gamma, i, i, x),$$

and if

$$f_\gamma(x) = \chi_\gamma * f \cdot (x) = \sum_{i,j=1}^{d_K(\gamma)} f\,\hat{}\,(\gamma, i, j) g(\gamma, i, j, x),$$

then $f_\gamma(x) \in L^1[G, K, N]$. It is immediately evident from Lemma 2.5 that $f_\gamma(kxk') = f_\gamma(x)$ for all $k, k' \in K$. Since $f(nxn^{-1}) = f(x)$ by assumption, and $\chi_\gamma(nxn^{-1}) = \chi_\gamma(x)$ because $\chi_\gamma(x)$ is a character, it follows from (2.1′) that $f_\gamma(nxn^{-1}) = f_\gamma(x)$.

Combining this result with (2.12) we see if $f \in L^1[G, K, N]$, then $f_\gamma\hat{}$ must be as described.

To prove the converse it is enough to show that the functions

$$\varphi_\gamma(c, c', x) = \sum_{\substack{i \in I(\gamma, c) \\ j \in I(\gamma, c')}} g(\gamma, i, j, x)\delta^\circ(i, j)$$

belong to $L^1[G, K, N]$. That $\varphi_\gamma(c, c', kxk') = \varphi_\gamma(c, c', x)$ for all $k, k' \in K$ follows from Lemma 2.5. It remains to show that $\varphi_\gamma(c, c', nxn^{-1}) = \varphi_\gamma(c, c', x)$ for $n \in N$. We have

$$\begin{aligned}\varphi_\gamma(c, c', nxn^{-1}) &= \sum_{\substack{i \in I(\gamma,c) \\ j \in I(\gamma,c')}} \sum_{r,s} g(\gamma, i, r, n) g(\gamma, r, s, x) \overline{g(\gamma, j, s, n)} \delta^\circ(i, j) \\ &= \sum_{r,s} g(\gamma, r, s, x) \sum_{\substack{i \in I(\gamma,c) \\ j \in I(\gamma,c')}} \delta^\circ(i, j) g(\gamma, i, r, n) \overline{g(\gamma, j, s, n)}.\end{aligned}$$

It is easy to see that

$$\sum_{\substack{i \in I(\gamma,c) \\ j \in I(\gamma,c')}} \delta^\circ(i, j) g(\gamma, i, r, n) \overline{g(\gamma, j, s, n)} = \delta^\circ(r, s)$$

if $r \in I(\gamma, c)$ and $s \in I(\gamma, c')$; otherwise this sum is zero. It follows that

$$\begin{aligned}\varphi_\gamma(c, c', nxn^{-1}) &= \sum_{\substack{r \in I(\gamma,c) \\ s \in I(\gamma,c')}} g(\gamma, r, s, x) \delta^\circ(r, s) \\ &= \varphi_\gamma(c, c', x),\end{aligned}$$

as desired.

We can now prove Lemma 2.2. Suppose that $f(px) = f(x)$ in $L^1[G]$ for all $f \in L^1[G, K, N]$. By Theorem 2.6,

$$\chi'_\gamma(x) = \sum_{i=1}^{d_K(\gamma)} g(\gamma, i, i, x)$$

belongs to $L^1[G, K, N]$, and therefore,

$$\chi'_\gamma(px) = \chi'_\gamma(x). \tag{2.13}$$

For $f \in L^1[K\backslash G]$ we see using Lemma 2.5 that

$$f_\gamma(x) = \chi'_\gamma * f \cdot (x) = \int_G \chi'_\gamma(xy^{-1}) f(y)\, dy. \tag{2.14}$$

Together (2.13) and (2.14) imply that $f_\gamma(px) = f_\gamma(x)$. Since this holds for all γ it follows that $f(px) = f(x)$. However, if $p \notin K$, we can easily construct a continuous function g in $L^1[K\backslash G]$ such that $g(px) \not\equiv g(x)$.

THEOREM 2.7. *A necessary and sufficient condition that $L^1[G, K, N]$ be commutative is that, for each fixed $\gamma \in \mathbf{A}_K$, the representations $\{U_N(\gamma, c, \cdot)\}_c$ be mutually inequivalent.*

Proof. This is an immediate consequence of Theorem 2.7.

We *henceforth assume* that $L^1[G, K, N]$ is *commutative.*

For $\alpha \in \mathbf{A}_K$ let

$$\chi(\alpha, a, x) = \sum_{i \in I(\alpha,a)} g(\alpha, i, i, x). \tag{2.15}$$

It is evident from the discussion we have just given that each $\chi(\alpha, a, x) \in L^1[G, K, N]$ and that if $f \in L^1[G, K, N]$, then

$$f(x) \sim \sum_{\alpha \in \mathbf{A}_K} \sum_a f\hat{\ }(\alpha, a)\, d(\alpha)\chi(\alpha, a, x),$$

where $f\hat{\ }(\alpha, a)$ is the common value of $f\hat{\ }(\alpha, i, i)$ for $i \in I(\alpha, a)$. The $\chi(\alpha, a, x)$'s look somewhat like characters. As we shall see they behave much like characters, and this is what we need.

LEMMA 2.8. *Given* (α, a) *there exists* (β, b) *such that*

$$\overline{\chi(\alpha, a, x)} = \chi(\beta, b, x).$$

Proof. Let $\alpha \in \mathbf{A}_K$. Recalling the definition of $\overline{U(\alpha, x)}$ acting on $\mathbf{H}(\alpha)$ we see that the set of those vectors $\xi \in \overline{\mathbf{H}(\alpha)}$ for which $\overline{U(\alpha, k)\xi} = \xi$ for all $k \in K$ is $\mathbf{H}_K(\alpha)$. Let $\overline{U_N(\alpha, n)}$ be the restriction of $\overline{U(\alpha, x)}$ to N. Then $\overline{U_N(\alpha, n)}$ acts irreducibly on $\mathbf{H}_K(\alpha, a)$. Thus $\overline{\chi(\alpha, a, x)} \in L^1[G, K, N]$. In addition $\overline{\chi(\alpha, a, n)}$, $n \in N$, is the character of the irreducible representation $\overline{U_N(\alpha, a, \cdot)}$ of N. Since $\mathbf{A}$ is a complete set of irreducible representations, there exists $\beta \in \mathbf{A}_K$ such that $\overline{U(\alpha, x)}$ is unitarily equivalent to $U(\beta, x)$. There are therefore complex constants $A(b)$ such that

$$\overline{\chi(\alpha, a, x)} = \sum_b A(b)\chi(\beta, b, x).$$

Let us restrict x to N. The functions $\overline{\chi(\alpha, a, n)}$, $\{\chi(\beta, b, n)\}_b$ are the characters of irreducible representations of N, and the $\{\chi(\beta, b, n)\}_b$ are mutually inequivalent. Thus $A(b)$ is 1 for some one b and is 0 for all the rest, as we wished to show.

Let $(\bar{\alpha}, \bar{a})$ be defined to be (β, b) so that

$$\overline{\chi(\alpha, a, x)} = \chi(\bar{\alpha}, \bar{a}, x).$$

Let us set, for $\alpha, \beta, \gamma \in \mathbf{A}_K$,

$$D(\alpha, a, \beta, b, \gamma, c) = \int_G \chi(\alpha, a, x)\chi(\beta, b, x)\chi(\gamma, c, x)\, dx. \tag{2.16}$$

THEOREM 2.9. *We have*

$$\chi(\alpha, a, x)\chi(\bar{\beta}, \bar{b}, x) = \sum_{\gamma, c} d(\gamma)\, d_K(\gamma, c)^{-1} D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c)\overline{\chi(\gamma, c, x)}, \tag{2.17}$$

where the sum on the right is finite. Moreover

$$D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c) \geqq 0. \tag{2.18}$$

Proof. It follows from the elementary theory of tensor products of representations that

$$\chi(\alpha, a, x)\overline{\chi(\beta, b, x)} = \sum_\gamma \sum_{i,j=1}^{d(\alpha)} A(\gamma, i, j)\overline{g(\gamma, i, j, x)}$$

where only those γ's appear for which $\overline{U(\gamma, \cdot)}$ is contained in the tensor product $U(\alpha, \cdot) \otimes \overline{U(\beta, \cdot)}$. In particular the sum we have just written down is finite. Moreover, since $\chi(\alpha, a, x)\overline{\chi(\beta, b, x)} \in L^1[G, K, N]$,

$$\chi(\alpha, a, x)\overline{\chi(\beta, b, x)} = \sum_{\gamma, c} A(\gamma, c)\overline{\chi(\gamma, c, x)}.$$

Multiplying both sides by $\chi(\gamma, c, x)$ and integrating over G we obtain (2.17).

It remains to prove (2.18). We recall that a continuous function $f(x)$ on G is "positive definite" if

$$\sum_{i,j=1}^{r} f(x_i x_j^{-1}) z_i \bar{z}_j \geqq 0$$

for all $x_1, \cdots, x_r$ in G and complex $z_1, \cdots, z_r$. As a special case of a very general theory, a detailed exposition of which can be found in [3, vol. II, § 34.10], it follows that f, assumed to be continuous on G, is positive definite if and only if for each $\alpha \in \mathbf{A}$ the matrix $[f\hat{\ }(\alpha, i, j)]_{i,j=1}^{d(\alpha)}$ is positive semidefinite. This implies that the $\chi(\alpha, a, x)$'s are positive definite. Since the product of positive definite functions is again positive definite, $\chi(\alpha, a, x)\chi(\beta, b, x)$ is positive definite, but this is possible only if (2.18) holds.

For future use we record the following fact, which apart from the roles of the indices was demonstrated in the course of the above proof.

COROLLARY 2.10. *$D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c) = 0$ unless $U(\beta, b)$ is contained in $U(\alpha, \cdot) \otimes U(\gamma, \cdot)$.*

COROLLARY 2.11. *We have*

$$d_K(\alpha, a)\, d_K(\beta, b) = \sum_{\gamma, c} d(\gamma) D(\alpha, \bar{a}, \bar{\beta}, b, \gamma, c). \tag{2.19}$$

Proof. Set $x = e$ in (2.17).

The following result, although not needed in what follows, is of some interest.

THEOREM 2.12. *We have*

$$d_K(\gamma, c) \int_N \chi(\gamma, c, xn^{-1}yn)\, dn = \chi(\gamma, c, x)\chi(\gamma, c, y)$$

for all $x, y \in G$.

Proof. This follows from (2.12).

We can, at last, state the principal result of the present paper. Let S be defined as at the end of § 1. Since $L \in L^1[G, K, N]$, where (as always) $L^1[G, K, N]$ is commutative,

$$L(x) \sim \sum_{\alpha \in \mathbf{A}_K} \sum_a d(\alpha) L\hat{\ }(\alpha, a) \chi(\alpha, a, x).$$

The assumption that $f \to f * L$ is a nonnegative operator on $L^2[G/K]$ is equivalent to the assumption $L\hat{\ }(\alpha, a) \geqq 0$. Let $\underset{\sim}{\mathbf{A}}_K$ be the collection of all the pairs (α, a) where $\alpha \in \mathbf{A}_K$. We make $\underset{\sim}{\mathbf{A}}_K$ into a discrete measure space by assigning the mass $d(\alpha) d_K(\alpha, a)$ to the point (α, a). We shall (essentially) show that if $L\hat{\ }(\alpha, a)$ behaves "regularly enough," then

$$N^+[\varepsilon, S] \sim |\{(x, (\alpha, a)) : |V(x)|^2 L\hat{\ }(\alpha, a) > \varepsilon\}|_{G \times \underset{\sim}{\mathbf{A}}_K}$$

as $\varepsilon \to 0+$, where $N^+[\varepsilon, S]$ is the number of eigenvalues of S greater than ε, and $|\{\cdot\}|_{G \times \underset{\sim}{\mathbf{A}}_K}$ is the measure of the set $\{\cdot\}$ in the product measure space $G \times \underset{\sim}{\mathbf{A}}_K$.

We conclude this section by briefly sketching a generalization of the ideas just developed. This material will not, however, be used in the sequel. Let G and K

be as before but let N be replaced by a compact group Ω of continuous automorphisms of G. The condition that N normalizes K becomes here the condition that $\omega K = K$ for all $\omega \in \Omega$. Fix ω in Ω. Since the mapping $x \to U(\alpha, \omega x)$ is, evidently, an irreducible unitary representation of G on $\mathbf{H}(\alpha)$, there is an index $\omega(\alpha) \in \mathbf{A}$ such that $U(\alpha, \omega x)$ and $U(\omega(\alpha), x)$ are unitarily equivalent. Consequently there exists a unique matrix $[W(\alpha, i, j, \omega)]$ in $SU(d(\alpha))$ such that

$$[g(\alpha, i, j, \omega x)] = [W(\alpha, i, j, \omega)]^{-1}[g(\omega(\alpha), i, j, x)][W(\alpha, i, j, \omega)]$$

for all $x \in G$. Let us now assume that $\omega(\alpha) = \alpha$ for all $\omega \in \Omega$. This will be the case if, for example, Ω is connected. We now have

$$[g(\alpha, i, j, \omega x)] = [W(\alpha, i, j, \omega)]^{-1}[g(\alpha, i, j, x)][W(\alpha, i, j, \omega)].$$

It is easy to verify that

$$[W(\alpha, i, j, \omega_1\omega_2)] = [W(\alpha, i, j, \omega_1)][W(\alpha, i, j, \omega_2)];$$

that is, $\omega \to [W(\alpha, i, j, \omega)]$ is a unitary representation of Ω on $\mathbf{H}(\alpha)$. The assumption $\omega K = K$ implies that for $\alpha \in \mathbf{A}_K$, $W(\alpha, i, j, \omega)$ leaves $\mathbf{H}_K(\alpha)$ invariant. $W(\alpha, i, j, \omega)$ acting on $\mathbf{H}_K(\alpha)$ can therefore be decomposed into a finite direct sum

$$[W(\alpha, i, j, \omega)] = \sum_a \oplus [W(\alpha, a, i, j, \omega)]$$

of irreducible representations. Exactly as before we find that if $L^1[G, K, \Omega]$ is the set of those functions $f \in L^1[G]$ for which

$$f(kxk') = f(x) \quad \text{in } L^1[G] \quad \text{for all } k, k' \in K,$$

and

$$f(\omega x) = f(x) \quad \text{in } L^1[G] \quad \text{for all } \omega \in \Omega,$$

then $L^1[G, K, \Omega]$ is commutative if and only if for each fixed $\alpha \in \mathbf{A}_K$ the representations $\{[W(\alpha, a, i, j, \omega)]\}_a$ are mutually inequivalent, etc.

It seems unlikely that such results as Theorem 2.7 and Theorem 2.9 are new, but I have not been able to find references for them.

3. Some examples. The only really simple examples of commutative algebras $L^1[G, K, N]$ are those of the form

$$L^1[G, \{e\}, G], \tag{3.1}$$

where G is any compact group. Clearly $\mathbf{H}_K(\alpha) = \mathbf{H}(\alpha)$ for all $\alpha \in \mathbf{A}$ so that $\mathbf{A} = \mathbf{A}_K$ and, since $N = G$, $U_N(\alpha, \cdot)$ acts irreducibly on $\mathbf{H}_K(\alpha)$. Thus there is only one $U_N(\alpha, a, \cdot)$ and the condition that, for α fixed, the $U_N(\alpha, a, \cdot)$ be mutually inequivalent is satisfied trivially. $L^1[G, \{e\}, G]$ consists of the "central functions" in $L^1[G]$.

A more sophisticated class of examples consists of those algebras

$$L^1[G, K, \{e\}], \quad \text{where } d_K(\alpha) = 0 \text{ or } 1 \quad \text{for all } \alpha \in \mathbf{A}. \tag{3.2}$$

Here $N = \{e\}$ and $U_N(\alpha, \cdot)$, $\alpha \in \mathbf{A}_K$, acts irreducibly on $\mathbf{H}_K(\alpha)$, because $\mathbf{H}_K(\alpha)$ is 1-dimensional. Thus, as above, there is only one $U_N(\alpha, a, \cdot)$, and the condition that, for $\alpha \in \mathbf{A}_K$ fixed, the $U_N(\alpha, a, \cdot)$ be mutually inequivalent, is again satisfied trivially.

In particular $L^1[G, K, \{e\}]$ is commutative whenever G/K is a compact symmetric space. Thus

$$L^1[SO(n), SO(n-1), \{e\}]$$

is commutative for $n = 3, 4, \cdots$.

It is not at all a simple matter to find examples different from those given in (3.1) and (3.2). A few such examples are given in Theorem 3.3 below.

LEMMA 3.1. *Let* $\boldsymbol{\theta}$ *be a bicontinuous mapping of* G *onto itself such that* $\boldsymbol{\theta}$ *is an anti-isomorphism (that is,* $\boldsymbol{\theta}(xy) = (\boldsymbol{\theta}y)(\boldsymbol{\theta}x)$*), and such that for each* $x \in G$ *there exist* $k_1, k_2 \in K$ *and* $n \in N$*, depending on* x*, with the property that*

$$\boldsymbol{\theta}x = k_1 n^{-1} x n k_2. \tag{3.3}$$

Then $L^1[G, K, N]$ *is commutative.*

Proof. This is a slight variant of a familiar result. We assert that $\boldsymbol{\theta}$ preserves Haar measure. For C a Borel set in G denote the Haar measure of C by $|C|$. We set $|C|_{\boldsymbol{\theta}} = |\boldsymbol{\theta}(C)|$. Then $|xC|_{\boldsymbol{\theta}} = |\boldsymbol{\theta}(xC)| = |\boldsymbol{\theta}(C)\boldsymbol{\theta}x| = |\boldsymbol{\theta}(C)| = |C|_{\boldsymbol{\theta}}$. Thus $|\cdot|_{\boldsymbol{\theta}}$ is a left invariant measure on G such that $|G|_{\boldsymbol{\theta}} = 1$. By the uniqueness of Haar measure, $|\boldsymbol{\theta}C| = |C|$. If $f \in L^1[G, K, N]$ is continuous, then (3.3) implies that $f(\boldsymbol{\theta}x) = f(x)$ for all $x \in G$. A simple approximation argument now shows that $f(\boldsymbol{\theta}x) = f(x)$ in $L^1[G]$ for all $f \in L^1[G, K, N]$.

Let $f, g \in L^1[G, K, N]$; then

$$\begin{aligned} f * g \cdot (x) = f * g(\boldsymbol{\theta}x) &= \int_G f([\boldsymbol{\theta}x]y^{-1})g(y)\,dy \\ &= \int_G f([\boldsymbol{\theta}x][\boldsymbol{\theta}z^{-1}])g(\boldsymbol{\theta}z)\,dz \quad (y = \boldsymbol{\theta}z) \\ &= \int_G f(\boldsymbol{\theta}[z^{-1}x])g(\boldsymbol{\theta}z)\,dz \\ &= \int_G f(z^{-1}x)g(z)\,dz = g * f \cdot (x). \end{aligned}$$

Let us take $G = SU(2)$, the set of all 2×2 unitary matrices with determinant 1. Every matrix in $SU(2)$ is of the form

$$\begin{bmatrix} w & z \\ -\bar{z} & \bar{w} \end{bmatrix},$$

where w and z are complex numbers such that $|w|^2 + |z|^2 = 1$, and conversely every such matrix is in $SU(2)$. Let $T(1)$ be the subgroup of $SU(2)$ consisting of the matrices

$$\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

THEOREM 3.2. $L^1[SU(2), \{e\}, T(1)]$ *is commutative.*

Proof. We assert that the assumptions of Lemma 3.1 hold where $\boldsymbol{\theta}$ is the map which sends each matrix in $SU(2)$ into its transpose. It is enough to verify that we can choose θ so that

$$\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \begin{bmatrix} w & z \\ -\bar{z} & \bar{w} \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} = \begin{bmatrix} w & e^{-2i\theta}z \\ -e^{2i\theta}\bar{z} & \bar{w} \end{bmatrix}$$

is equal to

$$\begin{bmatrix} w & -\bar{z} \\ z & \bar{w} \end{bmatrix}.$$

Let $Z(n)$ be the subgroup of $T(1)$ whose elements are

$$\begin{bmatrix} e^{2\pi ik/n} & 0 \\ 0 & e^{-2\pi ik/n} \end{bmatrix}, \qquad k = 0, 1, \cdots, n-1,$$

and let $Z'(4)$ be the subgroup whose elements are

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

By $\{T(1), Z'(4)\}$ we mean the smallest subgroup of $SU(2)$ containing $T(1)$ and $Z'(4)$.

THEOREM 3.3. *The subalgebras* $L^1[SU(2), K, N]$ *listed in Table* 1 *are commutative.*

Proof. Entry 1 comes from Theorem 3.2. Entries 2, 3, 4 and 5 are obtained from entry 1 and Scholium 2.3. Scholium 2.3 and entry 1 imply that $L^1[SU(2), T(1), T(1)]$ is commutative, but by (2.2) this is the same as $L^2[SU(2), T(1), e]$, which is entry 6. Entry 7 is obtained from entry 6 and Scholium 2.3. In this list entry 3 is a special case of (3.1) and entries 6 and 7 are special cases of (3.2).

The algebra $L^1[SU(2), Z(2), T(1)]$ is obviously commutative, but since $Z(2)$ is a normal subgroup of $SU(2)$ this is more advantageously considered as the assertion that $L^1[SU(2)/Z(2), \{e\}, T(1)]$ is commutative.

TABLE 1

	K	N
1	$\{e\}$	$T(1)$
2	$\{e\}$	$\{T(1), Z'(4)\}$
3	$\{e\}$	$SU(2)$
4	$Z(n), n \geqq 3$	$T(1)$
5	$Z(n), n \geqq 2$	$\{T(1), Z'(4)\}$
6	$T(1)$	$\{e\}$
7	$\{T(1), Z'(4)\}$	$\{e\}$

The irreducible unitary representations of $SU(2)$ are easy to write down. See [1], [3, vol. II] or [7]. We may take $\mathbf{A} = \{0, 1, \cdots\}$. $\alpha = 0$ corresponds to the trivial representation. For $\alpha > 0$ the representation $x \to U(\alpha, x)$ is obtained as follows. Let $\mathbf{H}(\alpha)$ be the complex linear space spanned by the homogeneous polynomials of degree α in two indeterminates z_1 and z_2. Let

$$\xi = \sum_{k=0}^{\alpha} A(k) z_1^k z_2^{\alpha-k}.$$

For $x \in SU(2)$,

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

we set

$$U(\alpha, x)\xi = \sum_{k=0}^{\alpha} A(k)[x_{11}z_1 + x_{21}z_2]^k [x_{12}z_1 + x_{22}z_2]^{\alpha-k}. \tag{3.4}$$

If we make $\mathbf{H}(\alpha)$ a Hilbert space by setting

$$\langle z_1^k z_2^{\alpha-k}, z_1^j z_2^{\alpha-j} \rangle = \delta(j,k)\frac{k!(\alpha-k)!}{\alpha!},$$

then $x \to U(\alpha, x)$ is an irreducible unitary representation of $SU(2)$.

Since there is only one representation of each dimension it is evident that

$$\bar{\alpha} = \alpha \quad \text{for all } \alpha \in \mathbf{A}. \tag{3.5}$$

It is not difficult to show that

$$U(\alpha, \cdot) \otimes U(\beta, \cdot) \sim \sum_{\gamma=|\alpha-\beta|}^{\alpha+\beta} \oplus\, U(\gamma, \cdot), \tag{3.6}$$

where "$\sim$" denotes unitary equivalence. See [3, vol. II, p. 135] and [7, p. 175]. The explicit unitary equivalence is given by the Clebsch–Gordan coefficients [7, Chap. III, § 8].

We now look more closely at three of the algebras listed in Theorem 3.3.

(a) $L^1[SU(2), \{e\}, SU(2)]$. This falls into the category (3.1). Here $\mathbf{A}_K = \mathbf{A}$ and there is only one a for each α. We therefore write $D(\alpha, \beta, \gamma)$ in place of $D(\alpha, a, \beta, b, \gamma, c)$. It follows from (3.5), (3.6) and (2.16) that

$$D(\alpha, \beta, \gamma) = \begin{cases} 1, & |\alpha - \beta| \leqq \gamma \leqq \alpha + \beta, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $L^1[SU(2), T(1), \{e\}]$. Let

$$x(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

Since

$$U(\alpha, x(\theta)) \sum_{k=0}^{\alpha} A(k) z_1^k z_2^{\alpha-k} = \sum_{k=0}^{\alpha} A(k)\, e^{i(2k-\alpha)\theta} z_1^k z_2^{\alpha-k}$$

it follows that $d_K(\alpha) = 0$ for $\alpha = 1, 3, 5, \cdots$, and $d_K(\alpha) = 1$ for $\alpha = 0, 2, 4, \cdots$. We set

$$A(m) = \frac{m!}{[(m/2)!2^{m/2}]^2}$$

if $m = 0, 2, 4, \cdots$. For other values of m let $A(m)$ be 0. For $\alpha, \beta, \gamma \in \mathbf{A}_K$ we have

$$D(\alpha, \beta, \gamma) = \frac{2A(-\alpha + \beta + \gamma)A(\alpha - \beta + \gamma)A(\alpha + \beta - \gamma)}{A(\alpha + \beta + \gamma)(\alpha + \beta + \gamma + 1)};$$

see [7, p. 186].

(c) $L^1[SU(2), \{e\}, T(1)]$. In this example $\mathbf{A}_K = \mathbf{A}$ and $\mathbf{H}_K(\alpha) = \mathbf{H}(\alpha)$. Since $T(1)$ is Abelian the irreducible subspaces of $\mathbf{H}_K(\alpha) = \mathbf{H}(\alpha)$ under the action of $U_{T(1)}(\alpha, \cdot)$ must be 1-dimensional. If $\mathbf{H}_K(\alpha, a)$ is the subspace spanned by $\xi(a) = z_1^a z_2^{\alpha - a}$, then $U_{T(1)}(\alpha, \cdot)$ acts irreducibly on $\mathbf{H}_K(\alpha, a)$, $a = 0, \cdots, \alpha$, so that $d_K(\alpha, a) = 1$ for all α and a. Since

$$U_{T(1)}(\alpha, a, x(\theta))\xi(a) = e^{i(2a-\alpha)\theta}\xi(a),$$

$U_{T(1)}(\alpha, a_1, \cdot)$ and $U_{T(1)}(\alpha, a_2, \cdot)$ are unitarily equivalent only if $a_1 = a_2$. $D(\alpha, a, \beta, b, \gamma, c)$ can be expressed in terms of the Clebsch–Gordan coefficients; see [7, (1), p. 185].

4. The basic estimates. For each $\varepsilon > 0$ let $\Pi(\varepsilon)$ be a finite subset of $\mathbf{\hat{A}}_K$, and let $P\hat{\ }(\varepsilon, \alpha, a)$ be the characteristic function of $\Pi(\varepsilon)$. We set

$$P(\varepsilon, x) = \sum_{\Pi(\varepsilon)} d(\alpha)\chi(\alpha, a) = \sum_{\mathbf{\hat{A}}_K} P\hat{\ }(\varepsilon, \alpha, a)\, d(\alpha)\chi(\alpha, a, x). \tag{4.1}$$

Clearly $P(\varepsilon, x) \in L^1[K\backslash G/K]$. Using (2.3′) and (2.3″) we see that if for some $(\alpha, a) \in \Pi(\varepsilon)$, $1 \leqq i \leqq d(\alpha)$, $j \in I(\alpha, a)$, then

$$\int_G P(\varepsilon, y^{-1}x)g(\alpha, i, j, y)\, dy = g(\alpha, i, j, x).$$

Otherwise this integral is zero.

It follows that the mapping $f \to f * P(\varepsilon)$ is a projection on $L^2[G/K]$ of rank $\Pi(\varepsilon)^{\#}$, where

$$\Pi(\varepsilon)^{\#} = \sum_{(\alpha,a)\in\Pi(\varepsilon)} d(\alpha)d_K(\alpha, a) = P(\varepsilon, e). \tag{4.2}$$

It is apparent from Theorem 2.9 that $|P(\varepsilon, x)|^2$ is equal to

$$\sum_{(\alpha,a),(\beta,b)\in\Pi(\varepsilon)} d(\alpha)\, d(\bar{\beta}) \cdot \sum_{(\gamma,c)\in\mathbf{\hat{A}}_K} D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c)\, d(\gamma)\, d_K(\gamma, c)^{-1}\overline{\chi(\gamma, c, x)} = \sum_{\mathbf{\hat{A}}_K} Q\hat{\ }(\varepsilon, \gamma, c)\, d(\gamma)\overline{\chi(\gamma, c, x)}, \tag{4.3}$$

where

$$Q\hat{\ }(\varepsilon, \gamma, c) = d_K(\gamma, c)^{-1} \sum_{(\alpha,a),(\beta,b)\in\Pi(\varepsilon)} d(\alpha)\, d(\bar{\beta})D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c). \tag{4.4}$$

Alternatively,

$$Q\hat{}(\varepsilon, \gamma, c) = d_K(\gamma, c)^{-1} \int_G |P(\varepsilon, x)|^2 \overline{\chi(\gamma, c, x)}\, dx. \tag{4.4'}$$

Let us agree that a measurable set Ω in G is "K right-invariant" if its characteristic function $\chi_\Omega \in L^\infty[G/K]$. If Ω is K right-invariant, then the mapping $f \to \chi_\Omega(x) f(x)$ is a projection on $L^2[G/K]$. We denote by $\Omega^\#$ the Haar measure of Ω. We exclude from consideration the case $\Omega^\# = 0$.

THEOREM 4.1. *Under the above assumptions if*

$$P_\Omega(\varepsilon) f \cdot (x) = \int_G \chi_\Omega(x) P(\varepsilon, y^{-1}x) \chi_\Omega(y) f(y)\, dy, \qquad f \in L^2[G/K],$$

then $0 \leqq P_\Omega(\varepsilon) \leqq 1$. *If* $\lambda(\varepsilon, 1) \geqq \lambda(\varepsilon, 2) \geqq \cdots$ *are the eigenvalues of* $P_\Omega(\varepsilon)$, *then*

(i) $0 \leqq \lambda(\varepsilon, n) \leqq 1, \qquad n = 1, 2, \cdots;$

(ii) $\sum_n \lambda(\varepsilon, n) = \Omega^\# \Pi(\varepsilon)^\#.$

If in addition,

$$\lim_{\varepsilon \to 0+} Q\hat{}(\varepsilon, \gamma, c)/\Pi(\varepsilon)^\# = 1 \quad \textit{for all } (\gamma, c) \in \underset{\sim}{\mathbf{A}}_K, \tag{4.5}$$

then

(iii) $\sum_n \lambda(\varepsilon, n)^2 = \Omega^\# \Pi(\varepsilon)^\# [1 - r_1(\varepsilon)],$

when $0 \leqq r_1(\varepsilon) \leqq 1$ *and where* $\lim_{\varepsilon \to 0+} r_1(\varepsilon) = 0$.

Proof. Since $P_\Omega(\varepsilon)$ is of the form EFE where E and F are projections on $L^2[G/K]$, it follows that $0 \leqq P_\Omega(\varepsilon) \leqq 1$. That (i) holds is an immediate consequence of this fact.

If we were working with $L^2[G]$ instead of $L^2[G/K]$, the results referred to below would be quite standard. That they hold here is due to the fact that

$$\chi_\Omega(x) P(\varepsilon, y^{-1}x) \chi_\Omega(y)$$

is invariant if x is replaced by xk_1 and if y is replaced by yk_2 for any $k_1, k_2 \in K$.

Because $P_\Omega(\varepsilon)$ is of finite rank we can (in view of the above remark) apply Mercer's theorem to obtain

$$\begin{aligned}\sum_n \lambda(\varepsilon, n) &= \int \chi_\Omega(x) P(\varepsilon, x^{-1}x) \chi_\Omega(x)\, dx \\ &= \Omega^\# P(\varepsilon, e) = \Omega^\# \Pi(\varepsilon)^\#.\end{aligned}$$

Thus (ii) is established. Similarly, using a standard result in the theory of integral equations (see [2, pp. 137–138]), we obtain

$$\begin{aligned}\sum_n \lambda(\varepsilon, n)^2 &= \int_G \int_G \chi_\Omega(x) |P(\varepsilon, y^{-1}x)|^2 \chi_\Omega(y)\, dx\, dy \\ &= \int_G \int_G \chi_\Omega(x) \chi_\Omega(xz^{-1}) |P(\varepsilon, z)|^2\, dx\, dz.\end{aligned}$$

If $\chi_\Omega^\vee(x) = \chi_\Omega(x^{-1})$, then

$$\int_G \chi_\Omega(x)\chi_\Omega(xz^{-1})\,dx = \chi_\Omega^\vee * \chi_\Omega \cdot (z).$$

Let

$$\chi_\Omega(x) = \sum_{\mathbf{A}_K} d(\alpha) \sum_{i=1}^{d(\alpha)} \sum_{j=1}^{d_K(\alpha)} \hat{\chi_\Omega}(\alpha, i, j) g(\alpha, i, j, x)$$

be the Fourier series for $\chi_\Omega(x)$. Replacing x by x^{-1} and taking conjugates we find that

$$\chi_\Omega^\vee(x) = \sum_{\mathbf{A}_K} d(\alpha) \sum_{j=1}^{d(\alpha)} \sum_{i=1}^{d_K(\alpha)} \overline{\hat{\chi_\Omega}(\alpha, j, i)} g(\alpha, i, j, x).$$

It follows from (2.5) that

$$\chi_\Omega^\vee * \chi_\Omega(x) = \sum_{\mathbf{A}_K} d(\alpha) \sum_{i_1, i_2 = 1}^{d_K(\alpha)} \left[\sum_{j=1}^{d(\alpha)} \overline{\hat{\chi_\Omega}(\alpha, j, i_1)} \hat{\chi_\Omega}(\alpha, j, i_2) \right] g(\alpha, i_1, i_2, x).$$

Using (4.3) we obtain

$$\sum_n \lambda(\varepsilon, n)^2 = \sum_{\mathbf{A}_K} d(\alpha) \sum_a \sum_{i \in I(\alpha, a)} \sum_{j=1}^{d(\alpha)} |\hat{\chi_\Omega}(\alpha, j, i)|^2 Q\hat{\ }(\varepsilon, \alpha, a).$$

Since

$$\begin{aligned} \Omega^\# &= \int_G |\chi_\Omega(x)|^2\,dx = \sum_{\mathbf{A}_K} d(\alpha) \sum_{j=1}^{d(\alpha)} \sum_{i=1}^{d_K(\alpha)} |\hat{\chi_\Omega}(\alpha, j, i)|^2 \\ &= \sum_{\mathbf{A}_K} d(\alpha) \sum_a \sum_{i \in I(\alpha, a)} \sum_{j=1}^{d(\alpha)} |\hat{\chi_\Omega}(\alpha, j, i)|^2, \end{aligned}$$

we have

$$\begin{aligned} [\Pi(\varepsilon)^\#]^{-1} \sum_n \lambda(\varepsilon, n)^2 &= \sum_{\mathbf{A}_K} d(\alpha) \sum_a \sum_{i \in I(\alpha, a)} \sum_{j=1}^{d(\alpha)} |\hat{\chi_\Omega}(\alpha, j, i)|^2 \left\{ 1 - \left[1 - \frac{Q\hat{\ }(\varepsilon, \alpha, a)}{\Pi(\varepsilon)^\#} \right] \right\} \\ &= \Omega^\# \{1 - r_1(\varepsilon)\}, \end{aligned}$$

where

$$\Omega^\# r_1(\varepsilon) = \sum_{\mathbf{A}_K} d(\alpha) \sum_a \sum_{i \in I(\alpha, a)} \sum_{j=1}^{d(\alpha)} |\hat{\chi_\Omega}(\alpha, j, i)|^2 \left[1 - \frac{Q\hat{\ }(\varepsilon, \alpha, a)}{\Pi(\varepsilon)^\#} \right].$$

It follows from (4.4) that

$$Q\hat{\ }(\varepsilon, \alpha, a) \geqq 0, \qquad (\alpha, a) \in \mathbf{A}_K.$$

Moreover, it follows from (4.4′) that

$$Q\hat{\ }(\varepsilon, \alpha, a) \leqq \int_G |P(\varepsilon, x)|^2\,dx = \sum_{\Pi(\varepsilon)} d(\alpha)\, d_K(\alpha, a) = \Pi(\varepsilon)^\#.$$

Thus

$$0 \leqq 1 - \frac{Q\hat{\ }(\varepsilon, \alpha, a)}{\Pi(\varepsilon)^{\#}} \leqq 1,$$

and by assumption (4.5),

$$\lim_{\varepsilon \to 0+} \frac{Q\hat{\ }(\varepsilon, \alpha, a)}{\Pi(\varepsilon)^{\#}} = 1.$$

The Lebesgue dominated convergence theorem implies that $r_1(\varepsilon) \to 0+$ as $\varepsilon \to 0+$.

We derive a simple easily verified sufficient condition for (4.5) to hold. Let us write $\beta \in \Pi_1(\varepsilon)$ if $(\beta, b) \in \Pi(\varepsilon)$ for every b. For $\varepsilon > 0$ and $\gamma \in \hat{\mathbf{A}}_K$ we define $\mathrm{Int}_\gamma[\Pi(\varepsilon)]$ to be all (α, a)'s in $\Pi(\varepsilon)$ such that each irreducible component of $U(\alpha, \cdot) \otimes U(\gamma, \cdot)$ is unitarily equivalent to some $U(\beta)$, $\beta \in \Pi_1(\varepsilon)$. Set

$$\mathrm{Bd}_\gamma[\Pi(\varepsilon)] = \Pi[\varepsilon] \backslash \mathrm{Int}_\gamma[\Pi(\varepsilon)].$$

If $(\alpha, a) \in \mathrm{Int}_\gamma[\Pi(\varepsilon)]$, then by Corollary 2.10, $D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c) = 0$ for $(\beta, b) \notin \Pi(\varepsilon)$ and every c. We have

$$Q\hat{\ }(\varepsilon, \gamma, c) = \textstyle\sum_1 + \sum_2,$$

where

$$\textstyle\sum_1 = d_K(\gamma, c)^{-1} \displaystyle\sum_{(\alpha,c) \in \mathrm{Int}_\gamma[\Pi(\varepsilon)]} d(\alpha) \sum_{(\beta,b) \in \Pi(\varepsilon)} d(\beta) D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c),$$

$$\textstyle\sum_2 = d_K(\gamma, c)^{-1} \displaystyle\sum_{(\alpha,c) \in \mathrm{Bd}_\gamma[\Pi(\varepsilon)]} d(\alpha) \sum_{(\beta,b) \in \Pi(\varepsilon)} d(\beta) D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c).$$

By the remark above and Corollary 2.11,

$$\begin{aligned} \textstyle\sum_1 &= d_K(\gamma, c)^{-1} \sum_{(\alpha,c) \in \mathrm{Int}_\gamma[\Pi(\varepsilon)]} d(\alpha) \sum_{(\beta,b) \in \hat{\mathbf{A}}_K} d(\beta) D(\alpha, a, \bar{\beta}, \bar{b}, \gamma, c) \\ &= d_K(\gamma, c)^{-1} \sum_{(\alpha,c) \mathrm{Int}_\gamma[\Pi(\varepsilon)]} d(\alpha)\, d_K(\alpha, a)\, d_K(\gamma, c) = \sum_{\mathrm{Int}_\gamma[\Pi(\varepsilon)]} d(\alpha)\, d_K(\alpha, c). \end{aligned}$$

Since $\sum_2 \geqq 0$ we have

$$Q\hat{\ }(\varepsilon, \gamma, c) \geqq \sum_{\mathrm{Int}_\gamma[\Pi(\varepsilon)]} d(\alpha)\, d_K(\alpha, c).$$

Referring to (4.2) we see that

$$1 - \frac{Q\hat{\ }(\varepsilon, \gamma, c)}{\Pi(\varepsilon)^{\#}} \leqq \sum_{\mathrm{Bd}_\gamma[\Pi(\varepsilon)]} d(\alpha)\, d_K(\alpha, c)/\Pi(\varepsilon)^{\#}.$$

We have thus proved the following.

THEOREM 4.2. *The condition* (4.5) *is satisfied if, for each* $\gamma \in \mathbf{A}_K$,

$$\lim_{\varepsilon \to 0+} \sum_{(\alpha,a) \in \mathrm{Bd}_\gamma[\Pi(\varepsilon)]} d(\alpha)\, d_K(\alpha, a)/\Pi(\varepsilon)^{\#} = 0.$$

Let $N^+[\delta, P_\Omega(\varepsilon)]$ be the number of the eigenvalues $\lambda(\varepsilon, k)$ of $P_\Omega(\varepsilon)$ greater than δ.

THEOREM 4.3. *If* (4.5) *holds and if* $r_1(\varepsilon)$ *is defined as in Theorem* 4.1, *then for* $r_1(\varepsilon) \leqq \frac{1}{2}$,

$$N^+[1 - r_1(\varepsilon)^{1/2}, P_\Omega(\varepsilon)] \gtrsim \Omega^\# \Pi(\varepsilon)^\# [1 - 4r_1(\varepsilon)^{1/2}]$$

and

$$N^+[r_1(\varepsilon)^{1/2}, P_\Omega(\varepsilon)] \lesssim \Omega^\# \Pi(\varepsilon)^\# [1 + 4r_1(\varepsilon)^{1/2}]$$

as $\varepsilon \to 0+$.

Note $a(\varepsilon) \lesssim b(\varepsilon)$ means that $\limsup_{\varepsilon \to 0+} a(\varepsilon)/b(\varepsilon) \leqq 1$, etc.

Proof. Let us define

$$S_1(\varepsilon) = \{k : \lambda(\varepsilon, k) < r_1(\varepsilon)^{1/2}\},$$
$$S_2(\varepsilon) = \{k : r_1(\varepsilon)^{1/2} \leqq \lambda(\varepsilon, k) \leqq 1 - r_1(\varepsilon)^{1/2}\},$$
$$S_3(\varepsilon) = \{k : 1 - r_1(\varepsilon)^{1/2} < \lambda(\varepsilon, k)\}.$$

We denote by $S_i(\varepsilon)^\#$ the number of elements in the set $S_i(\varepsilon)$.

Subtracting (iii) from (ii) in Theorem 4.1 we obtain

$$\sum_{S_2(\varepsilon)} \lambda(\varepsilon, k)[1 - \lambda(\varepsilon, k)] \leqq \sum_k \lambda(\varepsilon, k)[1 - \lambda(\varepsilon, k)] = r_1(\varepsilon)\Omega^\# \Pi(\varepsilon)^\#.$$

The terms in the sum on the left are nonnegative and exceed $\frac{1}{2}r_1(\varepsilon)^{1/2}$ if $r_1(\varepsilon)^{1/2} \leqq \frac{1}{2}$. Consequently,

$$\tfrac{1}{2}r_1(\varepsilon)^{1/2} S_2(\varepsilon)^\# \leqq r_1(\varepsilon)\Omega^\# \Pi(\varepsilon)^\#,$$
$$S_2(\varepsilon)^\# \leqq 2r_1(\varepsilon)^{1/2}\Omega^\# \Pi(\varepsilon)^\#.$$

We have

$$\sum_{S_1(\varepsilon)} \lambda(\varepsilon, k)^2 \leqq r_1(\varepsilon)^{1/2} \sum_k \lambda(\varepsilon, k) = r_1(\varepsilon)^{1/2}\Omega^\# \Pi(\varepsilon)^\#,$$
$$\sum_{S_2(\varepsilon)} \lambda(\varepsilon, k)^2 \leqq S_2(\varepsilon)^\# \leqq 2r_1(\varepsilon)^{1/2}\Omega^\# \Pi(\varepsilon)^\#,$$
$$\sum_{S_3(\varepsilon)} \lambda(\varepsilon, k)^2 \leqq S_3(\varepsilon)^\#.$$

Inserting these estimates in (iii) of Theorem 4.1 we find that, since

$$\sum_{S_3(\varepsilon)} \lambda(\varepsilon, k^2) = \Omega^\# \Pi(\varepsilon)^\# [1 - r_1(\varepsilon)] - \sum_{S_2(\varepsilon)} \lambda(\varepsilon, k)^2 - \sum_{S_1(\varepsilon)} \lambda(\varepsilon, k)^2,$$

we have

$$S_3(\varepsilon)^\# \geqq \Omega^\# \Pi(\varepsilon)^\# [1 - r_1(\varepsilon) - 3r_1(\varepsilon)^{1/2}] \geqq \Omega^\# \Pi^\# [1 - 4r_1(\varepsilon)^{1/2}].$$

This is the first assertion of our theorem.

We have

$$[1 - r_1(\varepsilon)^{1/2}] S_3(\varepsilon)^\# \leqq \sum_{S_3(\varepsilon)} \lambda(\varepsilon, k), \qquad 0 \leqq \sum_{S_1(\varepsilon)} \lambda(\varepsilon, k) + \sum_{S_2(\varepsilon)} \lambda(\varepsilon, k).$$

Inserting these in (ii) of Theorem 4.1 we obtain

$$S_3(\varepsilon)^\# \leqq \Omega^\# \Pi(\varepsilon)^\# [1 - r_1(\varepsilon)]^{-1/2} \leqq \Omega^\# \Pi(\varepsilon)^\# [1 + 2r_1(\varepsilon)^{1/2}]$$

if $r_1(\varepsilon) \leqq 1/2$. We now have

$$S_2(\varepsilon)^{\#} + S_3(\varepsilon)^{\#} \leqq \Omega^{\#}\Pi(\varepsilon)^{\#}[1 + 4r_1(\varepsilon)^{1/2}]$$

which is our second assertion.

Let Ω_1 and Ω_2 be disjoint measurable K right-invariant sets and let

$$P_{\Omega_1,\Omega_2}(\varepsilon) \leftrightarrow \chi_{\Omega_1}(x)P(\varepsilon, y^{-1}x)\chi_{\Omega_2}(y) + \chi_{\Omega_2}(x)P(\varepsilon, y^{-1}x)\chi_{\Omega_1}(y). \tag{4.6}$$

(The operator on the left corresponds to the kernel on the right.) Let $N[\delta, P_{\Omega_1,\Omega_2}(\varepsilon)]$ denote the number of eigenvalues of $P_{\Omega_1,\Omega_2}(\varepsilon)$ which are greater than δ in absolute value.

THEOREM 4.4. *If* (4.5) *holds, then, for* $\varepsilon, \delta > 0$,

$$N[\delta, P_{\Omega_1,\Omega_2}(\varepsilon)] = s(\varepsilon)\delta^{-2}\Pi(\varepsilon)^{\#},$$

where $s(\varepsilon) \to 0+$ *as* $\varepsilon \to 0+$.

Proof. Let $\{\lambda(\varepsilon, k)\}$ denote the eigenvalues of $P_{\Omega_1,\Omega_2}(\varepsilon)$. By a computation like the one used to obtain (iii) of Theorem 4.1 we find that

$$\sum_k \lambda(\varepsilon, k)^2 = \sum_{\mathbf{A}_K} d(\alpha)\, 2\mathrm{Re}\left[\sum_a \sum_{i\in I(\alpha,a)} \sum_{j=1}^{d(\alpha)} \overline{\hat{\chi_{\Omega_2}}(\alpha, j, i)}\hat{\chi_{\Omega_1}}(\alpha, j, i)\right] Q\hat{\ }(\varepsilon, \alpha, a),$$

which in turn implies that

$$\delta^2 N[\delta, P_{\Omega_1,\Omega_2}(\delta)]/\Pi(\varepsilon)^{\#} < s(\varepsilon),$$

where

$$s(\varepsilon) = \left|\sum_{\mathbf{A}_K} d(\alpha)\, 2\mathrm{Re}\left[\sum_a \sum_{i\in I(\alpha,a)} \sum_{j=1}^{d(\alpha)} \overline{\hat{\chi_{\Omega_2}}(\alpha, j, i)}\hat{\chi_{\Omega_2}}(\alpha, j, i)\right] \frac{Q\hat{\ }(\varepsilon, \alpha, a)}{\Pi(\varepsilon)^{\#}}\right|.$$

Since the series

$$\sum_{\mathbf{A}_K} d(\alpha)\, 2\mathrm{Re}\left[\sum_a \sum_{i\in I(\alpha,a)} \sum_{j=1}^{d(\alpha)} \overline{\hat{\chi_{\Omega_2}}(\alpha, j, i)}\hat{\chi_{\Omega_2}}(\alpha, j, i)\right]$$

converges absolutely and since its sum is

$$2\int_G \chi_{\Omega_2}(y)\chi_{\Omega_1}(y)\, dy = 0,$$

it follows from (4.5) and the Lebesgue dominated convergence theorem that $s(\varepsilon) \to 0$ as $\varepsilon \to 0+$.

COROLLARY 4.5. *If* (4.5) *holds, then there exists a function* $t(\varepsilon)$ *depending upon* $\Pi(\varepsilon), \Omega_1$ *and* Ω_2 *such that* $t(\varepsilon) \to 0+$ *as* $\varepsilon \to 0+$ *and such that*

$$N[t(\varepsilon), P_{\Omega_1,\Omega_2}(\varepsilon)] = o(\Pi(\varepsilon)^{\#}) \quad \text{as } \varepsilon \to 0+.$$

Proof. This follows immediately from Theorem 4.4.

5. The main theorem. We need various elementary inequalities connecting the eigenvalues of suitably related operators. For A a completely continuous self-adjoint operator on $L^2[G/K]$ and $\varepsilon > 0$ let $N^{\pm}(\varepsilon, A)$ be the number of eigenvalues of A which are $>\varepsilon$, $<-\varepsilon$, respectively.

Let the $A_j, j = 1, \cdots, n$, be completely continuous self-adjoint operators on $L^2[G/K]$ and let $\varepsilon_j \geqq 0, j = 1, \cdots, n$. It follows from the minimax characterization of the eigenvalues of such operators (see [2, pp. 132–134]) that

$$N^{\pm}\left[\sum_{j=1}^{n} \varepsilon_j, \sum_{j=1}^{n} A_j\right] \leqq \sum_{j=1}^{n} N^{\pm}[\varepsilon_j, A_j]. \tag{5.1}$$

It also follows that if $0 \leqq A_1 \leqq A_2$, then

$$N^{+}[\varepsilon, A_1] \leqq N^{+}[\varepsilon, A_2]. \tag{5.2}$$

For $V(x)$ and $V_1(x)$ nonnegative functions in $L^\infty[G/K]$ and $f \in L^2[G/K]$ let

$$M_V f \cdot (x) = V(x) f(x), \qquad M_{V_1} f \cdot (x) = V_1(x) f(x).$$

Let $L\hat{\ }(\alpha, a)$ and $L_0\hat{\ }(\alpha, a)$ be nonnegative functions on $\mathbf{A}_K$ which vanish at ∞; that is, given $\varepsilon > 0$ there exists a finite subset $\mathbf{A}_K(\varepsilon)$ of $\mathbf{A}_K$ with the property

$$L\hat{\ }(\alpha, a) < \varepsilon, \quad L_0\hat{\ }(\alpha, a) < \varepsilon \quad \text{if } (\alpha, a) \notin \mathbf{A}_K(\varepsilon).$$

The formula

$$R_{L\hat{}} f \cdot (x) = \sum_{\mathbf{A}_K} d(\alpha) \sum_a L\hat{\ }(\alpha, a) \sum_{j \in I(\alpha,a)} \sum_{i=1}^{d(\alpha)} f\hat{\ }(\alpha, i, j) g(\alpha, i, j, x),$$

where $f \in L^2[G/K]$, and the analogous formula for $R_{L_0\hat{}}$ define $R_{L\hat{}}$ and $R_{L_0\hat{}}$ as self-adjoint, nonnegative, completely continuous operators on $L^2[G/K]$. It is convenient to use the purely formal notation

$$T \leftrightarrow V(x) L(y^{-1}x) V(y)$$

to indicate that $T = M_V R_{L\hat{}} M_V$. If

$$T_1 \leftrightarrow V_1(x) L(y^{-1}x) V_1(y), \qquad T_0 \leftrightarrow V(x) L_0(y^{-1}x) V(y),$$

then T, T_0 and T_1 are all self-adjoint nonnegative, completely continuous operators on $L^2[G/K]$.

LEMMA 5.1. *If* $L\hat{\ }(\alpha, a) \leqq L_0\hat{\ }(\alpha, a)$ *for all* $(\alpha, a) \in \mathbf{A}_K$, *then* $N^+[\varepsilon, T] \leqq N^+[\varepsilon, T_0]$.

Proof. By (5.2) it is enough to show that $T \leqq T_0$. Given $\varphi \in L^2[G/K]$ let $\psi(x) = V(x)\varphi(x)$. Then $\psi \in L^2[G/K]$ and

$$\psi(x) = \sum_{\mathbf{A}_K} d(\alpha) \sum_{i=1}^{d_K(\alpha)} \sum_{j=1}^{d(\alpha)} \psi\hat{\ }(\alpha, j, i) g(\alpha, j, i, x),$$

where this series converges unconditionally in $L^2[G/K]$. From

$$\langle T\varphi | \varphi \rangle = \sum_{\mathbf{A}_K} d(\alpha) \sum_a L\hat{\ }(\alpha, a) \sum_{i \in I(\alpha,a)} \sum_{j=1}^{d(\alpha)} |\psi\hat{\ }(\alpha, j, i)|^2$$

and

$$\langle T_0\varphi | \varphi \rangle = \sum_{\mathbf{A}_K} d(\alpha) \sum_a L_0\hat{\ }(\alpha, a) \sum_{i \in I(\alpha,a)} \sum_{j=1}^{d(\alpha)} |\psi\hat{\ }(\alpha, j, i)|^2,$$

it is apparent that $\langle T\varphi | \varphi \rangle \leqq \langle T_0\varphi | \varphi \rangle$, etc.

LEMMA 5.2. *If* $V(x) \leqq V_1(x)$ *for all* $x \in G$, *then* $N^+[\varepsilon, T] \leqq N^+[\varepsilon, T_1]$.

Proof. If C is the operator on $L^2[G/K]$ defined by

$$Cf \cdot (x) = \frac{V(x)}{V_1(x)} f(x),$$

where 0/0 is defined to be 0, then C is a self-adjoint contraction, $\|Cf\| \leqq \|f\|$ for all $f \in L^2[G/K]$. We have

$$T = CT_1C,$$

where T_1 is positive and completely continuous. Our assertion is an immediate consequence of this and the minimax principle.

Let Ω be a measurable K right-invariant set in G and let $L^\frown(\alpha, a)$ be as above. Put

$$T_\Omega \leftrightarrow \chi_\Omega(x)L(y^{-1}x)\chi_\Omega(y). \tag{5.3}$$

Let us list some conditions on L:

(5.4)
- (i) if $\Pi(\varepsilon) = \{(\alpha, a) \in \mathbf{A}_K : L^\frown(\alpha, a) > \varepsilon\}$, then for each $(\gamma, c) \in \mathbf{A}_K$, $Q^\frown(\varepsilon, \gamma, c)/\Pi(\varepsilon)^\# \to 1$ as $\varepsilon \to 0+$;
- (ii) $\Pi(\varepsilon_1)^\# = o[\Pi(\varepsilon)]$ as $\varepsilon_1, \varepsilon \to 0+$ if $\varepsilon = o(\varepsilon_1)$;
- (iii) for $\theta > 0$ fixed, $\Pi(\theta\varepsilon)^\# = O[\Pi(\varepsilon)^\#]$ as $\varepsilon \to 0+$. By $a(\varepsilon) \gtrsim b(\varepsilon)$ as $\varepsilon \to 0+$ we mean that

$$\liminf_{\varepsilon \to 0+} a(\varepsilon)/b(\varepsilon) \geqq 1,$$

etc.

LEMMA 5.3. *If assumption* (i) *of* (5.4) *is satisfied, then for every* $0 < \delta < 1$ *we have*

$$N^+[\varepsilon, T_\Omega] \gtrsim \Omega^\# \Pi\left(\frac{\varepsilon}{1-\delta}\right)^\# \quad \textit{as } \varepsilon \to 0+.$$

Proof. For any $\varepsilon > 0$ we have

$$L^\frown(\alpha, a) \geqq \varepsilon P^\frown(\varepsilon, \alpha, a), \qquad (\alpha, a) \in \mathbf{A}_K,$$

where $P^\frown(\varepsilon, \alpha, a)$ is, as before, the characteristic function of $\Pi(\varepsilon)$. By Lemma 5.1.

$$T_\Omega \geqq \varepsilon P_\Omega(\varepsilon).$$

By (5.2) and Theorem 4.3,

$$N^+[\varepsilon(1-\delta), T_\Omega] \geqq N^+[\varepsilon(1-\delta), \varepsilon P_\Omega(\varepsilon)] = N^+[1-\delta, P_\Omega(\varepsilon)] \gtrsim \Pi(\varepsilon)^\# \Omega^\#.$$

An evident change of variable now gives the desired result.

LEMMA 5.4. *If assumptions* (i) *and* (ii) *of* (5.4) *are satisfied, then for every* $0 < \delta$ *we have*

$$N^+[\varepsilon, T_\Omega] \gtrsim \Omega^\# \Pi\left(\frac{\varepsilon}{1+\delta}\right)^\# \quad \textit{as } \varepsilon \to 0+.$$

Proof. Let $\varepsilon_1 > 0$ be a function of ε. We shall explain in the course of the argument how ε_1 is to be chosen. In all that follows

$$A = \operatorname*{l.u.b.}_{\mathbf{A}_K} L^\frown(\alpha, a).$$

Since if $0 < \varepsilon < \varepsilon_1 < A$ we have

$$L^{\wedge}(\alpha, a) \leqq \varepsilon + \varepsilon_1 P^{\wedge}(\varepsilon, \alpha, a) + AP^{\wedge}(\varepsilon_1, \alpha, a), \qquad \alpha, a \in \mathbf{A}_K,$$

it follows from Lemma 5.1 and (5.1) that for $\delta > 0$,

$$N^+[(1+\delta)\varepsilon, T_\Omega] \leqq N^+[\varepsilon, \varepsilon I] + N^+[\delta\varepsilon, \varepsilon_1 P_\Omega(\varepsilon)] + N^+[0, AP_\Omega(\varepsilon_1)]$$
$$\leqq N^+[\delta\varepsilon\varepsilon_1^{-1}, P_\Omega(\varepsilon)] + \Pi(\varepsilon_1)^{\#},$$

since $N^+[\varepsilon, \varepsilon I] = 0$, and since $P_\Omega(\varepsilon_1)$ has rank not exceeding $\Pi(\varepsilon_1)^{\#}$. If we now choose ε_1 to be a function of ε satisfying

$$\varepsilon = o(\varepsilon_1) \quad \text{as } \varepsilon \to 0+,$$

then it follows from assumption (ii) of (5.4) that

$$\Pi(\varepsilon_1)^{\#} = o[\Pi(\varepsilon)]^{\#} \quad \text{as } \varepsilon \to 0+.$$

Let $r(\varepsilon) = r_1(\varepsilon)^{1/2}$ where $r_1(\varepsilon)$ is as in Theorems 4.1 and 4.3. If

$$\varepsilon_1 = o(\varepsilon r(\varepsilon)^{-1}) \quad \text{as } \varepsilon \to 0+,$$

it follows that $r(\varepsilon) \leqq \delta\varepsilon\varepsilon_1^{-1}$ for ε sufficiently small, and thus

$$N^+[\delta\varepsilon\varepsilon_1^{-1}, P_\Omega(\varepsilon)] \leqq N^+[r(\varepsilon), P_\Omega(\varepsilon)] \lesssim \Omega^{\#}\Pi(\varepsilon)^{\#} \quad \text{as } \varepsilon \to 0+.$$

It is clear that we can choose ε_1 so that both of the conditions listed above are satisfied.

COROLLARY 5.5. *If assumptions* (i) *and* (ii) *are satisfied, then for every* $0 < \delta < 1$ *we have*

$$N^+[\varepsilon, T_\Omega] \gtrsim \Omega^{\#}\Pi\left(\frac{\varepsilon}{1-\delta}\right) \quad \textit{as } \varepsilon \to 0+,$$

$$N^+[\varepsilon, T_\Omega] \lesssim \Omega^{\#}\Pi\left(\frac{\varepsilon}{1+\delta}\right)^{\#} \quad \textit{as } \varepsilon \to 0+.$$

We have paused to prove this special case of our general theorem partly because of its intrinsic interest and partly because we are able to obtain it so very cheaply. Our main result is not a consequence of the arguments we have just given. But it is proved by similar (more complicated) arguments.

We begin by considering the case where

$$V(x) = \sum_{i=1}^{n} h_i \chi_{\Omega_i}(x); \tag{5.5}$$

here the $\{\Omega_i\}_{i=1}^n$ are disjoint measurable K right-invariant sets in G, and $h_1 > h_2 > \cdots > h_n > 0$. Let

$$T_V \leftrightarrow V(x)L(y^{-1}x)V(y). \tag{5.6}$$

THEOREM 5.6. *If* $V(x)$ *is given by* (5.5) *and* T_V *by* (5.6), *and if* L *satisfies the conditions* (5.4), *then for* $0 < \delta < 1$,

$$N^+[\varepsilon, T_V] \gtrsim \Psi\left[\frac{\varepsilon}{1-\delta}\right], \qquad N^+[\varepsilon, T_V] \lesssim \Psi\left[\frac{\varepsilon}{1+\delta}\right] \quad \textit{as } \varepsilon \to 0+,$$

where

$$\Psi(x) = |\{(x, \alpha, a): V(x)^2 L\hat{}(\alpha, a) > \varepsilon\}|_{G \times \mathbf{A}_K}.$$

Proof. Let us prove the first inequality for $n = 3$. If

$$R\hat{}(\varepsilon, \alpha, a) = \varepsilon h_1^{-2} P\hat{}(\varepsilon h_1^{-2}, \alpha, a) + \varepsilon(h_2^{-2} - h_1^{-2}) P\hat{}(\varepsilon h_2^{-2}, \alpha, a) + \varepsilon(h_3^{-2} - h_2^{-2}) \cdot P\hat{}(\varepsilon h_3^{-2}, \alpha, a),$$

then

$$L\hat{}(\alpha, a) \geqq R\hat{}(\varepsilon, \alpha, a), \qquad (\alpha, a) \in \mathbf{A}_K.$$

This can be verified by checking the cases:

$$0 < L\hat{}(\alpha, a) \leqq h_1^{-2}\varepsilon, \qquad h_1^{-2}\varepsilon < L\hat{}(\alpha, a) \leqq h_2^{-2}\varepsilon,$$
$$h_2^{-2}\varepsilon < L\hat{}(\alpha, a) \leqq h_3^{-2}\varepsilon, \qquad h_3^{-2}\varepsilon < L\hat{}(\alpha, a).$$

Consequently, if

$$R_V(\varepsilon) \leftrightarrow V(x) R(\varepsilon, y^{-1}x) V(y),$$

then by Lemma 5.1,

$$T_V \geqq R_V(\varepsilon).$$

If we set

$$R_i(\varepsilon) \leftrightarrow h_i^2 \chi_{\Omega_i}(x) R(\varepsilon, y^{-1}x) \chi_{\Omega_i}(y), \qquad i = 1, 2, 3,$$

then

$$R_V(\varepsilon) = \sum_{i=1}^{3} R_i(\varepsilon) + \sum_{\nu=1}^{3} \sum_{1 \leqq i < j \leqq 3} Q_{i,j}^{(\nu)}(\varepsilon),$$

where

$$Q_{i,j}^{(1)}(\varepsilon) = h_i h_j \varepsilon h_1^{-2} P_{i,j}(\varepsilon h_1^{-2}),$$
$$Q_{i,j}^{(2)}(\varepsilon) = h_i h_j \varepsilon (h_2^{-2} - h_1^{-2}) P_{i,j}(\varepsilon h_2^{-2}),$$
$$Q_{i,j}^{(3)}(\varepsilon) = h_i h_j \varepsilon (h_3^{-2} - h_2^{-2}) P_{i,j}(\varepsilon h_3^{-2}).$$

Here $P_{i,j}(\varepsilon) = P_{\Omega_i, \Omega_j}(\varepsilon)$ is defined as in (4.6). We have

$$T_V \geqq \sum_{i=1}^{3} R_i(\varepsilon) + \sum_{\nu=1}^{3} \sum_{1 \leqq i < j \leqq 3} Q_{i,j}^{(\nu)}(\varepsilon)$$

from which it follows that

$$\sum_{i=1}^{3} R_i(\varepsilon) \leqq T_V - \sum_{\nu=1}^{3} \sum_{1 \leqq i < j \leqq 3} Q_{i,j}^{(\nu)}(\varepsilon).$$

From this and from (5.1) we obtain

$$N^+[\varepsilon(1 - \delta), \sum_{i=1}^{3} R_i(\varepsilon)] \leqq N^+[\varepsilon(1 - 10\delta), T_V] + \sum_{\nu=1}^{3} \sum_{1 \leqq i, j \leqq 3} N[\varepsilon\delta, Q_{i,j}^{(\nu)}(\varepsilon)].$$

Using Theorem 4.4 and condition (iii) of (5.4) we have

$$N^+[\varepsilon\delta, Q_{i,j}^{(k)}(\varepsilon)] = o(\Pi(\varepsilon h_3^{-2})^{\#}) \quad \text{as } \varepsilon \to 0+$$

for $k = 1, 2, 3$, $1 \leqq i < j \leqq 3$.

Since $\sum_{i=1}^{3} R_i(\varepsilon)$ is direct,

$$N^+[\varepsilon(1-\delta), \sum_{i=1}^{3} R_i(\varepsilon)] = \sum_{i=1}^{3} N^+[\varepsilon(1-\delta), R_i(\varepsilon)].$$

It is not difficult to verify that

$$R\hat{}(\varepsilon, \alpha, a) \geqq \varepsilon h_i^{-2} P\hat{}(\varepsilon h_i^{-2}, \alpha, a), \qquad i = 1, 2, 3,$$

from which it follows by Lemma 5.1 that

$$R_i(\varepsilon) \geqq \varepsilon P_i(\varepsilon h_i^{-2}).$$

Here $P_i(\varepsilon) = P_{\Omega_i}(\varepsilon)$ as defined in Theorem 4.1. The argument used to prove Lemma 5.1 implies that

$$N^+[\varepsilon(1-\delta), R_i(\varepsilon)] \gtrsim \Omega_i^{\#} \Pi(\varepsilon h_i^{-2})^{\#}, \qquad i = 1, 2, 3.$$

Combining our results we have shown that

$$N^+[\varepsilon(1-10\delta), T_V] \gtrsim \sum_{i=1}^{3} \Omega_i^{\#} \Pi(\varepsilon h_i^{-2})^{\#} = \Psi(\varepsilon) \quad \text{as } \varepsilon \to 0+,$$

etc.

We now turn to the proof of the second inequality. We again take $n = 3$. Let

$$R\hat{}(\varepsilon, \alpha, a) = \varepsilon h_1^{-2} + \varepsilon(h_2^{-2} - h_1^{-2}) P\hat{}(\varepsilon h_1^{-2}, \alpha, a) + \varepsilon(h_3^{-2} - h_2^{-2}) P\hat{}(\varepsilon h_2^{-2}, \alpha, a)$$
$$+ (\varepsilon_1 - \varepsilon h_3^{-2}) P\hat{}(\varepsilon h_3^{-2}, \alpha, a) + A P\hat{}(\varepsilon_1, \alpha, a).$$

Here $A = \text{l.u.b. } L\hat{}(\alpha, a)$ and ε_1 is a function of ε such that $\varepsilon_1 \to 0+$, $\varepsilon = o(\varepsilon_1)$ as $\varepsilon \to 0+$; ε_1 will be specified precisely later. It is not difficult to check that

$$L\hat{}(\alpha, a) \leqq R\hat{}(\varepsilon, \alpha, a), \qquad (\alpha, a) \in \mathbf{A}_K.$$

Consequently, if

$$R_V(\varepsilon) \leftrightarrow V(x) R(\varepsilon, y^{-1}x) V(y),$$

then

$$T_V \leqq R_V(\varepsilon).$$

If we set

$$R_i(\varepsilon) \leftrightarrow h_i^2 \chi_{\Omega_i}(x) R(\varepsilon, y^{-1}x) \chi_{\Omega_i}(y), \qquad i = 1, 2, 3,$$

then

$$R_V(\varepsilon) = R_1(\varepsilon) + R_2(\varepsilon) + R_3(\varepsilon) + \sum_{\nu=1}^{4} \sum_{1 \leqq i < j \leqq 3} Q_{ij}^{(\nu)}(\varepsilon),$$

where

$$Q_{i,j}^{(1)}(\varepsilon) = h_i h_j \varepsilon(h_2^{-2} - h_1^{-2})P_{i,j}(\varepsilon h_1^{-2}),$$
$$Q_{i,j}^{(2)}(\varepsilon) = h_i h_j \varepsilon(h_3^{-2} - h_2^{-2})P_{i,j}(\varepsilon h_2^{-2}),$$
$$Q_{i,j}^{(3)}(\varepsilon) = h_i h_j(\varepsilon_1 - \varepsilon h_3^{-2})P_{i,j}(\varepsilon h_3^{-2}) \cdot$$
$$Q_{i,j}^{(4)} = h_i h_j A P_{i,j}(\varepsilon_1).$$

Note that the mixed term corresponding to εh_1^{-2} in $R\hat{}(\varepsilon, \alpha, a)$ is identically 0. Using (5.1) we see that

$$N^+[\varepsilon(1 + 10\delta), T_V] \leqq N^+\left[\varepsilon(1 + \delta), \sum_{i=1}^{3} R_i(\varepsilon)\right] + \sum_{\nu=1}^{3} \sum_{1 \leqq i < j \leqq 3} N^+[\varepsilon\delta, Q_{i,j}^{(\nu)}(\varepsilon)]$$
$$+ \sum_{1 \leqq i < j \leqq 3} N^+[0, Q_{i,j}^{(4)}(\varepsilon)].$$

Moreover, since the sum $R_1(\varepsilon) + R_2(\varepsilon) + R_3(\varepsilon)$ is direct,

$$N^+\left[\varepsilon(1 + \delta), \sum_{i=1}^{3} R_i(\varepsilon)\right] = \sum_{i=1}^{3} N^+[\varepsilon(1 + \delta), R_i(\varepsilon)].$$

By Theorem 4.4 and condition (iii) of (5.4),

$$N^+[\varepsilon\delta, h_i h_j \varepsilon(h_2^{-2} - h_1^{-2})P_{i,j}(\varepsilon h_1^{-2})] = o[\Pi(\varepsilon h_3^{-2})^{\#}],$$
$$N^+[\varepsilon\delta, h_i h_j \varepsilon(h_3^{-2} - h_2^{-2})P_{i,j}(\varepsilon h_2^{-2})] = o[\Pi(\varepsilon h_3^{-2})^{\#}].$$

By Corollary 4.5 and condition (iii) of (5.4) there exists a function $t_{i,j}^{(1)}(\varepsilon) \to 0+$ as $\varepsilon \to 0+$ such that if

(5.7) $$\varepsilon_1 = o(\varepsilon t_{i,j}^{(1)}(\varepsilon)^{-1}) \quad \text{as } \varepsilon \to 0+,$$

then

$$N^+[\varepsilon\delta, h_i h_j(\varepsilon_1 - \varepsilon h_3^{-2})P_{i,j}(\varepsilon h_3^{-2})] = o[\Pi(\varepsilon h_3^{-2})^{\#}].$$

Moreover, since the rank of $P_{i,j}(\varepsilon_1)$ does not exceed $2\Pi(\varepsilon_1)^{\#}$, it follows that

$$N^+[0, h_i h_j A P_{ij}(\varepsilon_1)] \leqq 2\Pi(\varepsilon_1)^{\#} = o[\Pi(\varepsilon h_3^{-2})^{\#}]$$

if

(5.8) $$\varepsilon = o(\varepsilon_1)$$

by condition (ii) of (5.4).

It is not hard to check that

$$h_k^2 R(\varepsilon, \alpha, a) \leqq \varepsilon + h_k^2 \varepsilon_1 P\hat{}(\varepsilon h_k^{-2}, \alpha, a) + h_k^2 A P\hat{}(\varepsilon_1, \alpha, a)$$

for $k = 1, 2, 3$. Arguing as in the proof of Lemma 5.4 we see that

$$N^+[\varepsilon(1 + \delta), R_k(\varepsilon)] \lesssim \Omega^{\#}\Pi(\varepsilon h_k^{-2})^{\#} \quad \text{as } \varepsilon \to 0+$$

provided

(5.9) $$\varepsilon_1 = O(\varepsilon t_k(\varepsilon)^{-1}) \quad \text{as } \varepsilon \to 0+,$$

where $t_k(\varepsilon) \to 0+$ as $\varepsilon \to 0+$, and provided (5.8) holds.

Let us choose, as we clearly may, ε_1 to be a function of ε such that (5.7), (5.8) and (5.9) all hold. Collecting results we see that

$$N^+[\varepsilon(1 + 10\delta), T_V] \lesssim \sum_{k=1}^{3} \Pi(\varepsilon h_k^{-2})^{\#}\Omega_k^{\#} = \Psi(\varepsilon) \quad \text{as } \varepsilon \to 0+.$$

THEOREM 5.7. *Let $V(x)$ be a nonnegative function in $L^\infty[G/K]$ and let $L^\wedge(\alpha, a)$ be a nonnegative function on $\mathbf{A}_K$ which vanishes at infinity. Assume further that* (5.4) *holds. If $T_V \leftrightarrow V(x)L(y^{-1}x)V(y)$, then for $0 < \delta < 1$:*

$$\text{(5.10)} \qquad \begin{aligned} &\text{(i)} \quad N^+[\varepsilon, T_V] \gtrsim \Psi\left(\frac{\varepsilon}{1-\delta}\right) \quad \textit{as } \varepsilon \to 0+; \\ &\text{(ii)} \quad N^+[\varepsilon, T_V] \lesssim \Psi\left(\frac{\varepsilon}{1+\delta}\right) \quad \textit{as } \varepsilon \to 0+. \end{aligned}$$

Proof. Given $\eta > 0$ there exist a measurable K right-invariant set Ω and a function $V_S(x)$ of the form (5.5) such that:

(a) $\quad \Omega^{\#} < \eta;$

(b) $\quad V(x) \leqq V(y), \qquad x \in \Omega, \quad y \notin \Omega;$

(c) $\quad V_S(x) \leqq V(x), \qquad x \in G;$

(d) $\quad V(x) \leqq (1 + \eta)V_S(x), \qquad x \notin \Omega.$

If we make use successively of (c), (5.2), Lemma 5.2, and Theorem 5.6 we find that

$$N^+[\varepsilon, T_V] \geqq N^+[\varepsilon, T_{V_S}] \gtrsim |\{(x, \alpha, a) : x \notin \Omega, V_S(x)^2 L^\wedge(\alpha, a) > \varepsilon(1 + \eta)\}|_{G \times \mathbf{A}_K}.$$

By (d),

$$\{(x, \alpha, a) : x \notin \Omega, V_S(x)^2 L^\wedge(\alpha, a) > \varepsilon(1 + \eta)\}$$
$$\supset \{(x, \alpha, a) : x \notin \Omega, V(x)^2 L^\wedge(\alpha, a) > \varepsilon(1 + \eta)^2\}.$$

Using (a) and (b) we find that

$$|\{(x, \alpha, a) : x \notin \Omega, V(x)^2 L^\wedge(\alpha, a) > \varepsilon(1 + \eta)^2\}|_{G \times \mathbf{A}_K}$$
$$\geqq (1 - \eta)|\{(x, \alpha, a) : V(x)^2 L^\wedge(\alpha, a) > \varepsilon(1 + \eta)^2\}|_{G \times \mathbf{A}_K},$$

from which it follows that

$$N^+[\varepsilon, T_V] \gtrsim (1 - \eta)\Psi(\varepsilon(1 + \eta)^2) \quad \text{as } \varepsilon \to 0+.$$

This proves our first assertion. To prove our second assertion we note that given $\eta > 0$ there exist a measurable K right-invariant set Ω and a function $V_S(x)$ of the form (5.5) such that:

(a) $\quad \Omega^{\#} < \eta,$

(b) $\quad V_S(x) \leqq V_S(y), \qquad x \in \Omega, \quad y \notin \Omega;$

(c) $\quad V(x) \leqq V_S(x), \qquad x \in G;$

(d) $\quad V_S(x)(1 - \eta) \leqq V(x), \qquad x \notin \Omega.$

If we use (c), (5.2), Lemma 5.2, and Theorem 5.6, we obtain

$$N^+[\varepsilon, T_V] \leqq N^+[\varepsilon, T_{V_S}] \lesssim |\{(x, \alpha, a): V_S(x)L\hat{}(\alpha, a) > \varepsilon(1 - \eta)\}|_{G \times \mathbf{A}_K}.$$

Using (a) and (b) we see that

$$|\{x, \alpha, a): V_S(x)^2 L\hat{}(\alpha, a) > \varepsilon(1 - \eta)\}|_{G \times \mathbf{A}_K}$$
$$\leqq (1 - \eta)^{-1} |\{(x, \alpha, a): x \notin \Omega,\ V_S(x)^2 L\hat{}(\alpha, a) > \varepsilon(1 - \eta)\}|_{G \times \mathbf{A}_K}.$$

By (d),

$$\{(x, \alpha, a): x \notin \Omega,\ V_S(x)^2 L\hat{}(\alpha, a) > \varepsilon(1 - \eta)\} \subset \{(x, \alpha, a): V(x)^2 L\hat{}(\alpha, a) > \varepsilon(1 - \eta)^2\},$$

and therefore,

$$N^+[\varepsilon, T_V] \lesssim (1 - \eta)^{-1} \Psi(\varepsilon(1 - \eta)^2) \quad \text{as } \varepsilon \to 0+.$$

If we take $G = \mathbf{T}_n$ and $K = \{0\}$, then Theorem 5.7 is very much the same as Widom's theorem mentioned in § 1, except that Widom requires V to be Riemann integrable and we do not.

It is not necessary to require that $V(x) \in L^\infty[G/K]$ be nonnegative. Suppose that $V(x)$ is complex-valued, and let

$$T_V \leftrightarrow V(x)L(y^{-1}x)\overline{V(y)}.$$

This operator is unitarily equivalent to

$$T_{|V|} \leftrightarrow |V(x)|L(y^{-1}x)|V(y)|$$

since

$$T_V = U T_{|V|} U^{-1},$$

where $U \cdot f(x) = u(x)f(x)$ and

$$u(x) = \begin{cases} V(x)/|V(x)| & \text{if } V(x) \neq 0, \\ 1 & \text{if } V(x) = 0. \end{cases}$$

Let us record our principal result in its most general form.

THEOREM 5.8. *Let $V \in L^\infty[G/K]$ and let $L\hat{}(\alpha, a)$ be a nonnegative function on $\mathbf{A}_K$ which vanishes at infinity. Assume further that* (5.4) *holds. Then for* $0 < \delta < 1$,

$$\text{(i)} \quad N^+[\varepsilon, T] \gtrsim \Psi\left(\frac{\varepsilon}{1 - \delta}\right) \quad \textit{as } \varepsilon \to 0+,$$

$$\text{(ii)} \quad N^+[\varepsilon, T] \lesssim \Psi\left(\frac{\varepsilon}{1 - \delta}\right) \quad \textit{as } \varepsilon \to 0+,$$

where

$$\Psi(x) = |\{(x, \alpha, a): |V(x)|^2 L\hat{}(\alpha, a) > \varepsilon\}|_{G \times \mathbf{A}_K}.$$

For each $\alpha \in \mathbf{A}_K$ let

$$[J\hat{}(\alpha)] = [J\hat{}(\alpha, i, j)]_{i,j = 1, \cdots, d_K(\alpha)}$$

be a matrix with complex entries such that:

(i) $[J\hat{\ }(\alpha)]$ is positive semidefinite;

(ii) $\|J(\alpha)\|$ vanishes at ∞.

Here $\|[J\hat{\ }(\alpha)]\|$ is the norm of $[J\hat{\ }(\alpha)]$ operating on the right on the $d_K(\alpha)$-dimensional Hilbert space of row vectors taken with the usual inner product. The formula

$$R_{J^\wedge} f \cdot (x) = \sum_{\mathbf{A}_K} d(\alpha) \sum_{i=1}^{d(\alpha)} \sum_{j=1}^{d_K(\alpha)} \left\{ \sum_{k=1}^{d_K(\alpha)} f\hat{\ }(\alpha, i, k) J\hat{\ }(\alpha, k, j) \right\} g(\alpha, i, j, x),$$

where $f \in L^2[G/K]$, defines $R_{J^\wedge}$ as a nonnegative completely continuous operator on $L^2[G/K]$. Let

$$S = M_V^* R_{J^\wedge} M_V,$$

where $V \in L^\infty[G/K]$. We indicate how, in some cases, it may be possible to obtain some information about $N^+[S, \varepsilon]$. Let $L\hat{\ }(\alpha, a)$ be as in Theorem 5.8, and for each $\alpha \in \mathbf{A}_K$, $1 \leqq i, j \leqq d_K(\alpha)$, let

$$L\hat{\ }(\alpha, i, j) = \delta(i, j) L\hat{\ }(\alpha, a) \quad \text{if } i \in I(\alpha, a).$$

$[L\hat{\ }(\alpha)] = [L\hat{\ }(\alpha, i, j)]$ is then a $d_K(\alpha) \times d_K(\alpha)$ matrix. Suppose that $L\hat{\ }(\alpha, a)$ satisfies conditions (5.4) and that for each $\alpha \in \mathbf{A}_K$ the matrix

$$[J\hat{\ }(\alpha, i, j) - L\hat{\ }(\alpha, i, j)]$$

is positive semidefinite. A simple computation shows that $R_{J^\wedge} \geqq R_{L^\wedge}$, which implies that $S \geqq T$. By (5.2) and Theorem 5.8 we see that

$$N^+[\varepsilon, S] \gtrsim \Psi\left(\frac{\varepsilon}{1 - \delta}\right) \quad \text{as } \varepsilon \to 0+$$

for each $\delta, 0 < \delta < 1$. If, on the other hand,

$$-[J\hat{\ }(\alpha, i, j) - L\hat{\ }(\alpha, i, j)]$$

is positive semidefinite, then

$$N^+[\varepsilon, S] \lesssim \Psi\left(\frac{\varepsilon}{1 + \delta}\right) \quad \text{as } \varepsilon \to 0+$$

for each $\delta, 0 < \delta < 1$, etc.

REFERENCES

[1] R. Coifman and G. Weiss, *Representations of compact groups and spherical harmonics*, Enseignement Math., 14 (1968), pp. 121–173.

[2] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953.

[3] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, vols. 1 and 2, Springer-Verlag, Berlin, 1963 and 1970.

[4] M. Kac, *On some connections between probability theory and integral equations*, Proc. Second Berkeley Symposium on Mathematical Statistics and Probability, Univ. of California Press, Berkeley, 1951, pp. 189–215.

[5] ———, *Distribution of eigenvalues of certain integral operators*, Michigan Math. J., 3 (1955–56), pp. 141–148.

[6] M. ROSENBLATT, *Some results on the asymptotic behavior of eigenvalues for a class of integral equations with translation kernels*, J. Math. Mech., 12 (1963), pp. 619–628.

[7] N. YA. VILENKIN, *Special functions and the theory of representations of groups*, Izdatel'stvo "Nauka", Moscow, 1965; English transl., Translations of Mathematical Monographs, vol. 22, Amer. Math. Soc., Providence, 1968.

[8] H. WIDOM, *Asymptotic behavior of the eigenvalues of certain integral equations*, Trans. Amer. Math. Soc., 109 (1963), pp. 278–295.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

THE EXISTENCE OF OSCILLATORY SOLUTIONS FOR A NONLINEAR DIFFERENTIAL EQUATION*

J. W. HEIDEL AND DON B. HINTON†

Abstract. For the differential equation $\ddot{y} + q(t)y^{\gamma} = 0$, where $q(t)$ is nonnegative and continuous and $0 < \gamma \neq 1$, a necessary and sufficient condition for the oscillation of all solutions is known. However, it is possible for oscillatory and nonoscillatory solutions to coexist. This paper gives sufficient conditions for the existence of oscillatory solutions in both the superlinear case $1 < \gamma$ and the sublinear case $0 < \gamma < 1$. In the former it is shown that, under certain conditions on $q(t)$, every solution with a zero is oscillatory. In the sublinear case three different sets of conditions are given for the existence of oscillatory solutions.

1. We consider the differential equation

$$\ddot{y} + q(t)y^{\gamma} = 0, \tag{1.1}$$

where the conditions

$$\begin{aligned} &q(t) \geqq 0, \quad \text{continuous on } (0, \infty), \\ &\gamma = \text{the quotient of odd, positive integers}, \end{aligned} \tag{1.2}$$

will always be assumed to hold.

A nontrivial solution $y(t)$ of (1.1) is said to be oscillatory if it has no "last" zero; that is, if $y(t_1) = 0$, then there is a $t_2 > t_1$ such that $y(t_2) = 0$. In the superlinear case $\gamma > 1$, this is not necessarily equivalent to $y(t)$ having arbitrarily large zeros since it is possible to have an oscillatory solution which is not extendable to ∞. The above definition of oscillation avoids this problem; the reader is referred to [14].

In the linear case $\gamma = 1$ the Sturm separation theorem implies that either all nontrivial solutions are oscillatory or they are all nonoscillatory. There are many sufficient conditions on $q(t)$ for the oscillation of all nontrivial solutions in the linear case and there are many necessary conditions, but there is no simple condition which is at the same time both necessary and sufficient.

The situation in the nonlinear case $\gamma \neq 1, \gamma > 0$ is just the opposite. It is possible to have both (nontrivial) oscillatory and nonoscillatory solutions of (1.1) for the same coefficent $q(t)$ and the same value of γ. However, there is a simple necessary and sufficient condition on $q(t)$ for the oscillation of *all* solutions of (1.1). Indeed, Atkinson [1] showed that if $\gamma > 1$, then all solutions of (1.1) are oscillatory if and only if

$$\int^{\infty} sq(s)\, ds = \infty.$$

* Received by the editors January 28, 1971, and in final revised form September 30, 1971.
† Mathematics Department, The University of Tennessee, Knoxville, Tennessee 37916.

Belohorec [2] showed that if $0 < \gamma < 1$, then all solutions of (1.1) are oscillatory if and only if

$$\int^{\infty} s^{\gamma} q(s)\, ds = \infty .$$

In this paper we are concerned with the question of existence of oscillatory solutions of (1.1). This question has been studied already by Jasny [18], Kurzweil [19], Kiguradze [20], Hinton [17], and Coffman and Wong [8], [9] in the case $\gamma > 1$ and by Belohorec [4], [5] and Coffman and Wong [9] for $0 < \gamma < 1$. No single necessary and sufficient condition has been given for the existence of oscillatory solutions in the nonlinear case. In fact, a reasonable conjecture [8, p. 366] has been shown to be invalid [14, Exs. 2, 3], [23].

2. The superlinear case $\gamma > 1$. Jasny and Kurzweil [18], [19] have shown that, under the hypotheses of the following theorem, every solution $y(t)$ such that $y(t_0) = 0$ and $|y'(t_0)|$ is sufficiently large, $t_0 > 0$, is oscillatory. Our purpose is to show that every solution with a zero is oscillatory. Moore and Nehari [21] raised the question as to whether it is possible for oscillatory solutions and nonoscillatory solutions with at least one zero on $(0, \infty)$ to coexist. Our theorem shows that this is not possible for a large class of coefficients.

THEOREM 2.1. *Suppose* $1 < \gamma$. *If* $q(t)t^{(\gamma+3)/2} > 0$ *and* $d/dt(q(t)t^{(\gamma+3)/2}) \geqq 0$ *on* $(0, \infty)$, *then every solution of* (1.1) *with a zero is oscillatory.*

Proof. We make the change of variables, $x = \log t$, $y(t) = t^{1/2}w(x)$ which transforms (1.1) into

$$w'' - \tfrac{1}{4}w + f(x)w^{\gamma} = 0, \qquad ' = d/dx, \tag{2.1}$$

where $f(x) = q(t)t^{(\gamma+3)/2}$. Clearly, the $(0, \infty)$ t-interval corresponds to the $(-\infty, \infty)$ x-interval. Define $G(w(x))$ by

$$G(w(x)) = \frac{w'(x)^2}{2} + \frac{f(x)}{\gamma + 1} w(x)^{\gamma+1} - \frac{w(x)^2}{8}.$$

Then $G(w(x)) = G(w(x_0) + (1/(\gamma + 1)) \int_{x_0}^{x} f'(u)w(u)^{\gamma+1}\, du$ along solutions of (2.1) which means that $G(w(x))$ is nondecreasing along solutions of (2.1).

Suppose that $y(t)$ is a nontrivial solution of (1.1) such that $y(t_0) = 0$ for some $t_0 > 0$. By uniqueness of initial value problems, valid for $\gamma \geqq 1$, $\dot{y}(t_0) \neq 0$. We may assume that $\dot{y}(t_0) > 0$ since the negative of $y(t)$ is also a solution of (1.1). Thus $y(t)$ is positive in some deleted right neighborhood of t_0 and we want to show that $y(t) = 0$ for some $t > t_0$.

Let $x_0 = \log t_0$. Then $w(x_0) = 0$ and $w'(x_0) > 0$. Also, $w(x)$ is positive in some deleted right neighborhood of x_0. Define

$$\alpha(x) = \left[\frac{\gamma + 1}{8f(x)}\right]^{1/(\gamma-1)} \quad \text{and} \quad \beta(x) = \left[\frac{1}{4f(x)}\right]^{1/(\gamma-1)}.$$

Note that $0 < \beta(x) < \alpha(x)$ for all x and that α, β are nonincreasing.

Our problem is to show that $w(x) = 0$ for some $x > x_0$. Suppose first that $w'(x_1) = 0$ for some $x_1 > x_0$. Since $G(w(x))$ is nondecreasing along solutions of (2.1) we must have $G(w(x_1)) \geqq G(w(x_0)) > 0$. Hence $w(x_1) > \alpha(x_1)$. This means that

$w''(x_1) < 0$ since $w''(x) < 0$ for $w(x) > \beta(x)$. Let x_2 be the first point greater than x_1 such that $w(x_2) = \alpha(x_2)$. Then $w'(x_2) = c < 0$ and $w'(x) < c$ in some deleted right neighborhood of x_2. We assert that $w'(x) < c$ to the right of x_2 as long as $w(x) > 0$. Suppose to the contrary that $w'(x_3) = c$, where $w(x) > 0$ for $x_2 < x \leqq x_3$ and $w'(x) < c$ for $x_2 < x < x_3$. Then $w''(x_3) \geqq 0$. Thus $w(x_3) \leqq \beta(x_3)$. But then $G(w(x_3)) < c^2/2 = G(w(x_2))$. This contradicts the fact that $G(w(x))$ is nondecreasing. Thus $w'(x) < c$ as long as $w > 0$. Hence $w(x_4) = 0$ for some $x_4 > x_0$.

The remaining case is that $w'(x) > 0$ for all $x \geqq x_0$. Since $w'' > 0$ for $w < \beta$, there is an $x_1 > x_0$ such that $w(x_1) = \beta(x_1)$. Letting $g(x) = f(x)w(x)^{\gamma-1} - \frac{1}{4}$ we see that $g(x) > 0$ and increasing for $x > x_1$. Now $w(x)$ is a positive solution of

$$w'' + g(x)w = 0. \tag{2.2}$$

But all solutions of (2.2) are oscillatory since $g(x) > 0$ and nondecreasing. This eliminates the second case and shows that $y(t)$ must have another zero to the right of t_0. By repeating the process $y(t)$ has no last zero.

Remark. Coffman and Wong [9] have also established Theorem 2.1. However, we include our proof since it is more direct and is closely related to the proofs in the next section.

Remark. Kiguradze [20] has given a shorter proof of the Jasny-Kurzweil result and his proof actually establishes our result if $\lim_{t\to\infty} q(t)t^{(\gamma+3)/2} = \infty$ as $t \to \infty$. We do not require this condition, and our proof is entirely different from his.

Remark. The sharpness of the condition $d/dt(q(t)t^{(\gamma+3)/2}) \geqq 0$ is indicated by a result of Nehari [22] which says that (1.1) has no nontrivial oscillatory solutions if $q(t)\,(t \log t)^{(\gamma+3)/2}$ is nonincreasing (see also [20] and [7]).

3. The sublinear case $0 < \gamma < 1$. Although in the superlinear case all of the abovementioned results (including ours) were known to Fowler [10] in 1930 for the "Emden-Fowler" equation $q(t) = t^\sigma$, our results in this section have not been anticipated for any special cases. In fact, the question of the existence of oscillatory solutions of (1.1) (when not all solutions are oscillatory) has not been treated at all except for one "borderline" case considered by Belohorec [4] (see also [9]), to be mentioned below.

It is known [15] that if there exist nonoscillatory solutions, then there exist nonoscillatory solutions with at least one zero. Hence the restriction below that the initial slope be sufficiently small in absolute value cannot be omitted.

THEOREM 3.1. *Suppose that* $0 < \gamma < 1$. *If* $q(t)t^{(\gamma+3)/2} > 0$ *and* $d/dt(q(t)t^{(\gamma+3)/2}) \geqq 0$ *on* $(0, \infty)$ *and* $\lim_{t\to\infty} t\, d/dt(q(t)t^{(\gamma+3)/2}) = \infty$, *then every solution* $y(t)$ *of* (1.1) *such that* $y(t_0) = 0$ *and* $|y'(t_0)|$ *is sufficiently small is oscillatory.*

Proof. We make the same change of variables and transformation as in Theorem 2.1 and obtain (2.1):

$$w'' - \tfrac{1}{4}w + f(x)w^\gamma = 0. \tag{3.1}$$

As before, $f'(x) \geqq 0$ so that $G(w(x))$ is nondecreasing along solutions of (3.1). Again define $\beta(x) = (4f(x))^{1/(1-\gamma)}$. This time $\beta(x)$ is nondecreasing.

We assert that if $w(x_0) = 0$ and

$$\frac{w'(x_0)^2}{2} < Kf(x_0)^{2/(1-\gamma)}, \qquad K = 4^{(1+\gamma)/(1-\gamma)} \cdot \frac{(1-\gamma)}{2(1+\gamma)},$$

then $|w(x)| < \beta(x)$ for $x \geqq x_0$. To see this consider

$$G(w(x)) = G(w(x_0)) + \int_{x_0}^{x} f'(u)\frac{w(u)^{\gamma+1}}{\gamma+1}\,du.$$

As long as $|w(x)| \leqq \beta(x)$, then $w^{\gamma+1} < (4f)^{(1+\gamma)/(1-\gamma)}$. Hence

$$\begin{aligned} G(w(x)) &\leqq w'(x_0)^2/2 - Kf(x_0)^{2/(1-\gamma)} + Kf(x)^{2/(1-\gamma)} \\ &< Kf(x)^{2/(1-\gamma)}. \end{aligned}$$

Suppose that $|w(x)| = \beta(x)$ for some $x > x_0$ and let $x_1 > x_0$ be the first such point. Then

$$\begin{aligned} G(w(x_1)) &= \frac{w'(x_1)^2}{2} + \frac{f(x_1)}{\gamma+1}(4f(x_1))^{(1+\gamma)/(1-\gamma)} - \frac{(4f(x_1))^{2/(1-\gamma)}}{8} \\ &= \frac{w'(x_1)^2}{2} + Kf(x_1)^{2/(1-\gamma)} \\ &\geqq Kf(x_1)^{2/(1-\gamma)}. \end{aligned}$$

But this contradicts $G(w(x)) < Kf(x)^{2/(1-\gamma)}$ for $x_0 \leqq x \leqq x_1$. This proves the assertion.

We now assert that if $w(x_0) = 0$ and $w'(x_0)^2/2 < Kf(x_0)^{2/(1-\gamma)}$, then $w(x)$ is oscillatory. Suppose not. Then $w(x)$ has a last zero, say at $x = x_1$. We may assume that $0 < w(x)$ for $x > x_1$ (otherwise consider $-w(x)$). Also, $w(x) < \beta(x)$ from above. Thus from (3.1) we have $w''(x) < 0$ for $x > x_1$ and $w'(x) > 0$ for $x > x_1$. Hence $\lim_{x\to\infty} w'(x) = L < \infty$ exists. But $\lim_{x\to\infty} \beta'(x) = \infty$. Thus there exist numbers $x_2 \geqq x_1$ and $0 < c < 1$ such that $w(x) < c\beta(x)$ for $x \geqq x_2$.

Transforming back to t variables we obtain

$$y(t)/t^{1/2} < c(4q(t)t^{(\gamma+3)/2})^{1/(1-\gamma)}$$

for large t. This is equivalent to

$$t^2q(t)y(t)^{\gamma-1} > c^{\gamma-1}/4 = (1+\varepsilon)/4$$

for some $\varepsilon > 0$ or

$$q(t)y(t)^{\gamma-1} > \frac{1+\varepsilon}{4t^2}$$

for large t. Thus $y(t)$ must be an oscillatory solution of

$$\ddot{y} + (q(t)y(t)^{\gamma-1})y = 0.$$

This contradiction proves the theorem.

THEOREM 3.2. *Suppose* $0 < \gamma < 1$. *If* $q(t)t^{(\gamma+3)/2} > 0$, $d/dt(q(t)t^{(\gamma+3)/2}) \geqq 0$, *and* $q(t)t^{(\gamma+3)/2} \leqq k < \infty$ *for* $t > 0$, *then every solution* $y(t)$ *of* (1.1) *such that* $y(t_0) = 0$ *and* $|y'(t_0)|$ *is sufficiently small is oscillatory.*

Proof. Proceeding as in Theorem 3.1 we see that if $w(x_0) = 0$ and $w'(x_0)^2/2 < Kf(x_0)^{2/(1-\gamma)}$, then not only is $|w(x)| < \beta(x)$ for $x \geqq x_0$, but also

$$G(w(x)) < Kf(x)^{2/(1-\gamma)} - \varepsilon, \qquad x \geqq x_0,$$

where $0 < \varepsilon < Kf(x_0)^{2/(1-\gamma)} - w'(x_0)^2/2$. Thus

$$w(x)^{\gamma+1}\left(\frac{f(x)}{\gamma+1} - \frac{w(x)^{1-\gamma}}{8}\right) < Kf(x)^{2/(1-\gamma)} - \varepsilon. \tag{3.2}$$

Since $|w(x)| < \beta(x)$, we have

$$\frac{f(x)}{\gamma+1} - \frac{w(x)^{1-\gamma}}{8} \geqq \frac{1-\gamma}{2(1+\gamma)}f(x). \tag{3.3}$$

Therefore, dividing (3.2) by (3.3), we have

$$\begin{aligned} w(x)^{\gamma+1} &< (4f(x))^{(1+\gamma)/(1-\gamma)} - \varepsilon\frac{2(1+\gamma)}{1-\gamma}\frac{1}{k} \\ &< \alpha(4f(x))^{(1+\gamma)/(1-\gamma)}, \end{aligned}$$

where

$$\alpha = \frac{(4k)^{(1+\gamma)/(1-\gamma)} - \varepsilon[2(1+\gamma)/k(1-\gamma)]}{(4k)^{(1+\gamma)/(1-\gamma)}} < 1.$$

Hence $|w(x)| < \alpha^{1/(\gamma+1)}(4f(x))^{1/(1-\gamma)} = \alpha^{1/(\gamma+1)}\beta(x)$ and $\alpha^{1/(\gamma+1)} < 1$.

Thus, proceeding as in Theorem 3.1 we see that $w(x)$ must be oscillatory.

COROLLARY. *The sublinear Emden-Fowler equation* ($0 < \gamma < 1$, $q(t) = t^\sigma$) *has oscillatory solutions if* $\sigma \geqq -(\gamma + 3)/2$.

Remark. It is interesting to note that Belohorec [4] has shown that if $d/dt(q(t)t^{(\gamma+3)/2}) \leqq 0$ and $q(t)t^{(\gamma+3)/2} \geqq k_1 > 0$ for $t > 0$, then (1.1) has oscillatory solutions.

Remark. These results on the existence of oscillatory solutions are sharp since Brunovsky [6] has shown that if $q(t) = t^\sigma$, $\sigma < -(\gamma + 3)/2$, then all nontrivial solutions of (1.1) are nonoscillatory. Belohorec [3], [4], Gollwitzer [11] and the first author [12] have obtained nonoscillation theorems for (1.1) in the general case.

4. A Liouville transformation. In this section we make a change of variables of (1.1) which reduces to the Liouville transformation in case $\gamma = 1$. By this means, we obtain an additional criterion for the existence of oscillatory solutions of (1.1) in the sublinear case. For a large class of oscillatory solutions our method of proof will yield an asymptotic formula for the number of zeros. The more general equation with positive coefficients,

$$(r(t)\dot{y})^{\cdot} + q(t)y^\gamma = 0, \tag{4.1}$$

is considered since the transformation is equally applicable to it.

If y is a solution of (4.1), define $H(y(t))$ by

$$H(y(t)) = (y(t)/\eta(t))^{\gamma+1} + [(\gamma+1)/2]r(t)^2[\eta(t)\dot{y}(t) - y(t)\dot{\eta}(t)]^2,$$

where

$$\eta(t) = [r(t)q(t)]^{-1/(\gamma+3)}.$$

THEOREM 4.1. *Suppose* $0 < \gamma < 1$, r and q *are positive with continuous second derivatives on a ray* $[a, \infty)$, $K = \int_a^\infty |\eta(r\dot{\eta})^{\cdot}|\,dt < \infty$, *and* $\int_a^\infty 1/(r\eta^2)\,dt = \infty$. *Then* *there is a positive number L such that if y is a nontrivial solution of* (4.1) *satisfying* $H(y(a)) < L$, *then y will have the following two properties*:

(i) $H(y(t))$ *has a positive limit as* $t \to \infty$;

(ii) *y is oscillatory and if* $N(t)$ *denotes the number of zeros of y on* $[a, t]$, *then* $N(t)/\int_a^t 1/(r\eta^2)\,dt$ *has a positive limit as* $t \to \infty$.

Proof. We make the change of variables $x = \int_a^t 1/(r\eta^2)\,dt$ and $y(t) = \eta(t)w(x)$. Equation (4.1) transforms into

$$w'' + A(x)w + w^\gamma = 0, \qquad ' = d/dx, \tag{4.2}$$

where $A(x) = [r\eta^3(r\dot\eta)^{\cdot}](t)$. Hence

$$\int_0^\infty |A(x)|\,dx = \int_a^\infty |\eta(r\dot\eta)^{\cdot}|\,dt < \infty.$$

For (4.2), define the polar coordinates ρ and θ by

$$\begin{aligned} \rho\sin\theta &= (\operatorname{sgn} w)|w|^{(\gamma+1)/2}, \\ \rho\cos\theta &= [(\gamma+1)/2]^{1/2}w'. \end{aligned} \tag{4.3}$$

Since $\rho^2 = w^{\gamma+1} + (\gamma+1)w'^2/2$, it follows that ρ is differentiable and

$$2\rho\rho' = (\gamma+1)w^\gamma w' + (\gamma+1)w'w'' = -(\gamma+1)Aww'. \tag{4.4}$$

From $|w'| \leqq [2/(\gamma+1)]^{1/2}\rho$ and $|w| \leqq \rho^{2/(\gamma+1)}$, equation (4.4) is of the form $\rho' = B\rho^{2/(\gamma+1)}$, where $B = -(\gamma+1)Aww'/2\rho^{(\gamma+3)/(\gamma+1)}$. An integration of $\rho' = B\rho^{2/(\gamma+1)}$ yields

$$\left(\frac{1+\gamma}{1-\gamma}\right)\left[\frac{1}{\rho(0)^{(1-\gamma)/(1+\gamma)}} - \frac{1}{\rho(x)^{(1-\gamma)/(1+\gamma)}}\right] = \int_0^x B(\xi)\,d\xi. \tag{4.5}$$

From (4.5) and

$$\int_0^\infty |B(\xi)|\,d\xi \leqq [(\gamma+1)/2]^{1/2}\int_0^\infty |A(\xi)|\,d\xi = [(\gamma+1)/2]^{1/2}K,$$

it follows that if $\rho(0)$ is sufficiently small, then ρ is bounded on $[0, \infty)$; in fact, if

$$\rho(0) < \left[\frac{1-\gamma}{1+\gamma}\left(\frac{1+\gamma}{2}\right)^{1/2}K\right]^{(\gamma+1)/(\gamma-1)} \equiv L^{1/2},$$

then ρ has a positive limit at ∞. Since $\rho(x)^2 = H(y(t))$, we have established property (i).

When $w(x) \neq 0$, we have from (4.3) that θ is differentiable and

$$\begin{aligned} \rho'\sin\theta + \rho\theta'\cos\theta &= (\gamma+1)|w|^{(\gamma-1)/2}w'/2, \\ \rho'\cos\theta - \rho\theta'\sin\theta &= [(\gamma+1)/2]^{1/2}w''. \end{aligned} \tag{4.6}$$

Solving (4.6) for θ' yields

$$\begin{aligned} \theta' &= \{[(\gamma+1)/2]|w|^{(\gamma-1)/2}w'\cos\theta + [(\gamma+1)/2]^{1/2}(Aw + w^\gamma)\sin\theta\}/\rho \\ &= [(\gamma+1)/2]^{1/2}\{|w|^{(\gamma-1)/2}\rho\cos^2\theta + Aw\sin\theta + |w|^{(\gamma-1)/2}\rho\sin^2\theta\}/\rho \\ &= [(\gamma+1)/2]^{1/2}\{|\rho\sin\theta|^{(\gamma-1)/(\gamma+1)} + A(w/\rho)\sin\theta\}. \end{aligned}$$

Therefore, θ satisfies the equation

$$|\sin\theta|^{(1-\gamma)/(1+\gamma)}\theta' = [(\gamma+1)/2]^{1/2}\rho^{(\gamma-1)/(\gamma+1)} + C, \tag{4.7}$$

where $C = [(\gamma+1)/2]^{1/2}A(w/\rho)\sin\theta|\sin\theta|^{(1-\gamma)/(1+\gamma)}$.

If $\rho(0) < L^{1/2}$, then w/ρ is bounded and $\int_0^\infty |C(x)|\,dx < \infty$; hence we have from (4.7) that $\theta(x) \to \infty$ as $x \to \infty$. Choosing $0 \leqq \theta(0) < 2\pi$ and defining x_k by $\theta(x_k) = k\pi$, $k = 2, 3, \cdots$, we have from (4.7) that

$$\begin{aligned}\int_{x_k}^{x_{k+1}} |\sin\theta|^{(1-\gamma)/(1+\gamma)}\theta'\,dx &= [(\gamma+1)/2]^{1/2}\int_{x_k}^{x_{k+1}} \rho^{(\gamma-1)/(\gamma+1)}\,dx \\ &\quad + \int_{x_k}^{x_{k+1}} C\,dx \\ &= \int_0^\pi |\sin u|^{(1-\gamma)/(1+\gamma)}\,du \equiv I.\end{aligned}$$

Thus

$$nI = [(\gamma+1)/2]^{1/2}\int_{x_2}^{x_{n+2}} \rho^{(\gamma-1)/(\gamma+1)}\,dx + \int_{x_2}^{x_{n+2}} C\,dx. \tag{4.8}$$

Since $\rho(x) \to \rho(\infty)$ as $x \to \infty$, from (4.8) we obtain

$$\frac{IN(t)}{[(\gamma+1)/2]^{1/2}\rho(\infty)^{(\gamma-1)/(\gamma+1)}x} \to 1 \quad \text{as } x \to \infty;$$

hence property (ii) is established.

Remark. For the coefficients $r = t^\alpha$ and $q = t^\sigma$ with $t \geqq a > 0$, Theorem 4.1 is applicable if $2(\alpha + \sigma)/(\gamma + 3) > \alpha - 1$. For $\alpha = 0$, Theorem 3.1 applies if $2\sigma(\gamma + 3) > -1$ and Theorem 3.2 applies for $2\sigma(\gamma + 3) = -1$.

Remark. The second author has discussed the existence of oscillatory solutions for the superlinear case under similar conditions, but with different techniques, in [17]. An asymptotic formula for the distribution of zeros of (4.1) in the superlinear case was obtained in [16].

Added in proof. Using the techniques of Theorems 3.1 and 3.2, Mr. Kuo-liang Chiou [24] has succeeded in establishing the following unifying result: if $0 < \gamma < 1$ and if $q(t)t^{(\gamma+3)/2} > 0$ and $d/dt(q(t)t^{(\gamma+3)/2}) \geqq 0$, then every solution of (1.1) such that $y(t_0) = 0$ and $|y(t_0)|$ is sufficiently small is oscillatory.

REFERENCES

[1] F. V. Atkinson, *On second order nonlinear oscillations*, Pacific J. Math., 5 (1955), pp. 643–647.

[2] S. Belohorec, *Oscillatory solutions of certain nonlinear differential equations of the second order*, Mat.-Fyz. Casopis Sloven. Akad. Vied., 11 (1961), no. 4, pp. 250–255 (in Czechoslovakian).

[3] ———, *Nonoscillatory solutions of a certain nonlinear differential equation of the second order*, Ibid., 12 (1962), no. 4, pp. 253–262 (in Czechoslovakian).

[4] ———, *On some properties of the equation $y''(x) + f(x)y^\alpha(x) = 0$, $0 < \alpha < 1$*, Ibid., 17 (1967), no. 1, pp. 10–19.

[5] ———, *A criterion for oscillation and nonoscillation*, Acta Fac. Rerum Natur. Univ. Comenian., 20 (1969), pp. 75–79.

[6] P. BRUNOVSKÝ, *On Emden-Fowler's equation in the case $n < 1$*, Mat.-Fyz. Casopis Sloven. Akad. Vied., 12 (1962), no. 1, pp. 60–81 (in Czechoslovakian).

[7] KUO-LIANG CHIOU, *A second order nonlinear oscillation theorem*, SIAM J. Appl. Math., 21 (1971), pp. 221–224.

[8] C. V. COFFMAN AND J. S. W. WONG, *On a second order nonlinear oscillation problem*, Trans. Amer. Math. Soc., 147 (1970), pp. 357–366.

[9] ———, *Oscillation and nonoscillation of solutions of generalized Emden-Fowler equations*, Carnegie Mellon Rep. 69–39, Carnegie-Mellon Univ., Pittsburgh, 1969.

[10] R. H. FOWLER, *Further studies of Emden's and similar differential equations*, Quart. J. Math., 2 (1931), pp. 259–288.

[11] H. E. GOLLWITZER, *Nonoscillation theorems for a nonlinear differential equation*, Proc. Amer. Math. Soc., 26 (1970), pp. 78–84.

[12] J. W. HEIDEL, *A nonoscillation theorem for a nonlinear second order differential equation*, Ibid., 22 (1969), pp. 485–488.

[13] ———, *A short proof of Atkinson's oscillation theorem*, SIAM Rev., 11 (1969), pp. 389–390.

[14] ———, *Uniqueness, continuation and nonoscillation for a second order nonlinear differential equation*, Pacific J. Math., 32 (1970), pp. 715–721.

[15] ———, *Rate of growth of nonoscillatory solutions for the differential equation $\ddot{y} + q(t)|y|\gamma \operatorname{sgn} y = 0$, $0 < \gamma < 1$*, Quart. Appl. Math., 28 (1971), pp. 601–606.

[16] DON HINTON, *Some stability conditions for a nonlinear differential equation*, Trans. Amer. Math. Soc., 139 (1969), pp. 349–358.

[17] ———, *An oscillation criterion for solutions of $(ry')' + qy^\gamma = 0$*, Michigan Math. J., 16 (1969), pp. 349–352.

[18] M. JASNY, *On the existence of an oscillating solution of the nonlinear differential equation of the second order $y'' + f(x)y^{2n-1} = 0$, $f(x) > 0$*, Casopis Pest. Mat., 85 (1960), pp. 78–83 (in Russian).

[19] J. KURZWEIL, *A note on oscillatory solutions of the equation $y'' + f(x)y^{2n-1} = 0$*, Ibid., 85 (1960), pp. 357–358 (in Russian).

[20] I. T. KIGURADZE, *On the condition for oscillation of solutions of the differential equation $u'' + a(t)|u|^n \operatorname{sgn} u = 0$*, Ibid., 87 (1962), pp. 492–495 (in Russian).

[21] R. A. MOORE AND Z. NEHARI, *Nonoscillation theorems for a class of nonlinear differential equations*, Trans. Amer. Math. Soc., 93 (1959), pp. 30–52.

[22] Z. NEHARI, *A nonlinear oscillation problem*, J. Differential Equations, 5 (1969), pp. 452–460.

[23] KUO-LIANG CHIOU, *A nonoscillation theorem for the superlinear case of the second order differential equation $y'' + yF(y^2x) = 0$*, to appear.

[24] ———, *The existence of oscillatory solutions for the equation $\ddot{y} + qy^r = 0$, $0 < r < 1$*, Proc. Amer. Math. Soc., to appear.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

EXPLICIT EVALUATION OF CERTAIN POLYNOMIALS*

L. CARLITZ†

Abstract. For real nonzero k, put

$$S_m(k) = \frac{1}{2}\sum_{n=0}^{\infty} \alpha_n^{-m-2}, \qquad T_m(k) = \frac{1}{2}\sum_{n=0}^{\infty} \beta_n^{-2m-2},$$

where α_n runs through the nonzero roots of $\tan \alpha = k\alpha$ and β_n runs through the roots of $k \cot \beta + \beta = 0$. Liron [2, pp. 105–107] showed that $S_m(k) = (k-1)^{-m-1}P_{m+1}(k)$, where $P_{m+1}(k)$ is a polynomial in k of degree $m+1$. He showed also that $T_m(k)$ is a polynomial in k of degree $m+1$. In the present paper it is shown that the coefficients of $P_{m+1}(k)$ and $T_m(k)$ can be expressed simply in terms of tangent coefficients of higher order.

1. Introduction. For real nonzero k, put

$$S_m(k) = -\sum_{n=0}^{\infty} \alpha_n^{-m-2}, \tag{1.1}$$

where α_n runs through the nonzero roots of

$$\tan \alpha = k\alpha. \tag{1.2}$$

Liron [2] showed that

$$\sum_{m=0}^{\infty} S_m(k)t^{2m} = \frac{1}{2kt^2} + \frac{1}{2}\left(1 - \frac{k-1}{k^2t^2}\right)\frac{k \sin t}{kt \cos t - \sin t}, \qquad k \neq 1, \tag{1.3}$$

and

$$\sum_{m=0}^{\infty} S_m(1)t^{2m} = \frac{3}{2t^2} + \frac{\sin t}{2(t \cos t - \sin t)}. \tag{1.4}$$

He showed also that

$$S_m(k) = (k-1)^{-m-1}P_{m+1}(k), \qquad k \neq 1, \tag{1.5}$$

where $P_{m+1}(k)$ is a polynomial of degree $m+1$ in k, with rational coefficients, and

$$P_{m+1}(1) = 3^{-m-1}. \tag{1.6}$$

While $P_{m+1}(k)$ was exhibited as a determinant of order $m+1$, explicit formulas for the coefficients were not obtained.

Next put

$$T_m(k) = \frac{1}{2}\sum_{n=0}^{\infty} \beta_n^{-2m-2}, \tag{1.7}$$

* Received by the editors April 29, 1971, and in revised form September 16, 1971.

† Department of Mathematics, Duke University, Durham, North Carolina 27706. This work was supported in part by the National Science Foundation under Grant GP-17031.

where β_n runs through the roots of

$$k \cot \beta + \beta = 0. \tag{1.8}$$

Liron [2] showed that

$$\sum_{m=0}^{\infty} T_m(k) t^{2m} = \frac{k-1}{2t^2} - \frac{1}{2}\left(1 + \frac{k^2-k}{t^2}\right)\frac{\cos t}{k \cos t + t \sin t} \tag{1.9}$$

and that $T_m(k)$ is a polynomial of degree $m + 1$ in k with rational coefficients. Again $T_m(k)$ was exhibited as a determinant of order $m + 1$ but explicit formulas for the coefficients were not obtained.

In the present note we show that the coefficients of $P_{m+1}(k)$ and $T_m(k)$ can be expressed simply in terms of tangent coefficients of higher order. As an application we get simple generating functions for $S'_m(k)$ and $T'_m(k)$; see (4.1) and (4.2) below.

2. The polynomial $T_m(k)$. It is convenient to begin with $T_m(k)$. Put

$$\tan^r t = \sum_{n=r}^{\infty} T_n^{(r)} \frac{t^n}{n!} \tag{2.1}$$

and, in particular,

$$\tan t = \sum_{n=1}^{\infty} T_n \frac{t^n}{n!}, \tag{2.2}$$

so that $T_n^{(1)} = T_n$. The coefficients $T_n^{(r)}$ are rational integers. Indeed, by a result due to Hurwitz [1, p. 345] they satisfy

$$T_n^{(r)} \equiv 0 \pmod{r!}, \qquad n \geqq r.$$

Now

$$\begin{aligned} \frac{\cos t}{k \cos t + t \sin t} &= \frac{1}{k}\left(\frac{1}{1 + k^{-1} t \tan t}\right) \\ &= \sum_{r=0}^{\infty} (-1)^r k^{-r-1} t^r \tan^r t \\ &= \sum_{r=0}^{\infty} (-1)^r k^{-r-1} \sum_{n=r}^{\infty} T_n^{(r)} \frac{t^{n+r}}{n!} \\ &= \sum_{n=0}^{\infty} t^n \sum_{2r \leqq n} (-1)^r \frac{k^{-r-1}}{(n-r)!} T_{n-r}^{(r)}. \end{aligned} \tag{2.3}$$

Thus,

$$\begin{aligned} &\frac{k-1}{2t^2} - \frac{1}{2}\left(1 + \frac{k^2-k}{t^2}\right)\frac{\cos t}{k \cos t + t \sin t} \\ &\qquad = \frac{k-1}{2t^2} - \frac{1}{2}\left(\frac{1}{k} + \frac{k-1}{t^2}\right) \sum_{n=0}^{\infty} t^n \sum_{2r \leqq n} (-1)^r \frac{k^{-r}}{(n-r)!} T_{n-r}^{(r)} \\ &\qquad = -\frac{1}{2k} \sum_{n=0}^{\infty} t^n \sum_{2r \leqq n} (-1)^r \frac{k^{-r}}{(n-r)!} T_{n-r}^{(r)} \\ &\qquad\quad -\frac{k-1}{2} \sum_{n=0}^{\infty} t^n \sum_{2r \leqq n+2} (-1)^r \frac{k^{-r}}{(n-r+2)!} T_{n-r+2}^{(r)}. \end{aligned}$$

Clearly only even powers of t need be considered. We therefore have

$$-\frac{1}{2k}\sum_{n=0}^{\infty} t^{2n}\sum_{r=0}^{n}(-1)^r\frac{k^{-r}}{(2n-r)!}T_{2n-r}^{(r)}$$
$$-\frac{k-1}{2}\sum_{n=0}^{\infty} t^{2n}\sum_{r=0}^{n+1}(-1)^r\frac{k^{-r}}{(2n-r+2)!}T_{2n-r+2}^{(r)}.$$

Comparison with (1.9) now yields

$$\begin{aligned} T_n(k) &= -\frac{1}{2k}\sum_{r=0}^{n}(-1)^r\frac{k^{-r}}{(2n-r)!}T_{2n-r}^{(r)} \\ &\quad -\frac{1}{2}(k-1)\sum_{r=0}^{n+1}(-1)^r\frac{k^{-r}}{(2n-r+2)!}T_{2n-r+2}^{(r)} \\ &= \frac{1}{2}\sum_{r=0}^{n+1}(-1)^r\frac{k^{-r}}{(2n-r+2)!}\{(2n-r+2)(T_{2n-r+1}^{(r-1)}+T_{2n-r+1}^{(r+1)})+T_{2n-r+2}^{(r)}\}. \end{aligned} \tag{2.4}$$

Returning to (2.1) and differentiating gives

$$r(\tan^{r-1} t + \tan^{r+1} t) = \sum_{n=r-1}^{\infty} T_{n+1}^{(r)}\frac{t^n}{n!},$$

so that

$$r(T_n^{(r-1)} + T_n^{(r+1)}) = T_{n+1}^{(r)}. \tag{2.5}$$

Therefore, comparing (2.5) with (2.4), we get

$$\begin{aligned} T_n(k) &= (n+1)\sum_{r=0}^{n+1}(-1)^r\frac{k^{-r}}{r(2n-r+2)!}T_{2n-r+2}^{(r)} \\ &= (n+1)\sum_{r=0}^{n+1}(-1)^{n-r+1}\frac{k^{-n(n-r+1)}}{(n-r+1)(n+r+1)!}T_{n+r+1}^{(n-r+1)}. \end{aligned} \tag{2.6}$$

Since, by (2.1),

$$T_n^{(n)} = n!, \tag{2.7}$$

it follows from (2.6) that the coefficient of k^{-n-1} in $T_n(k)$ is equal to $(-1)^{n+1}$. To get the constant term in $T_n(k)$ we note that, in the right member of (2.4), the term $T_{2n-r+1}^{(r-1)}$ is to be ignored when $r = 0$. We find that the constant term is equal to

$$\frac{1}{2(2n+1)!}T_{2n+1}. \tag{2.8}$$

Since

$$T_{2n+1} \equiv 0 \pmod 2, \qquad n \geqq 1,$$

it follows from (2.3), (2.6) and (2.8) that

$$\frac{(2n+1)!}{n+1}T_n(k) \tag{2.9}$$

has integral coefficients.

3. The polynomial $S_n(k)$. Turning next to $S_n(k)$, we put

$$(\tan t - t)^r = \sum_{n=3r}^{\infty} U_n^{(r)} \frac{t^n}{n!}. \tag{3.1}$$

The $U_n^{(r)}$ are rational integers, and by Hurwitz's lemma,

$$U_n^{(r)} \equiv 0 \pmod{r!}, \qquad n \geqq 3r. \tag{3.2}$$

Now, for $k \neq 1$,

$$\frac{\sin t}{kt\cos t - \sin t} = \frac{\tan t}{kt - \tan t} = \frac{\tan t}{(k-1)t - (\tan t - t)}$$

$$= \frac{\tan t}{t} \sum_{r=0}^{\infty} (k-1)^{-r-1} t^{-r} (\tan t - t)^r$$

$$= \frac{1}{k-1}\left\{1 + k \sum_{r=1}^{\infty} (k-1)^{-r} t^{-r} (\tan t - t)^r\right\}$$

$$= \frac{1}{k-1}\left\{1 + k \sum_{n=1}^{\infty} t^{2n} \sum_{r=1}^{n} \frac{(k-1)^{-r}}{(2n+r)!} U_{2n+r}^{(r)}\right\}.$$

Thus,

$$\frac{1}{2kt^2} + \frac{1}{2}\left(1 - \frac{k-1}{k^2t^2}\right) \frac{k \sin t}{kt\cos t - \sin t}$$

$$= \frac{1}{2kt^2} + \frac{1}{2}\left(\frac{k}{k-1} + \frac{1}{kt^2}\right)\left\{1 + k \sum_{n=1}^{\infty} t^{2n} \sum_{r=1}^{n} \frac{(k-1)^{-r}}{(2n+r)!} U_{2n+r}^{(r)}\right\}$$

$$= \frac{1}{2}\frac{k}{k-1} + \frac{1}{2}\left(\frac{k^2}{k-1} - \frac{1}{t^2}\right) \sum_{n=1}^{\infty} t^{2n} \sum_{r=1}^{n} \frac{(k-1)^{-r}}{(2n+r)!} U_{2n+r}^{(r)}$$

$$= \frac{1}{2}\frac{k}{k-1} - \frac{1}{6}\frac{1}{k-1} + \frac{1}{2}\sum_{n=1}^{\infty} t^{2n}\left\{k^2 \sum_{r=1}^{n} \frac{(k-1)^{-r-1}}{(2n+r)!} U_{2e+r}^{(r)} - \sum_{r=1}^{n+1} \frac{(k-1)^{-r}}{(2n+r+2)!} U_{2n+r+2}^{(r)}\right\}.$$

Therefore, by (1.3),

$$S_0(k) = \frac{1}{6}\left(\frac{3k-1}{k-1}\right), \qquad k \neq 1,$$

while, for $n \geqq 1$,

$$S_n(k) = \frac{1}{2} k^2 \sum_{r=1}^{n} \frac{(k-1)^{-r-1}}{(2n+r)!} U_{2n+r}^{(r)} - \frac{1}{2} \sum_{r=1}^{n+1} \frac{(k-1)^{-r}}{(2n+r+2)!} U_{2n+r+2}^{(r)}.$$

After a little manipulation this becomes

$$S_n(k) = \frac{1}{2} \sum_{r=0}^{n+1} (k-1)^{-r} \left\{\frac{U_{2n+r-1}^{(r-1)}}{(2n+r-1)!} + 2\frac{U_{2n+r}^{(r)}}{(2n+r)!} + \frac{U_{2n+r+1}^{(r+1)}}{(2n+r+1)!} - \frac{U_{2n+r+2}^{(r)}}{(2n+r+2)!}\right\}. \tag{3.3}$$

Now it follows on differentiating (3.1) that

$$r\left\{\frac{U_n^{(r+1)}}{n!} + 2\frac{U_{n-1}^{(r)}}{(n-1)!} + \frac{U_{n-2}^{(r-1)}}{(n-2)!}\right\} = \frac{U_{n+1}^{(r)}}{n!}. \tag{3.4}$$

Therefore, (3.3) reduces to

$$S_n(k) = (n+1)\sum_{r=0}^{n+1}(k-1)^{-r}\frac{U_{2n+r+2}^{(r)}}{r(2n+r+2)!}, \qquad k \neq 1, \quad n \geqq 1. \tag{3.5}$$

The constant term ($r = 0$) is

$$\frac{1}{2(2n+1)!}U_{2n+1}^{(1)} = \frac{1}{2(2n+1)!}T_{2n+1}. \tag{3.6}$$

It follows from (1.5) and (3.5) that

$$P_{n+1}(1) = \frac{1}{(3n+3)!}U_{3n+3}^{(n+1)}.$$

Also it follows at once from (3.1) and $T_3 = 2$ that

$$U_{3n+3}^{(n+1)} = \frac{(3n+3)!}{3^{n+1}},$$

and therefore,

$$P_{n+1}(1) = 3^{-n-1},$$

as proved by Liron in a different way.

It is also clear from (3.6) and (3.2) that

$$\frac{(3n+3)!}{(n+1)!}P_{n+1}(k) \tag{3.7}$$

has integral coefficients. This result may be compared with (2.9).

4. Generating functions for $T_n'(k)$ and $S_n'(k)$. Differentiation of (2.6) with respect to k gives

$$kT_n'(k) = -(n+1)\sum_{r=1}^{n+1}(-1)^r\frac{k^{-r}}{(2n-r+2)!}T_{2n-r+2}^{(r)}.$$

Thus,

$$\begin{aligned} k\sum_{n=0}^{\infty}T_n'(k)\frac{t^{2n+2}}{n+1} &= -\sum_{n=1}^{\infty}t^{2n}\sum_{r=1}^{n}(-1)^r\frac{k^{-r}}{(2n-r)!}T_{2n-r}^{(r)} \\ &= -\sum_{n=1}^{\infty}t^n\sum_{0<2r\leqq n}(-1)^r\frac{k^{-r}}{(n-r)!}T_{n-r}^{(r)} \\ &= \sum_{r=1}^{\infty}(-1)^{r-1}k^{-r}\sum_{n=0}^{\infty}T_n^{(r)}\frac{t^n}{n!}. \end{aligned}$$

It follows that

$$k \sum_{n=0}^{\infty} T_n'(k) \frac{t^{2n+2}}{n+1} = \frac{t \sin t}{k \cos t + t \sin t}. \tag{4.1}$$

Similarly, differentiation of (3.5) gives

$$(k-1)S_n'(k) = -(n+1) \sum_{r=0}^{n+1} \frac{(k-1)^{-r}}{(2n+r+2)!} U_{2n+r+2}^{(r)}.$$

Thus,

$$\begin{aligned}(k-1) \sum_{n=0}^{\infty} S_n'(k) \frac{t^{2n+2}}{n+1} &= -\sum_{n=1}^{\infty} t^{2n} \sum_{r=1}^{n} \frac{(k-1)^{-r}}{(2n+r)!} U_{2n+r}^{(r)} \\ &= -\sum_{n=1}^{\infty} t^{n} \sum_{0<2r\leqq n} \frac{(k-1)^{-r}}{(n+r)!} U_{n+r}^{(r)} \\ &= -\sum_{r=1}^{\infty} (k-1)^{-r} t^{-r} \sum_{n=3r}^{\infty} U_n^{(r)} \frac{t^n}{n!}.\end{aligned}$$

A little manipulation leads to

$$(k-1) \sum_{n=0}^{\infty} S_n'(k) \frac{t^{2n+2}}{n+1} = \frac{t \cos t - \sin t}{kt \cos t - \sin t}. \tag{4.2}$$

REFERENCES

[1] A. HURWITZ, *Über die Entwickelungskoeffizienten der lemniscatischen Funktionen*, Math. Ann., 51 (1899), pp. 196–226 (= *Mathematische Werke*, vol. II, Birkhaüser, Basel, 1933, pp. 342–373).

[2] N. LIRON, *Some infinite sums*, this Journal, 2 (1971), pp. 105–112.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

PERTURBING UNIFORM ULTIMATE BOUNDED DIFFERENTIAL SYSTEMS*

STEPHEN R. BERNFELD†

Abstract. We obtain results on the eventual uniform boundedness and eventual uniform ultimate boundedness of solutions of the differential equation $\dot{x} = f(t, x) + g(t, x)$ given that solutions of the equation $\dot{x} = f(t, x)$ are uniformly bounded and uniformly ultimately bounded. By assuming various regularity conditions on f we obtain admissible classes of g such that the boundedness properties are preserved. By use of various examples the admissible classes of g are shown to be "maximal."

1. Introduction. A great deal of research has been done concerning the perturbation of uniform asymptotically stable differential systems. Recently, Strauss and Yorke [5] have generalized many of the earlier results concerned with this question by considering systems in which the perturbation terms are diminishing.

Our purpose here is to obtain results concerning the perturbation of uniformly bounded (hereafter called UB) and uniformly ultimately bounded (hereafter called UUB) differential systems. In particular, we shall prove theorems on the eventual uniform boundedness (hereafter called EvUB) and eventual uniform ultimate boundedness (hereafter called EvUUB) of solutions of the differential equation

$$\dot{x} = f(t, x) + g(t, x), \tag{P}$$

given that the solutions of the equation

$$\dot{x} = f(t, x) \tag{E}$$

are UB and UUB, where f and g satisfy various conditions. We shall always assume $f, g : [0, \infty) \times R^n \to R^n$ are continuous.

Yoshizawa [6], using Lyapunov functions, has proved that if system (E) is UB and UUB and f satisfies a uniform Lipschitz condition, then system (P) is EvUB and EvUUB if g is either "integrable" or approaches zero as $t \to \infty$. Using techniques similar to those used by Strauss and Yorke [4], [5], we generalize this result by assuming f satisfies either a more general Lipschitz condition (not necessarily uniform) or an inner product condition, and that g satisfies a condition called "strongly diminishing," special cases of which include those considered by Yoshizawa. Examples are provided which show that the conditions on g cannot be easily weakened.

For the case when $f(t, x) = A(t)x$, we prove that if $g(t, x) = g_1(t, x) + h(t)$, where g_1 is strongly absolutely diminishing and $h(t)$ satisfies a condition widely used by Massera and Schaffer [3] ($\sup_{t \in [0, \infty)} \int_t^{t+1} |h(s)| \, ds < \infty$), then solutions of (P) are EvUB and EvUUB. An example is provided which shows the conditions on h and g_1 cannot be easily weakened.

* Received by the editors June 4, 1971, and in revised form October 11, 1971.

† Department of Mathematics, University of Missouri, Columbia, Missouri 65201.

Our results are applied to systems with small parameters as well as to second order equations with a forcing term.

2. Preliminaries. Let R^n denote Euclidean n-space and let $|\cdot|$ denote any n-dimensional norm. We use $\|\cdot\|$ for the Euclidean norm and $\langle x, y\rangle$ for the inner product of $x, y \in R^n$. We shall assume throughout that the right-hand side of every differential equation is continuous on $D_S = \{(t, x): t \geqq 0, |x| > S, S > 0\}$. For $(t_0, x_0) \in D_S$ we denote by $x(t, t_0, x_0)$ that solution satisfying $x(t_0, t_0, x_0) = x_0$. (For clarity, we shall often use $y(t, t_0, x_0)$ to be a solution of (P) and $x(t, t_0, x_0)$ for (E).)

DEFINITION 2.1. The solutions of (E) are *eventually uniformly bounded* (EvUB) if for each $\alpha > 0$ there exist $\gamma = \gamma(\alpha) \geqq 0$ and $\beta(\alpha)$ such that

$$|x(t, t_0, x_0)| < \beta(\alpha) \quad \text{for } |x_0| < \alpha \quad \text{and} \quad t \geqq t_0 \geqq \gamma.$$

The solutions are *uniformly bounded* (UB) if $\gamma(\alpha) \equiv 0$.

DEFINITION 2.2. The solutions of (E) are *eventually uniformly ultimately bounded* (EvUUB) if there exists a $B > 0$ such that for each $\alpha > 0$ there exist $\gamma = \gamma(\alpha) \geqq 0$ and $T(\alpha) \geqq 0$ such that

$$|x(t, t_0, x_0)| < B \quad \text{for } |x_0| < \alpha, \quad t_0 \geqq \gamma \quad \text{and} \quad t \geqq t_0 + T(\alpha).$$

The solutions are *uniformly ultimately bounded* (UUB) if $\gamma(\alpha) \equiv 0$ for all α.

DEFINITION 2.3. Let $h: [0, \infty) \to R^n$ be continuous. Then h is *absolutely diminishing* if

$$\int_t^{t+1} |h(s)|\, ds \to 0 \quad \text{as } t \to \infty.$$

DEFINITION 2.4. Let $h: [0, \infty) \to R^n$ be continuous. Then h is *diminishing* if

$$\sup_{0 \leqq u \leqq 1} \left| \int_t^{t+u} h(s)\, ds \right| \to 0 \quad \text{as } t \to \infty.$$

DEFINITION 2.5. Let $g: [0, \infty) \times R^n \to R^n$ be continuous. Then g is *strongly absolutely diminishing* if

$$\int_t^{t+1} |g(s, x(s))|\, ds \to 0 \quad \text{as } t \to \infty$$

for each bounded continuous function $x(s)$.

We notice that if for all m sufficiently large there exists an absolutely diminishing function $h_m(t)$ such that $|g(t, x)| \leqq h_m(t)$ for $t \geqq 0$ and $|x| \leqq m$, then g is absolutely diminishing.

DEFINITION 2.6. Let $g: [0, \infty) \times R^n \to R^n$ be continuous. Then g is *strongly diminishing* if

$$\sup_{0 \leqq u \leqq 1} \left| \int_t^{t+u} g(s, x(s))\, ds \right| \to 0 \quad \text{as } t \to \infty$$

for each bounded continuous function $x(s)$.

DEFINITION 2.7. Let $\overline{C_0(x)}$ be the class of continuous functions $f(t, x)$ defined on $[0, \infty) \times R^n$ with range in R^n such that for each $\alpha > 0$ there exists $L(\alpha) > 0$ such that

$$|f(t, x) - f(t, y)| \leqq L(\alpha)|x - y| \tag{2.1}$$

for all $t \geqq 0$ and all $|x| \leqq \alpha$, $|y| \leqq \alpha$. We say f is a *Lipschitz function* if $f \in \overline{C_0(x)}$.

DEFINITION 2.8. Let $\mathscr{M}$ be the set of continuous functions $h: [0, \infty) \to R^n$ satisfying

$$\sup_{t \in [0,\infty)} \int_t^{t+1} |h(s)|\, ds < \infty,$$

and let $\overline{\mathscr{M}}$ be the set of continuous functions $h \in [0, \infty) \to R^n$ satisfying

$$\sup_{\substack{0 \leqq u \leqq 1 \\ t \in [0,\infty)}} \left| \int_t^{t+u} h(s)\, ds \right| < \infty.$$

We shall also consider the following condition.

(H_1) There exists $K > 0$ such that for each continuous bounded function $x(t) \in D_S$, $t \geqq 0$,

$$\limsup_{t \to \infty} \frac{\int_t^{t+1} |g(s, x(s))|\, ds}{|x(t)|} < K.$$

We observe that if $|g(t, x)| \leqq h(t)|x|$, where $h(t) \in \mathscr{M}$, then g satisfies (H_1); or if g is strongly absolutely diminishing, then g satisfies (H_1).

Definitions 2.1, 2.2, and 2.7 may also be found in [6], Definitions 2.3 and 2.4 in [5], and the definition of $\mathscr{M}$ in [3].

We now present some lemmas which will help simplify the proofs of our main results. Lemma 2.9 is an inequality of the Gronwall type and Lemma 2.10 is analogous to Lemma 2 of [1, p. 102]. We shall omit the proofs of these two lemmas.

LEMMA 2.9. *Let $r(t)$, $p(t)$, $u(t)$ be continuous for $t \geqq t_0$; let $c \geqq 0$ and $u(t) \geqq 0$; and let*

$$r(t) \leqq c + \int_{t_0}^t [u(s)r(s) + p(s)]\, ds.$$

Then,

$$r(t) \leqq c \exp\left(\int_{t_0}^t u(s)\, ds\right) + \int_{t_0}^t p(s) \exp\left(\int_s^t u(m)\, dm\right) ds.$$

LEMMA 2.10. *Let $x(s)$ be a continuous bounded function. Assume*

$$\limsup_{t \to \infty} \int_t^{t+1} |g(s, x(s))|\, ds \leqq k$$

for some $k > 0$. Then for each $\varepsilon > 0$ there exists $t_0(\varepsilon)$ such that for any $c > 0$,

$$\sup_{t \geqq t_0} e^{-ct} \int_{t_0}^t e^{cs}|g(s, x(s))|\, ds \leqq \frac{k_1(\varepsilon)\, e^{2c}}{e^c - 1},$$

where $k_1(\varepsilon) = k + \varepsilon$.

LEMMA 2.11. *If $g(t, x)$ is a strongly diminishing function, then for each $\alpha > 0$,*

$$\sup_{0 \leqq u \leqq 1} \left| \int_t^{t+u} g(s, x(s))\, ds \right| \to 0 \quad \text{as } t \to \infty$$

uniformly for all continuous bounded functions $x(t)$, $t \geqq 0$, satisfying $|x(t)| \leqq \alpha$.

Proof. Suppose the conclusion is false. Then there exist an $\alpha > 0$, a sequence of points $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$, a sequence of points $\{u_n\}$ where $u_n \to u_0 \leqq 1$ as $n \to \infty$, a sequence of continuous functions $\{y_n(s)\}$ where $|y_n(s)| \leqq \alpha$ and an $\varepsilon > 0$ such that

$$\left| \int_{t_n}^{t_n+u_n} g(s, y_n(s))\, ds \right| > \varepsilon.$$

We may assume $t_{n+1} - t_n > 1$. Construct the continuous bounded function

$$\hat{y}(t) = \begin{cases} y_1(t) & \text{for } t \in [0, t_1], \\ y_n(t) & \text{for } t \in [t_n, t_n + u_n], \\ y_n(t_n + u_n) + \dfrac{(t - t_n - u_n)}{(t_{n+1} - t_n - u_n)}(y_{n+1}(t_{n+1}) - y_n(t_n + u_n)) & \\ \qquad & \text{for } t \in [t_n + u_n, t_{n+1}]. \end{cases}$$

Hence,

$$\left| \int_{t_n}^{t_n+u_n} g(s, \hat{y}(s))\, ds \right| = \left| \int_{t_n}^{t_n+u_n} g(s, y_n(s))\, ds \right| > \varepsilon,$$

which is a contradiction, since from the hypothesis we have

$$\lim_{n \to \infty} \left| \int_{t_n}^{t_n+u_n} g(s, \hat{y}(s))\, ds \right| = 0.$$

Using similar techniques as in Lemma 2.11 we can prove the following lemma.

LEMMA 2.12. *Assume $g(t, x)$ satisfies* (H_1). *Then there exists $K > 0$ such that for each $\alpha > 0$ there exists $t_0(\alpha)$ such that for $t \geqq t_0(\alpha)$,*

$$\int_t^{t+1} |g(s, x(s))|\, ds \leqq K|x(t)|$$

for all $|x(t)| \leqq \alpha$.

In the following lemma we omit the proof.

LEMMA 2.13. *If solutions are UB and EvUUB, then they are UUB.*

We now give a characterization of UB and UUB in terms of Lyapunov functions.

THEOREM 2.14 (Yoshizawa [6, p. 107]). *Assume $f(t, x)$ is locally Lipschitz in x (for each point (t, x) there exists a neighborhood where the Lipschitz condition* (2.1) *holds). Then solutions of* (E) *are UB if and only if there exists a continuous function $V: D_S \to R$ satisfying*

(2.2) $\qquad$ *V is locally Lipschitz;*

(2.3) $\qquad$ $a(|x|) \leqq V(t, x) \leqq b(|x|)$, *where* $a(\cdot)$, $b(\cdot)$ *are continuous increasing functions and* $a(r) \to \infty$ *as* $r \to \infty$;

$$\dot{V}(t,x) = \limsup_{h \to 0^+} \frac{V(t+h, x+hf(t,x)) - V(t,x)}{h} \leqq 0. \tag{2.4}$$

Moreover, solutions are UUB if and only if V satisfies, in addition,

$$\dot{V}(t,x) \leqq -cV(t,x) \quad \text{for some } c > 0. \tag{2.5}$$

Moreover, if $f \in \overline{C_0(x)}$, *then* $V \in \overline{C_0(x)}$.

It has also been proved [6, p. 92] that if $f(t,x) = A(t)x$, then solutions of (E) are UB and UUB if and only if there exists a continuous $V: D_S \to R$ satisfying (2.3) with $a(|x|) = |x|$, (2.5) and

$$|V(t,x) - V(t,y)| \leqq \Gamma|x - y| \tag{2.6}$$

for some $\Gamma > 0$ and all x, y satisfying $|x|, |y| \geqq S$.

3. Perturbed linear systems.

THEOREM 3.1. *Assume solutions of*

$$\dot{x} = A(t)x \tag{L}$$

are UB and UUB. Consider the perturbed system

$$\dot{x} = A(t)x + \gamma(t,x) + g(t,x) + h(t). \tag{PL}$$

Assume γ *is continuous on* D_S *and satisfies*

$$|\gamma(t,x)| \leqq L(t)|x|,$$

where

$$\limsup_{\tau \to \infty} \frac{1}{\tau} \int_{t_0}^{t_0+\tau} L(s)\, ds < Q$$

for sufficiently small Q *and large* t_0. *Assume g is continuous on* D_S *and g satisfies* (H_1) *for sufficiently small K. Assume* $h \in \mathscr{M}$. *Then the solutions of* (PL) *are EvUB and EvUUB.*

Proof. From the remarks after Theorem 2.14 we have the existence of a function V satisfying (2.3), (2.5) and (2.6). Hence, we have

$$\begin{aligned} \dot{V}_{(\mathrm{PL})}(t,x) &\leqq \dot{V}_{(\mathrm{L})}(t,x) + \Gamma|\gamma(t,x)| + \Gamma|g(t,x)| + \Gamma|h(t)| \\ &\leqq -cV(t,x) + \Gamma L(t)|x| + \Gamma|g(t,x)| + \Gamma|h(t)| \\ &\leqq (-c + \Gamma L(t))V(t,x) + \Gamma|g(t,x)| + \Gamma|h(t)|. \end{aligned} \tag{3.1}$$

Let $x(t, t_0, x_0)$ be a solution of (PL) and define, for $t \geqq t_0$,

$$r(t, t_0, r_0) = V(t, x(t, t_0, x_0)).$$

Hence, from (3.1) we have

$$\dot{r}(t) \leqq (-c + \Gamma L(t))r(t) + \Gamma|g(t, x(t))| + \Gamma|h(t)|. \tag{3.2}$$

Pick T_1 so large that for $t_0 > T_1$ and $\tau > T_1$, $(1/\tau)\int_{t_0}^{t_0+\tau} L(s)\,ds \leqq Q$, where we assume $Q < c/\Gamma$. Define $-\mu = Q\Gamma - c$; and hence, for $t > t_0 + T_1$ and $t_0 > T_1$, we have

$$\frac{1}{t - t_0}\int_{t_0}^{t} (-c + \Gamma L(s))\, ds \leqq -\mu.$$

Therefore,

$$(3.3) \qquad \int_{t_0}^{t} (-c + \Gamma L(s))\, ds \leqq -\mu(t - t_0).$$

From (3.2), (3.3), using the variation of constants formula for linear systems, we obtain

$$(3.4) \quad r(t, t_0, r_0) \leqq e^{-\mu(t-t_0)} r_0 + \Gamma e^{-\mu t} \int_{t_0}^{t} e^{\mu s}|g(s, x(s))|\, ds + \Gamma e^{-\mu t}\int_{t_0}^{t} |h(s)|\, e^{\mu s}\, ds.$$

Using (2.3), we have from (3.4),

$$(3.5) \quad \begin{aligned} |x(t, t_0, x_0)| \leqq{}& e^{-\mu(t-t_0)} b(|x_0|) + \Gamma e^{-\mu t}\int_{t_0}^{t} e^{\mu s}|g(s, x(s))|\, ds \\ &+ \Gamma e^{-\mu t}\int_{t_0}^{t} |h(s)|\, e^{\mu s}\, ds. \end{aligned}$$

Pick any $\alpha > S$, and from Lemma 2.12 there exists $T_2 = T_2(\alpha)$ such that for $t \geqq T_2$,

$$(3.6) \qquad \int_{t}^{t+1} |g(s, x(s))|\, ds \leqq K|x(t)|$$

for all continuous functions $x(t)$ such that $|x(t)| \leqq \alpha$. For $t_0 \geqq \max(T_2(3b(\alpha)), T_1)$ and for $|x_0| \leqq \alpha$, we have from (3.5), using Lemma 2.10 and (3.6), that for as long as $|x(t, t_0, x_0)| \leqq 3b(\alpha)$,

$$(3.7) \quad |x(t, t_0, x_0)| \leqq e^{-\mu(t-t_0)} b(|x_0|) + \frac{2K\Gamma e^{2\mu}}{e^{\mu} - 1} \sup_{t \geqq s \geqq t_0} |x(s, t_0, x_0)| + \frac{\Gamma K_1 e^{2\mu}}{e^{\mu} - 1},$$

where $K_1 = \sup_{t \geqq 0}\int_t^{t+1} |h(s)|\, ds$. We may assume that α is so large that $\Gamma K_1 e^{2\mu}/(e^{\mu} - 1) < b(\alpha)/2$. Pick K so small that $K < (e^{\mu} - 1)/(4\Gamma e^{2\mu})$. Hence, from (3.7) we have, for as long as $|x(t, t_0, x_0)| \leqq 3b(\alpha)$,

$$\sup_{t_0 \leqq s \leqq t} |x(s, t_0, x_0)| \left(1 - \frac{2K\Gamma e^{2\mu}}{e^{\mu} - 1}\right) \leqq b(\alpha) + \frac{\Gamma K_1 e^{2\mu}}{e^{\mu} - 1},$$

or

$$(3.8) \qquad |x(t, t_0, x_0)| < 2b(\alpha) + b(\alpha) = 3b(\alpha).$$

Hence (3.8) holds for all $t \geqq t_0(\alpha) = \max(T_2(3b(\alpha)), T_1)$. This implies solutions are EvUB. There exists $T_3 = T_3(\alpha) > 0$ such that $e^{-\mu T_3(\alpha)} b(\alpha) < 1$ for all α. Hence, for all $t \geqq t_0 + T_3$ we have, from (3.8),

$$|x(t, t_0, x_0)| \leqq 2 + \frac{2\Gamma K_1 e^{2\mu}}{e^{\mu} - 1},$$

which implies solutions are EvUUB, thus proving the theorem.

We now consider perturbed systems with small parameters. Such systems [2] often arise in mechanical processes where one is interested in preserving the process under small perturbations. From the proof of Theorem 3.1 we may state the following result which extends the work of Yoshizawa [6, p. 132] to systems with boundedness properties.

COROLLARY 3.2. *Assume solutions of* (L) *are UB and UUB. Consider the perturbed equation*

$$(\mathrm{P}_\varepsilon) \qquad \dot{x} = A(t)x + g(t, x, \varepsilon),$$

where ε *is an m-vector. If*

$$|g(t, x, \varepsilon)| \leqq h(t, \varepsilon)|x|$$

and if there exists a $K > 0$ *such that*

$$\int_t^{t+1} |h(s, \varepsilon)|\, ds \leqq K|\varepsilon|,$$

then there exists an $\varepsilon_0 > 0$ *such that for all* $|\varepsilon| < \varepsilon_0$ *solutions of* (P_ε) *are EvUB and EvUUB.*

In Theorem 3.1 we observed that we could perturb (L) by a function $h(t)$ satisfying $\sup_{t \geqq 0} \int_t^{t+1} |h(s)|\, ds < \infty$. A natural question is whether we can weaken the condition on $h(t)$ so as to satisfy $h(t) \in \overline{\mathscr{M}}$; that is,

$$(3.9) \qquad \sup_{\substack{0 \leqq u \leqq 1 \\ t \geqq 0}} \left| \int_t^{t+u} h(s)\, ds \right| < \infty,$$

and still have solutions of

$$(\mathrm{PLH}) \qquad \dot{x} = A(t)x + h(t)$$

be EvUB and EvUUB. The following example, based on one given by Strauss and Yorke [5, p. 472], shows that Theorem 3.1 does not hold for $h(t)$ satisfying (3.9).

Example 3.3. Consider the linear system

$$\dot{x} = A(t)x, \qquad A(t) = \begin{pmatrix} -1 & e^t \\ -e^t & -1 \end{pmatrix}.$$

Then a fundamental matrix satisfies

$$X(t) = e^{-t}\begin{pmatrix} \sin e^t & -\cos e^t \\ \cos e^t & \sin e^t \end{pmatrix}, \qquad X^{-1}(t) = e^t\begin{pmatrix} \sin e^t & \cos e^t \\ -\cos e^t & \sin e^t \end{pmatrix}.$$

Also, $|X(t)X^{-1}(s)| \leqq Ke^{-(t-s)}$ for some $K > 0$ and for all $t \geqq s \geqq 0$. Define

$$h(t) = \begin{pmatrix} e^{t/2} \sin e^t \\ e^{t/2} \cos e^t \end{pmatrix}.$$

It is not difficult to show $h(t)$ satisfies (3.9).

We now consider the solution through $(t_0, 0)$, $x(t, t_0, 0)$ of (PLH). For any $t_0 > 0$ and $t \geqq t_0$,

$$x(t, t_0, 0) = X(t) \int_{t_0}^t X^{-1}(s)h(s)\, ds.$$

Hence,

$$x(t, t_0, 0) = \frac{2}{3}(e^{t/2} - e^{-(t-3t_0/2)})\begin{pmatrix}\sin e^t \\ \cos e^t\end{pmatrix}.$$

Therefore $\limsup_{t\to\infty} |x(t, t_0, 0)| = \infty$; that is, solutions are not even eventually bounded.

We observe that $h(t)$ is diminishing in Example 3.3. In fact, a reasonable conjecture is that we can perturb (L) by a function $h(t)$ satisfying (3.9) if and only if we can perturb (L) by a diminishing function $h(t)$. Moreover, with the same techniques as in Lemma 4.1 in [5], we can show that if $A(t) \in \mathcal{M}$, that is, $\sup_{t\geqq 0}\int_t^{t+1} |A(s)|\,ds < \infty$, then a necessary condition on $h(t)$ for solutions of (PLH) to be EvUB and EvUUB is that $h(t)$ satisfy (3.9); and in Corollary 4.3 we show that a sufficient condition is that $h(t)$ be diminishing. Hence, if our previous conjecture is true, then if $A(t) \in \mathcal{M}$ and solutions of (L) are UB and UUB, then solutions of (PLH) are EvUB and EvUUB if and only if $h(t)$ satisfies (3.9)

4. Perturbing nonlinear systems. In this section we consider the equation (E) and the perturbed equation (P) and we assume $f(t, x)$ satisfies either a generalized Lipschitz condition or an inner product condition. In order to prove solutions of (P) are EvUB and EvUUB when solutions of (E) are UB and UUB we essentially show that on compact intervals each solution of (P) stays close to a solution of (E). This technique has been used by Strauss and Yorke previously [4], [5] in the case of stability.

THEOREM 4.1. *Assume solutions of* (E) *are UB and UUB and that for each* $\alpha > 0$ *there exists* $L(\alpha) > 0$ *such that f satisfies, on* D_S,

$$|f(t, x) - f(t, y)| \leqq \lambda(t)L(\alpha)|x - y|, \tag{4.1}$$

for all $|x| \leqq \alpha$ *and* $|y| \leqq \alpha$, *where* $\lambda(t) \in \mathcal{M}$. *If* $g(t, x)$ *is strongly diminishing on* D_S, *then solutions of* (P) *are EvUB and EvUUB. Moreover, if solutions of* (P) *are UB, then they are UUB.*

Proof. Define

$$\Lambda(t_0) = \sup_{T \geqq t_0} \int_T^{T+1} \lambda(s)\,ds.$$

There exists an $M > 0$ such that $\Lambda(t_0) \leqq M$ for all $t_0 \geqq 0$. Hence, we have for $t \geqq t_0$,

$$\int_{t_0}^{t} \lambda(s)\,ds = \int_{t_0}^{t_0+1} \lambda(s)\,ds + \int_{t_0+1}^{t_0+2} \lambda(s)\,ds + \cdots + \int_{t_0+m}^{t} \lambda(s)\,ds$$
$$\leqq (t - t_0 + 1)\Lambda(t_0) \leqq M(t - t_0 + 1).$$

For any $\alpha > S$ define

$$G_\alpha(t) = \sup_{\substack{0\leqq u\leqq 1\\ T\geqq t\\ |x(s)|\leqq\alpha}} \left| \int_T^{T+u} g(s, x(s))\,ds \right|,$$

and we conclude from Lemma 2.11 that $G_\alpha(t) \downarrow 0$ as $t \to \infty$. Hence, for all $x(t)$ such that $|x(t)| \leqq \alpha$ and for $t \geqq t_0 \geqq 0$, we have

$$\left|\int_{t_0}^{t} g(s, x(s))\, ds\right| \leqq \left|\int_{t_0}^{t_0+1} g(s, x(s))\, ds\right| + \left|\int_{t_0+1}^{t_0+2} g(s, x(s))\, ds\right| + \cdots + \left|\int_{t_0+m}^{t} g(s, x(s))\, ds\right|$$
$$\leqq G_\alpha(t_0)(t - t_0 + 1).$$

Let $x(t, t_0, x_0)$ be a solution of (E) with $|x_0| < \alpha$. Then there exists $\beta(\alpha)$ such that $|x(t, t_0, x_0)| < \beta(\alpha)$ for $t \geqq t_0$, and there exist a $B > 0$ and a $T_0(\alpha)$ such that $|x(t, t_0, x_0)| < B$ for $t \geqq t_0 + T_0(\alpha)$. Define

$$B_1 = B_1(\alpha) = \max(\beta(\alpha) + 1, \beta(B + 1) + 1).$$

Let $y(t, t_0, x)$ be any solution of (P). Then for as long as $|y(t, t_0, x_0)| \leqq B_1(\alpha)$ on the interval $[t_0, t_0 + \tau]$ for some $\tau > 0$, we have, using (4.1),

$$|y(t, t_0, x_0) - x(t, t_0, x_0)|$$
$$= \left|\int_{t_0}^{t} (f(s, y(s, t_0, x_0)) + g(s, y(s, t_0, x_0)))\, ds - \int_{t_0}^{t} f(s, x(s, t_0, x_0))\, ds\right|$$
$$\leqq \int_{t_0}^{t} \lambda(s)L(B_1)|y(s, t_0, x_0) - x(s, t_0, x_0)|\, ds + (t - t_0 + 1)G_{B_1}(t_0)$$
$$= G_{B_1}(t_0) + \int_{t_0}^{t} (\lambda(s)L(B_1)|y(s, t_0, x_0) - x(s, t_0, x_0)| + G_{B_1}(t_0))\, ds.$$

Using Lemma 2.9 we conclude

$$|y(t, t_0, x_0) - x(t, t_0, x_0)|$$
$$\leqq G_{B_1}(t_0) \exp\left(L(B_1) \int_{t_0}^{t} \lambda(s)\, ds\right) + \int_{t_0}^{t} G_{B_1}(t_0) \exp\left(L(B_1) \int_{s}^{t} \lambda(u)\, du\right) ds.$$
$$\leqq G_{B_1}(t_0) \exp(L(B_1)(t - t_0 + 1)M) + \int_{t_0}^{t} G_{B_1}(t_0) \exp(L(B_1)(t - t_0 + 1)M)\, ds.$$

Hence, for as long as $|y(t, t_0, x_0)| \leqq B_1$ for $t \in [t_0, t_0 + \tau]$,

$$|y(t, t_0, x_0) - x(t, t_0, x_0)| \leqq G_{B_1}(t_0) \exp(L(B_1)(\tau + 1)M) + \tau G_{B_1}(t_0) \exp(L(B_1)(\tau + 1)M) = (1 + \tau)G_{B_1}(t_0) \exp(L(B_1)(\tau + 1)M). \tag{4.2}$$

Choose $\tau = \tau(\alpha) = T_0(B_1(\alpha))$. We may pick $T_1 = T_1(\alpha)$ so large that for $t \geqq T_1$ we have

$$(1 + \tau)G_{B_1}(t) \exp(L(B_1)(\tau + 1)M) < 1. \tag{4.3}$$

Pick $t_0 \geqq T_1(\alpha)$, and thus for as long as $|y(t, t_0, x_0)| \leqq B_1(\alpha)$ in the interval $[t_0, t_0 + \tau]$, we have, using (4.2) and (4.3),

$$\begin{aligned} |y(t, t_0, x_0)| &\leqq |y(t, t_0, x_0) - x(t, t_0, x_0)| + |x(t, t_0, x_0)| \\ &< 1 + \beta(\alpha) \leqq B_1(\alpha). \end{aligned} \tag{4.4}$$

Hence, $|y(t, t_0, x_0)| \leqq B_1(\alpha)$ for all $t \in [t_0, t_0 + \tau]$. Since we may assume $T_0(B_1(\alpha)) \geqq T_0(\alpha)$, we have $|x(t_0 + \tau, t_0, x_0)| < B$, and using (4.4) we conclude $|y(t_0 + \tau, t_0, x_0)| \leqq 1 + B \leqq B_1(\alpha)$. Let $x_1 = y(t_0 + \tau, t_0, x_0)$. Then since $|x_1| \leqq 1 + B$, we have $|x(t, t_0 + \tau, x_1)| < \beta(B + 1) \leqq B_1(\alpha)$ for $t \geqq t_0 + \tau$. For as long as $|y(t, t_0 + \tau, x_1)| \leqq B_1(\alpha)$ on the interval $[t_0 + \tau, t_0 + 2\tau]$, we have, using (4.2) and (4.3),

$$\begin{aligned} |y(t, t_0 + \tau, x_1)| &\leqq |y(t, t_0 + \tau, x_1) - x(t, t_0 + \tau, x_1)| + |x(t, t_0 + \tau, x_1)| \\ &< 1 + \beta(B + 1) \leqq B_1(\alpha). \end{aligned}$$

Hence, $|y(t, t_0 + \tau, x_1)| \leqq B_1(\alpha)$ for all $t \in [t_0 + \tau, t_0 + 2\tau]$. Since $\tau = T_0(B_1(\alpha)) \geqq T_0(B + 1)$, we have $|x(t_0 + 2\tau, t_0 + \tau, x_1)| < B$. Thus, we have

$$\begin{aligned} |y(t_0 + 2\tau, t_0 + \tau, x_1)| &\leqq |y(t_0 + 2\tau, t_0 + \tau, x_1) - x(t_0 + 2\tau, t_0 + \tau, x_1)| \\ &\quad + |x(t_0 + 2\tau, t_0 + \tau, x_1)| \\ &\leqq 1 + B. \end{aligned}$$

Let m be any positive integer and assume $|y(t, t_0, x_0)| \leqq B_1(\alpha)$ for $t \in [t_0, t_0 + m\tau]$ and $|y(t_0 + m\tau, t_0, x_0)| \leqq B + 1 \leqq B_1(\alpha)$. Let $x_m = y(t_0 + m\tau, t_0, x_0)$, and thus $|x(t, t_0 + m\tau, x_m)| < \beta(B + 1)$ for $t \geqq t_0 + m\tau$. For as long as $|y(t, t_0 + m\tau, x_m)| \leqq B_1(\alpha)$ on the interval $[t_0 + m\tau, t_0 + (m + 1)\tau]$ we have

$$\begin{aligned} |y(t, t_0 + m\tau, x_m)| &\leqq |y(t, t_0 + m\tau, x_m) - x(t, t_0 + m\tau, x_m)| + |x(t, t_0 + m\tau, x_m)| \\ &\leqq 1 + \beta(B + 1) \leqq B_1(\alpha). \end{aligned} \tag{4.5}$$

Hence, $|y(t, t_0 + m\tau, x_m)| \leqq B_1(\alpha)$ for all $t \in [t_0 + m\tau, t_0 + (m + 1)\tau]$; moreover,

$$\begin{aligned} &|y(t_0 + (m + 1)\tau, t_0 + m\tau, x_m)| \\ &\qquad \leqq |y(t_0 + (m + 1)\tau, t_0 + m\tau, x_m) - x(t_0 + (m + 1)\tau, t_0 + m\tau, x_m)| \\ &\qquad\quad + |x(t_0 + (m + 1)\tau, t_0 + m\tau, x_m)| \\ &\qquad \leqq 1 + B. \end{aligned}$$

Thus, by induction we have $|y(t, t_0, x_0)| \leqq B_1(\alpha)$ for $t \in [t_0 + m\tau]$ for all $m > 0$; that is, $|y(t, t_0, x_0)| \leqq B_1(\alpha)$ for $t \geqq t_0 \geqq T_1(\alpha)$. Hence solutions of (P) are EvUB.

We also observe from (4.5) that for $t \in [t_0 + m\tau, t_0 + (m + 1)\tau]$ for all $m \geqq 1$, that is, for $t \geqq t_0 + \tau$, we have

$$|y(t, t_0, x_0)| = |y(t, t_0 + m\tau, x_m)| \leqq 1 + \beta(B + 1).$$

Hence, for any $\alpha > S > 0$ and for $|x_0| < \alpha$, we have for $t_0 \geqq T_1(\alpha)$ and $t \geqq t_0 + T_0(B_1(\alpha))$ that $|y(t, t_0, x_0)| \leqq 1 + \beta(B + 1)$. Hence, solutions of (P) are EvUUB. If, moreover, solutions of (P) are UB, then by Lemma 2.13 solutions are UUB, thus completing the proof of Theorem 4.1.

Remark. We observe that if $\int_0^\infty |g(s, x(s))|\, ds < \infty$ for each bounded continuous function $x(t)$ or if $|g(t, x(t))| \to 0$ as $t \to \infty$ for each bounded continuous function $x(t)$, then $g(t, x)$ is strongly diminishing. Moreover if $f(t, x) \in \overline{C_0(x)}$, then $f(t, x)$ satisfies (4.1). Hence as a special case of Theorem 4.1 we have obtained the following result of Yoshizawa [6, p. 127].

COROLLARY 4.2. *If solutions of* (E) *are UB and UUB, where* $f \in \overline{C_0(x)}$, *and if* $\int_{t_0}^\infty |g_1(t, x(t))|\, dt < \infty$ *for each bounded continuous function* $x(t)$ *and if* $|g_2(t, x(t))| \to 0$ *as* $t \to \infty$ *for each bounded continuous function* $x(t)$, *then solutions of*

$$\dot{x} = f(t, x) + g_1(t, x) + g_2(t, x)$$

are EvUB and EvUUB.

In proving this result Yoshizawa used Lyapunov functions. However, it does not seem that a Lyapunov function can be used alone to prove Theorem 4.1 for two reasons. First, the condition (4.1) on f may not necessarily imply that there exists a Lyapunov function $V(t, x)$ satisfying (4.1) as it does in the case for $f \in \overline{C_0(x)}$. Second, since $g(t, x)$ is strongly diminishing and not absolutely strongly diminishing, we are not able to obtain a differential inequality in terms of a Lyapunov function. We now apply Theorem 4.1 to (L).

COROLLARY 4.3. *Assume solutions of* (L) *are UB and UUB and that* $\sup_{t \in [0,\infty)} \int_t^{t+1} |A(s)|\, ds < \infty$. *If* $g(t, x)$ *is strongly diminishing, then solutions of*

$$\dot{x} = A(t)x + g(t, x)$$

are EvUB and EvUUB.

As we pointed out in Example 3.3 we cannot perturb (L) by a diminishing function $h(t)$ without restrictions on $A(t)$.

We now apply Theorem 4.1 to a forced Lienard equation. Various examples of forced Lienard equations are presented in [6] where the forcing term is integrable, and using Lyapunov functions, one is able to conclude that the forced equation is UB and UUB. In our next example we weaken the conditions of our forcing term and conclude that solutions of the forced equation are EvUB and EvUUB. This type of "trade-off" sometimes may be desirable.

Example 4.4. Consider the Lienard equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

where $f(x), g(x)$ are Lipschitz, $F(x) = \int_0^x f(u)\, du \to {}^{+\infty}_{-\infty}$, as $x \to {}^{+\infty}_{-\infty}$, and where $xg(x) \geqq 0$. Then, under these conditions, it is known that solutions are UB and UUB (see [6, p. 41]). It has been shown [6, p. 41] that if we consider

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t),$$

where $\int_0^\infty p(s)\, ds < \infty$, then solutions are UB and UUB. From Theorem 4.1 we may weaken the condition on $p(t)$ by assuming p is diminishing and thus conclude that solutions are EvUB and EvUUB.

Remark. In Theorem 4.1 we have seen that a sufficient condition for solutions of

$$(\mathrm{P_H}) \qquad \dot{x} = f(t, x) + h(t)$$

to be EvUB and EvUUB when solutions of (E) are UB and UUB is for $h(t)$ to be diminishing. It would be of interest to see whether solutions of $(\mathrm{P_H})$ are EvUB and

EvUUB if $h(t) \in \overline{\mathscr{M}}$, assuming of course solutions of (E) are UB and UUB and f satisfies (4.1). In the following example we show we cannot necessarily even perturb (E) with a function $h(t) \in \mathscr{M}$.

Example 4.5. Consider the scalar equation

$$\dot{x} = f(x), \tag{4.6}$$

where

$$f(x) = \begin{cases} -1/x & \text{for} \quad x > 1, \\ -x & \text{for} \quad x \leqq 1; \end{cases}$$

and we see $f \in \overline{C_0(x)}$ (in fact the Lipschitz constant is 1 for all points in the space). For $x_0 \leqq 1$ we have all solutions approaching zero uniformly. For $x > 1$ we have, for any point (t_0, x_0),

$$x(t, t_0, x_0) = \sqrt{x_0^2 - 2(t - t_0)} \quad \text{for} \quad t_0 \leqq t \leqq t_0 + (x_0^2 - 1)/2,$$

and

$$x(t, t_0, x_0) = \exp\left[-\left(t - \left(t_0 + \frac{x_0^2 - 1}{2}\right)\right)\right] \text{ for } \quad t \geqq t_0 + \frac{x_0^2 - 1}{2}.$$

Hence, solutions of (4.6) are UB and UUB for a bound 1. We now perturb (4.6) by $h(t) \equiv 1$. Consider then,

$$\begin{aligned} \dot{x} &= -\frac{1}{x} + 1, \qquad x > 1, \\ \dot{x} &= -x + 1, \qquad x \leqq 1. \end{aligned} \tag{4.7}$$

For any point (t_0, x_0), $t_0 \in [0, \infty)$, $x_0 > 1$, we see that $\dot{x}$ is increasing, and hence solutions of (4.7) are unbounded for $x_0 > 1$.

We observe that Theorem 4.1 does not include the cases where $f(t, x) = \psi(t)x^{2n+1}$, $n = 0, 1, 2$, where $\psi(t)$ is negative but is not contained in $\mathscr{M}$. For example, we cannot apply either Theorems 3.1 or 4.1 to the scalar function $f(t, x) = -t^p x^{2n+1}$, where $p > 0$.

We now present a theorem which will allow us to perturb (E) when f satisfies the following inner product condition: for each $\alpha > 0$ there exists $L(\alpha) > 0$ such that

$$\langle x - y, f(t, x) - f(t, y)\rangle \leqq \lambda(t)L(\alpha)\|x - y\|^2 \tag{4.8}$$

for $\|x\| \leqq \alpha$, $\|y\| \leqq \alpha$ and $\lambda \in \mathscr{M}$. A similar condition for $\lambda(t) \equiv 1$ and $L(\alpha) \equiv L$ has been used by Strauss and Yorke [5] for the case of stability. For the case in which $f(t, x) = -t^p x^{2n+1}$, we see f satisfies (4.10) by letting $\lambda(t) \equiv 0$.

THEOREM 4.6. *Assume solutions of* (E) *are UB and UUB, where f satisfies* (4.8). *Then, if g is strongly absolutely diminishing, solutions of* (P) *are E_vUB and E_vUUB.*

Proof. Much of the proof is similar to that of Theorem 4.1 and we shall omit it.

Remark. In Example 3.3 we observe that $f(t, x) = A(t)x$ satisfies the inner product condition since $\langle x, A(t)x \rangle = -\|x\|^2$. Hence, Theorem 4.6 is not true for diminishing functions.

We have assumed throughout that solutions of the unperturbed system are UB and UUB. In Theorems 4.1 and 4.6, we may, however, only need to assume that solutions of the unperturbed system are EvUB and EvUUB in order to conclude the identical results. The proof of this result imitates the proof given in Theorem 4.1 in that we now show solutions of the perturbed system are eventually close to solutions of the unperturbed systems. Such conditions on the unperturbed system have been considered by Strauss and Yorke [5] in the case of stability.

REFERENCES

[1] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.

[2] A. Halanay, *Differential Equations*, Academic Press, New York and London, 1966.

[3] J. Massera and J. Schäffer, *Linear differential equations and functional analysis*, Ann. of Math., 67 (1958), pp. 517–573.

[4] A. Strauss and J. Yorke, *Perturbation theorems for ordinary differential equations*, J. Differential Equations, 3 (1967), pp. 15–30.

[5] ———, *Perturbing uniform asymptotically stable nonlinear systems*, Ibid., 6 (1969), pp. 452–483.

[6] T. Yoshizawa, *Stability Theory by Liapunov's Second Method*, Math. Soc. Japan, Tokyo, 1966.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

SEMIGROUPS AND ASYMPTOTIC STABILITY OF NONLINEAR DIFFERENTIAL EQUATIONS*

C. V. PAO†

Abstract. The purpose of this paper is to extend a well-known theorem of Lyapunov on matrices to unbounded linear operators in an infinite-dimensional Banach space by using semigroup theory. This extension is used for the study of stability problems of nonlinear differential equations. In the special case of a Hilbert space, a direct extension of Lyapunov's theorem for a bounded operator is obtained; and in the general case of a Banach space the idea of equivalent semi-inner products is introduced. Applications are given to a class of initial boundary value problems in a bounded domain in R^n under nonlinear perturbations.

1. Introduction. Let X be a complex (or real) Banach space with norm $|\cdot|$. In case X is a Hilbert space, we denote it by H. If X is the finite-dimensional space C^n (or R^n), Lyapunov's theorem states that an $n \times n$ matrix A is stable (i.e., the eigenvalues of A all have negative real parts) if and only if the matrix equation $A^*P + PA = -I$ has a unique solution P which is Hermitian positive definite, where A^* is the adjoint of A (for example, see [1, p. 245]). If we consider A as an operator on C^n, it can be shown that a stable matrix is a bounded operator which generates a strongly continuous group $\{T_t; -\infty < t < \infty\}$ such that $|T_t| \leqq Me^{-\beta t}$ for $t \geqq 0$, where M, β are positive constants. In a recent paper [2], Datko extended this result partially to an infinite-dimensional Hilbert space by considering A as an infinitesimal generator of a strongly continuous semigroup. In this paper we extend, on the one hand, Lyapunov's theorem fully in a Hilbert space by considering A as an infinitesimal generator of a group and, on the other hand, we extend similar results in a Banach space when A is unbounded and generates a semigroup. We do this by using the notion of an equivalent semi-inner product in X (cf. [3] or [7, p. 250]). This latter consideration improves considerably the result given in [5]. This improvement is crucial in studying the stability problem of the nonlinear differential equation

$$du/dt = Au + f(t, u), \qquad t \in R^+ \equiv [0, \infty), \tag{1.1}$$

where f is a nonlinear mapping on $R^+ \times G$ to X with G an open subset of X.

It is well known that if A is a stable matrix and if $|f(t, x)| = o(|x|)$ uniformly in t as $|x| \to 0$, then the zero solution of (1.1) is asymptotically stable. We extend this result to a Banach space for an unbounded operator A which generates a semigroup (of class C_0). In the meantime, we weaken the condition on f to include a larger class of nonlinear perturbations. Finally we give an application of the results on the abstract differential equation (1.1) to the following initial boundary value problem:

$$\partial u/\partial t = Lu + f(t, x, u), \tag{1.2}$$

$$u(t, x') = 0 \quad \text{on } R^+ \times \partial\Omega, \tag{1.3}$$

$$u(0, x) = \psi(x) \quad \text{in } \Omega, \tag{1.4}$$

* Received by the editors May 27, 1971, and in revised form October 5, 1971.

† Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607.

where Ω is a bounded open subset of R^n with smooth boundary $\partial\Omega$, ψ and f are given functions and

$$Lu = \sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + c(x)u, \qquad x \in \Omega.$$

Conditions on the coefficients of L and the function f are given to ensure the stability of the system (1.2)–(1.4) in the space $L^2(\Omega)$.

2. Semigroups and positive operators. A semi-inner product $[x, y]$ defined for $x, y \in X$ is a complex (or real) number such that:

(i) $[x + y, z] = [x, z] + [y, z]$,
(ii) $[\alpha x, y] = \alpha[x, y]$, $\alpha \in C^1$,
(iii) $[x, x] = |x|^2$,
(iv) $|[x, y]| \leqq |x|\,|y|$.

Any Banach space can be made into a semi-inner product space. In general, semi-inner product is not unique. If X is a Hilbert space, it is unique and coincides with the inner product in X. Two semi-inner products $[\cdot,\cdot]$, $[\cdot,\cdot]_e$ defined on the same vector space X are said to be equivalent if the norms $|\cdot|$, $|\cdot|_e$ are equivalent, where $|x|_e = [x, x]_e^{1/2}$. The space X_e (i.e., X equipped with $|\cdot|_e$) is called an equivalent space to X. Thus X_e is a Banach space if and only if X is. A linear operator A with domain $D(A)$ and range $R(A)$ both in X is said to be dissipative with respect to $[\cdot,\cdot]$ if $\operatorname{Re}[Ax, x] \leqq 0$, $x \in D(A)$; and it is said to be strictly dissipative if there exists a constant $\beta > 0$ such that $\operatorname{Re}[Ax, x] \leqq -\beta|x|^2$, $x \in D(A)$. A bounded operator P on a Hilbert space H with inner product $(\cdot,\cdot)$ is said to be nonnegative if $(x, Px) \geqq 0$, $x \in X$; and it is called positive definite if there exists a constant $\delta > 0$ such that $(x, Px) \geqq \delta|x|^2$, $x \in X$. It is known that P is symmetric positive definite if and only if $(x, y)_e \equiv (x, Py)$ defines an equivalent inner product on H such that

$$\delta|x|^2 \leqq |x|_e^2 \leqq \gamma|x|^2, \qquad x \in H, \tag{2.1}$$

where $|P| \leqq \gamma$ (cf. [6]).

THEOREM 2.1. *Let A be the infinitesimal generator of a strongly continuous group $\{T_t; -\infty < t < \infty\}$ of class C_0 in a (complex) Hilbert space H. Then $|T_t| \leqq Me^{-\beta t}$, $t \geqq 0$, for some positive constants M, β if and only if there exists a unique symmetric positive definite bounded operator P on H such that*

$$(PAx, x) + (x, PAx) = -(x, x), \qquad x \in D(A). \tag{2.2}$$

If, in addition, A is a bounded operator on H, then

$$A^*P + PA = -I. \tag{2.3}$$

Proof. Necessity. Assume that $|T_t| \leqq Me^{-\beta t}$ for $t \geqq 0$. Define

$$(x, y)_e = \int_0^\infty (T_t x, T_t y)\, dt, \qquad x, y \in H. \tag{2.4}$$

It is readily seen that $(x, y)_e$ is sesquilinear, $(x, y)_e = \overline{(y, x)}_e$ and

$$|(x, y)_e| \leqq \gamma|x|\,|y|, \tag{2.5}$$

where $\gamma = M^2/2\beta$. Since $\{T_t\}$ is a group, there exist positive constants K, ω such that $|T_{-t}| \leqq Ke^{\omega|t|}$. Thus

$$|x| = |T_{-t}T_t x| \leqq Ke^{\omega|t|}|T_t x|$$

which implies that $|T_t x| \geqq K^{-1}e^{-\omega t}|x|$ for $t \geqq 0$. It follows from (2.4), (2.5) that

$$\delta|x|^2 \leqq |x|_e^2 \leqq \gamma|x|^2, \qquad x \in H, \tag{2.6}$$

where $\delta = (2\omega K^2)^{-1}$. By the Lax-Milgram theorem (for example, see [7, p. 92]) there exists a unique bounded linear operator P with $|P| \leqq \gamma$ such that

$$(x, y)_e = (x, Py), \qquad x, y \in H.$$

The condition $(x, y)_e = \overline{(y, x)}_e$ and (2.6) imply that P is symmetric and positive definite. To show (2.2) we observe that for $t \geqq 0$,

$$\begin{aligned} (T_t x, T_t x)_e - (x, x)_e &= \lim_{b\to\infty}\left[\int_0^b (T_{s+t}x, T_{s+t}x)\,ds - \int_0^b (T_s x, T_s x)\,ds\right] \\ &= \lim_{b\to\infty}\left[\int_b^{b+t} (T_s x, T_s x)\,ds - \int_0^t (T_s x, T_s x)\,ds\right] \\ &= -\int_0^t (T_s x, T_s x)\,ds. \end{aligned} \tag{2.7}$$

It follows from the relation

$$(Ax, x)_e + (x, Ax)_e = \lim_{t\downarrow 0}\,[(T_t x, T_t x)_e - (x, x)_e] \tag{2.8}$$

that $(Ax, x)_e + (x, Ax)_e = -(x, x)$. Hence (2.2) is proved.

Sufficiency. By hypothesis, $(x, y)_e \equiv (x, Py)$ defines an inner product in H satisfying (2.1). From (2.2), (2.1),

$$2\,\mathrm{Re}\,(Ax, x)_e = -|x|^2 \leqq -\gamma^{-1}|x|_e^2, \qquad x \in D(A),$$

which shows that the operator $A_1 = A + \gamma^{-1}I$ is dissipative in H_e. Since A is the infinitesimal generator of a strongly continuous semigroup in H, there exists a real constant ω such that any number λ with $\mathrm{Re}\,\lambda > \omega$ is in the resolvent set of A. In particular, $R(\alpha I - A_1) = H$ if $\alpha > \omega + \gamma$. This fact together with the dissipative property of A_1 imply that $R(I - A_1) = H$ (cf. [8]). Hence, A_1 is the infinitesimal generator of a contraction semigroup $\{S_t; t \geqq 0\}$ on H_e (cf. [3]). Let $T_t = e^{-t/\gamma}S_t$. Then $\{T_t; t \geqq 0\}$ is a strongly continuous semigroup with infinitesimal generator A in H_e and satisfies

$$|T_t x|_e \leqq e^{-t/\gamma}|x|_e.$$

It follows from (2.1) that

$$|T_t x| \leqq (\gamma/\delta)e^{-t/\gamma}|x|, \qquad x \in H.$$

The equivalence relation between $|\cdot|_e$ and $|\cdot|$ ensures that A is the infinitesimal generator of $\{T_t; t \geqq 0\}$ in H. This proves the sufficiency.

To show (2.3) we observe from (2.2) that

$$((A^*P + PA + I)x, x) = 0, \qquad x \in H.$$

It is easily shown by using the above equation and the symmetric property of $B \equiv A^*P + PA + I$ that $B = 0$. This completes the proof of the theorem.

COROLLARY. *If A is unbounded and is the infinitesimal generator of a semigroup $\{T_t; t \geqq 0\}$ of class C_0 in H such that $|T_t| \leqq Me^{-\beta t}$, then there exists an inner product $(\cdot, \cdot)_e$ in H such that*

$$|x|_e^2 \leqq (M^2/2\beta)|x|^2, \qquad x \in H,$$

and

$$2 \operatorname{Re}(Ax, x)_e = -|x|^2, \qquad x \in D(A). \tag{2.9}$$

Proof. Define $(x, y)_e$ by (2.4). Then (2.5) holds and if $(x, x)_e = 0$, we have, by the continuity of $|T_t x|$ in t, $|T_t x| = 0$ for all $t > 0$. Letting $t \downarrow 0$ gives $x = 0$. Thus $(\cdot, \cdot)_e$ defines an inner product on H. From (2.7), (2.8) we obtain (2.9).

Remark 2.1. By using a different approach, the second part of Theorem 2.1 for bounded operators has been shown by Wong [9], and the corollary to the same theorem is essentially the same as the main theorem given by Datko [2]. However, our proof seems to be simpler than that given in [2].

THEOREM 2.2. *Let A be a bounded operator which generates a strongly continuous semigroup $\{T_t; t \geqq 0\}$ of class C_0 in H. Then $|T_t| \leqq Me^{-\beta t}$, $t \geqq 0$, for some positive constants M, β if and only if there exists a symmetric positive definite operator P on H such that*

$$(A^*Px, x) + (PAx, x) \leqq -\mu|x|^2, \qquad x \in H, \tag{2.10}$$

for any given μ with $0 < \mu < 1$.

Proof. The sufficiency follows from the proof of Theorem 2.1. To show the necessity, we define

$$(x, y)' = \int_0^\infty (T_t x, T_t y)\, dt + \delta_0(x, y), \tag{2.11}$$

where $\delta_0 > 0$ is to be chosen. Since (2.5), (2.6) hold with $\gamma = (M^2/2\beta) + \delta_0$, there exists a symmetric positive definite operator P on H with $|P| \leqq \gamma$ such that $(x, y)' = (x, Py)$. By using the relations (2.7), (2.8) and (2.11) we obtain

$$2 \operatorname{Re}(Ax, x)' = -|x|^2 + 2\delta_0 \operatorname{Re}(Ax, x) \leqq -(1 - 2\delta_0|A|)|x|^2.$$

Therefore the condition (2.10) follows by taking $\delta_0 = (1 - \mu)/2|A|$.

THEOREM 2.3. *Let A be the infinitesimal generator of a strongly continuous semigroup $\{T_t; t \geqq 0\}$ of class C_0 in a Banach space X. Then $|T_t| \leqq Me^{-\beta t}$, $t \geqq 0$, for some positive constants M, β if and only if there exists an equivalent semi-inner product $[\cdot, \cdot]_e$ in X such that*

$$|x| \leqq |x|_e \leqq M|x|, \qquad x \in X, \tag{2.12}$$

and

$$\operatorname{Re}[Ax, x]_e \leqq -\beta|x|_e^2, \qquad x \in D(A). \tag{2.13}$$

Proof. Necessity. Assume that $|T_t| \leqq Me^{-\beta t}$. Let $S_t = e^{\beta t}T_t$. Then $|S_t| \leqq M$ and $\{S_t; t \geqq 0\}$ is also a semigroup of class C_0 in X. Define

$$|x|_e = \sup_{t \geqq 0} |S_t x|, \qquad x \in X. \tag{2.14}$$

It is easily seen that $|\cdot|_e$ is a norm in X and satisfies (2.12). Moreover, for each $t \geqq 0$,

$$(2.15) \qquad |S_t|_e = \sup_{|x|_e=1} |S_t x|_e = \sup_{|x|_e=1} (\sup_{\tau \geqq 0} |S_{\tau+t}x|) \leqq \sup_{|x|_e=1} (|x|_e) = 1.$$

Thus by the equivalence relation (2.12), $\{S_t; t \geqq 0\}$ is a semigroup of class C_0 in X_e and by (2.15) it is a contraction semigroup. Since

$$\lim_{t \downarrow 0} t^{-1}(S_t x - x) = \lim_{t \downarrow 0} t^{-1}\{e^{\beta t}(T_t x - x) + (e^{\beta t} - 1)x\} = Ax + \beta x, \qquad x \in D(A),$$

in X we see from (2.12) that the above limit holds in X_e and thus the infinitesimal generator of $\{S_t; t \geqq 0\}$ in X_e is $(A + \beta I)$. Hence, for each $x \in D(A)$,

$$\begin{aligned} \operatorname{Re}\,[(A + \beta I)x, x]_e &= \lim_{t \downarrow 0} t^{-1} \operatorname{Re}\,[S_t x - x, x]_e = \lim_{t \downarrow 0} t^{-1}(\operatorname{Re}\,[S_t x, x]_e - [x, x]_e) \\ &\leqq \lim_{t \downarrow 0} t^{-1}(|S_t x|_e |x|_e - |x|_e^2) \leqq 0, \end{aligned}$$

where we have used the fact that $|S_t x|_e \leqq |x|_e$. Notice that for each fixed $x \in X_e$ the semi-inner product $[y, x]_e$ is continuous in y (cf. [7, p. 250]). The above inequality shows that (2.13) holds. Thus the necessity is proved.

Sufficiency. If (2.13) holds, then $(A + \beta I)$ is dissipative in X_e. Since A is an infinitesimal generator, $D(A)$ is dense in X_e and $R(\alpha I - (A + \beta I)) = X$ for $\alpha > \omega + \beta$. Thus $R(I - (A + \beta I)) = X_e$ (cf. [8]). This shows that $(A + \beta I)$ generates a contraction semigroup $\{S_t; t \geqq 0\}$ in X_e. Let $T_t = e^{-\beta t}S_t$. It is easily seen that the infinitesimal generator of $\{T_t; t \geqq 0\}$ in X_e is A and satisfies

$$|T_t x|_e \leqq e^{-\beta t}|x|_e.$$

By the equivalence relation (2.12), $\{T_t; t \geqq 0\}$ is a semigroup in X such that $|T_t x| \leqq Me^{-\beta t}|x|$ for $x \in X$. This completes the proof of the theorem.

Remark 2.2. The norm $|\cdot|_e$ defined in (2.14) was given in [10] and was used in [4] for the investigation of contraction groups. Equation (2.4) was introduced in [4].

3. Stability of nonlinear equations. By a solution of (1.1) we mean an X-valued function $u(t)$ with values in $D(A)$ such that $u(t)$ is Lipschitz continuous on R^+ and the strong derivative $du(t)/dt$ exists and satisfies (1.1) for almost all values of t in R^+. The definitions of stability and asymptotic stability are in the sense of Lyapunov (e.g. see [6]). Let G be an open subset of X containing the zero vector and let $f(t, u)$ be a (nonlinear) mapping from $R^+ \times G$ into X. Throughout this section we always assume that for each $u_0 \in D(A) \cap G$ there exists a local solution $u(t)$ to (1.1) with $u(0) = u_0$ which can be continued so long as it remains in $D(A) \cap G$.

THEOREM 3.1. *Let A be a (possibly unbounded) linear operator which is the infinitesimal generator of a strongly continuous semigroup $\{T_t; t \geqq 0\}$ of class C_0 in X such that $|T_t| \leqq Me^{-\beta t}$ for some positive constants M, β. Let $|\cdot|_e$ be the equivalent norm defined in* (2.14). *Assume that $f(t, 0) = 0$. If there exists a continuous real-valued function $k(t)$ on R^+ satisfying*

$$(3.1) \qquad \limsup_{t \to \infty} \frac{1}{t} \int_0^t k(s)\, ds < \beta$$

and if

$$\text{(3.2)} \qquad \operatorname{Re}[f(t,u),u]_e \leqq k(t)|u|_e^2, \qquad u \in G, \quad t \in R^+,$$

where $[\cdot,\cdot]_e$ *is any semi-inner product consistent with the norm* $|\cdot|_e$, *then the zero solution of* (1.1) *is asymptotically stable.*

Proof. The existence of an equivalent semi-inner product $[\cdot,\cdot]_e$ satisfying (2.12) and (2.13) follows from Theorem 2.3. Let $u(t)$ be a solution of (1.1) with $u(0) = u_0$. Then from the relation $||u(s)|_e - |u(t)|_e| \leqq |u(s) - u(t)|_e$ for s, t in R^+ we see that $|u(t)|_e$ is also Lipschitz continuous in R^+, and thus it is almost everywhere differentiable in $(0, \infty)$. Let $t \in (0, \infty)$ be a fixed point at which $u(t)$ and $|u(t)|$ are differentiable. Then for every $s \in (0, \infty)$,

$$\text{(3.3)} \quad \operatorname{Re}[u(s) - u(t), u(t)]_e = \operatorname{Re}[u(s), u(t)]_e - [u(t), u(t)]_e \leqq |u(t)|_e(|u(s)|_e - |u(t)|_e).$$

By dividing (3.3) by $s - t > 0$ and letting $s \downarrow t$ we have

$$\text{(3.4)} \qquad \operatorname{Re}[u'(t), u(t)]_e \leqq |u(t)|_e \frac{d}{dt}|u(t)|_e.$$

Similarly if we divide (3.3) by $s - t < 0$ and let $s \uparrow t$, then

$$\text{(3.5)} \qquad \operatorname{Re}[u'(t), u(t)]_e \geqq |u(t)|_e \frac{d}{dt}|u(t)|_e.$$

It follows from (3.4), (3.5) that

$$\text{(3.6)} \qquad |u(t)|_e \frac{d}{dt}|u(t)|_e = \operatorname{Re}[u'(t), u(t)]_e$$

for almost all $t \in (0, \infty)$. Substituting $u' = Au + f(t, u)$ and using the relations (2.13), (3.2) we obtain from (3.6) that

$$\text{(3.7)} \qquad \frac{d}{dt}|u(t)|_e^2 \leqq -2(\beta - k(t))|u(t)|_e^2.$$

After transforming the right term to the left, multiply by $\exp(2\int_0^t (\beta - k(s))\,ds)$ and following by integration from 0 to t, we obtain

$$\exp\left(2\int_0^t (\beta - k(s))\,ds\right)|u(t)|_e^2 - |u(0)|_e^2 \leqq 0.$$

This inequality together with (2.12) imply that

$$\text{(3.8)} \qquad \begin{aligned} |u(t)| \leqq |u(t)|_e &\leqq \exp\left(-\int_0^t (\beta - k(s))\,ds\right)|u_0|_e \\ &\leqq M\exp\left(-\int_0^t (\beta - k(s))\,ds\right)|u_0|. \end{aligned}$$

The above inequalities hold for as long as $|u(t)|$ remains in G. By the assumption (3.1),

$$\sigma \equiv \limsup_{t \to \infty} \frac{1}{t}\int_0^t (\beta - k(s))\,ds > 0.$$

Thus for some finite value T,

$$\int_0^t (\beta - k(s))\, ds \geqq \frac{\sigma}{2} t \quad \text{for } t \geqq T. \tag{3.9}$$

Therefore, if $|u_0|$ is sufficiently small, then by (3.8) and (3.9), $|u(t)|$ can be made arbitrarily small for all $t \geqq 0$ and thus remains in G. This shows that $u(t)$ can be continued to R^+ and the zero solution is stable. Since (3.8) and (3.9) also imply that $|u(t)| \to 0$ as $t \to \infty$, the zero solution is asymptotically stable. This proves the theorem.

Remark 3.1. If $|f(t, u)| = o(|u|)$ uniformly in t as $|u| \to 0$, then there exists $\delta > 0$ such that $|f(t, u)| \leqq (2M)^{-1}\beta|u|$ for $|u| \leqq \delta$. Since

$$\operatorname{Re}\,[f(t, u), u]_e \leqq |f(t, u)|_e |u|_e \leqq M|f(t, u)|\,|u|_e \leqq (\beta/2)|u|_e^2,$$

if we take $k(t) = \beta/2$, then f satisfies all the conditions in Theorem 3.1, where the subset G may be replaced by $G \cap \{u; |u| < \delta\}$. It should be noted that the function $k(t)$ in Theorem 3.1 is not necessarily nonnegative. In particular if $\limsup k(t) \leqq 0$ as $t \to \infty$, then (3.1) holds.

Example. As an application of Theorem 3.1 as well as a demonstration to the formulation of an infinitesimal generator from a partial differential operator, we consider the initial boundary value problem (1.2)–(1.4). Let $\bar{\Omega}$ be the closure of Ω and let $C^k(\Omega)$ (respectively, $C^k(\bar{\Omega})$) be the set of all k-times continuously (respectively, uniformly continuously) differentiable functions in Ω. Let G be an open subset of $L^2(\Omega)$ containing the zero vector. We assume that:

(i) $a_{ij}(x) \in C^1(\Omega)$, $c(x) \in C^0(\bar{\Omega})$ and there exists a constant $c_0 > 0$ such that

$$\operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\bar{\xi}_j \geqq c_0 \sum_{i=1}^{n} |\xi_i|^2, \qquad x \in \bar{\Omega}, \tag{3.10}$$

where $\xi = (\xi_1, \cdots, \xi_n) \in C^n$.

(ii) $f(t, x, u)$ is defined for $t \in R^+$, $x \in \Omega$, $u \in G$ and with values in $L^2(\Omega)$ for each t. Moreover there exists a continuous real function $k(t)$ on R^+ such that

$$\operatorname{Re} \int_\Omega f(t, x, u(x))\overline{u(x)}\, dx \leqq k(t) \int_\Omega |u(x)|^2\, dx, \qquad t \in R^+, \quad u \in G. \tag{3.11}$$

In order to formulate (1.2)–(1.4) into an abstract Cauchy problem (1.1), we set $V = \{u \in C^2(\bar{\Omega});\ u = 0 \text{ on } \partial\Omega\}$ and let $\bar{V}$ be the closure of V in $H^2(\Omega)$ (see [11, p. 57] for the definition of $\mathrm{H}^2(\Omega)$). Define two operators A, f with $D(A) = \bar{V}$ and $D(f) = R^+ \times G$ by

$$(Au)(x) = (Lu)(x), \qquad u \in D(A),$$

$$(f(t, u))(x) = f(t, x, u(x)), \qquad t \in R^+, \quad u \in G.$$

Then by (i), (ii), A is a densely defined linear operator with domain and range both in $L^2(\Omega)$ and f is a mapping on $R^+ \times G$ into $L^2(\Omega)$. With this definition, the system (1.2)–(1.4) is formulated as an abstract equation (1.1) in $L^2(\Omega)$ with the initial condition $u(0) = \psi$. To show that A is an infinitesimal generator, we perform integration by parts for $(Lu, u)_0$ and use (3.10) to obtain

$$\operatorname{Re}(Lu, u)_0 \equiv \operatorname{Re} \int_\Omega (Lu)(x) \cdot \bar{u}(x)\, dx \leqq - \int_\Omega \left(c_0 \sum_{i=1}^n |u_{x_i}|^2 - c(x)|u|^2 \right) dx$$

for $u \in V$. Using the inequality

$$\sum_{i=1}^n \int_\Omega |u_{x_i}|^2\, dx \geqq \gamma \int_\Omega |u|^2\, dx, \qquad u \in V,$$

where $\gamma > 0$ depends only on Ω, we see that

$$\operatorname{Re}(Lu, u)_0 \leqq -(\gamma c_0 - \min_{x \in \bar{\Omega}} c(x))|u|_0^2, \qquad u \in V. \tag{3.12}$$

Let

$$\beta = \gamma c_0 - \min_{x \in \bar{\Omega}} c(x) \tag{3.13}$$

and assume, for convenience, that $\beta \geqq 0$ (this assumption can be removed). Then by the definition of $\bar{V}$, for each $u \in D(A)$ there is a sequence $\{u_n\}$ in V such that $u_n \to u$ in $H^2(\Omega)$. Thus $u_n \to u$ in $L^2(\Omega)$ and by (i), $Lu_n \to Au$ in $L^2(\Omega)$. It follows from (3.12) that for each $u \in D(A)$,

$$\operatorname{Re}(Au, u)_0 = \lim_{n \to \infty} \operatorname{Re}(Lu_n, u_n)_0 \leqq - \lim_{n \to \infty} \beta|u_n|_0^2 = -\beta|u|_0^2. \tag{3.14}$$

It is known that if the assumption (i) holds and if $\partial\Omega$ is sufficiently smooth, then there exists a real constant λ_0 such that $R(\lambda I - A) = L^2(\Omega)$ for all $\lambda \geqq \lambda_0$ (cf. [11, p. 75]). This condition and (3.14) imply that $R(I - A) = L^2(\Omega)$. Therefore, A generates a contraction semigroup $\{T_t; t \geqq 0\}$ of class C_0 in $L^2(\Omega)$ with $|T_t| \leqq e^{-\beta t}$ (cf. [7, p. 250]). As an application of Theorem 3.1 we obtain the following results: Under the conditions (i), (ii) and some smoothness condition on $\partial\Omega$, the zero solution of the initial boundary value problem (1.2)–(1.4) is asymptotically stable provided that $f(t, x, 0) = 0$ and the function $k(t)$ in (ii) satisfies (3.1) with β given by (3.13). In particular, if $f(t, x, u) \equiv 0$, then for each $\psi \in \bar{V}$ there exists a unique solution to the linear system (1.2)–(1.4) in L^2, and if, in addition, $\beta > 0$, then the zero solution of the linear system is exponentially asymptotically stable.

REFERENCES

[1] R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.

[2] R. Datko, *Extending a theorem of A. M. Lyapunov to Hilbert space*, J. Math. Anal. Appl., 32 (1970), pp. 610–616.

[3] G. Lumer and R. S. Phillips, *Dissipative operators in a Banach space*, Pacific J. Math., 11 (1961), pp. 679–698.

[4] W. G. Vogt, M. M. Eisen and G. R. Buis, *Contraction groups and equivalent norms*, Nagoya Math. J., 34 (1968), pp. 1–3.

[5] R. Datko, *An extension of a theorem of A. M. Lyapunov to semi-groups of operators*, J. Math. Anal. Appl., 24 (1968), pp. 290–295.

[6] C. V. Pao, *The existence and stability of solutions of nonlinear operator differential equations*, Arch. Rational Mech. Anal., 35 (1969), pp. 16–29.

[7] K. Yosida, *Functional Analysis*, Springer-Verlag, Berlin, 1966.

[8] S. Oharu, *Note on the representation of semi-groups of nonlinear operators*, Proc. Japan Acad., 42 (1967), pp. 1149–1154.

[9] J. S. W. WONG, *A remark of a theorem of Lyapunov*, Canad. Math. Bull., 13 (1970), pp. 141–143.
[10] W. FELLER, *On the generation of unbounded semi-groups of bounded linear operators*, Ann. of Math., 58 (1953), pp. 166–174.
[11] A. FRIEDMAN, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

SIAM J. MATH. ANAL.
Vol. 3, No. 2, May 1972

SUMS OF CONVOLUTION OPERATORS*

STAN K. KRANZLER†

Abstract. Let Ω be an open set in R_n and let $\mathscr{E}(\Omega)$ denote the space of infinitely differentiable functions on Ω. Necessary and sufficient conditions are exhibited for a family $\{\Omega_i\}_{i=1}^N$ of open sets in R_n and a family $\{S_i\}_{i=1}^N \subset \mathscr{E}'(R_n)$ in order that the convolution equation

$$\sum_{i=1}^{N} S_i * u_i = f$$

have a solution $(u_1, u_2, \cdots, u_N)$ in $\bigoplus_{i=1}^N \mathscr{E}(\Omega_i)$ for every f in $\mathscr{E}(\Omega)$.

A simple example and a geometrical interpretation of the condition on the family $\{\Omega_i\}_{i=1}^N$ is provided.

1. Introduction. Consider the convolution equation

$$S * u = f, \tag{1.1}$$

where $S \in \mathscr{E}'(R_n)$ and f is a distribution. Ehrenpreis [5] has obtained necessary and sufficient conditions on the distribution S in order that (1.1) have a solution u in $\mathscr{D}'(R_n)$ (respectively $\mathscr{D}'_F(R_n), \mathscr{E}(R_n)$) for every f in $\mathscr{D}'(R_n)$ (respectively $\mathscr{D}'_F(R_n), \mathscr{E}(R_n)$). Hörmander [7] has extended this result by giving necessary and sufficient conditions on a pair (Ω_1, Ω_2) of open sets in R_n and on $S \in \mathscr{E}'(R_n)$ for the existence of a solution u in $\mathscr{D}'(\Omega_2)$ (respectively $\mathscr{D}'_F(\Omega_2), \mathscr{E}(\Omega_2)$) for every f in $\mathscr{D}'(\Omega_1)$ (respectively $\mathscr{D}'_F(\Omega_1), \mathscr{E}(\Omega_1)$).

The question naturally arises: What can one say about convolution equations of the following form:

$$\sum_{i=1}^{N} S_i * u_i = f, \tag{1.2}$$

where $\{S_i\}_{i=1}^N$ is a finite set in $\mathscr{E}'(R_n)$ and f is a distribution? We shall exhibit necessary and sufficient conditions on the family $\{S_i\}_{i=1}^N$ and open sets $\{\Omega\}_{i=0}^N$ in R_n in order that (1.2) have a solution $(u_1, u_2, \cdots, u_n)$ in $\bigoplus_{i=1}^N \mathscr{E}(\Omega_i)$ for all f in $\mathscr{E}(\Omega_0)$.

Before proceeding to the main work we first introduce the following notation and definitions. For each compact set K in the open set $\Omega \subset R_n$ and each multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ where the α_i are nonnegative integers we denote by $\mathscr{D}(K)$ the set of all functions in $C^\infty(\Omega)$ whose supports lie completely within K and are provided with the topology determined by the semi-norms

$$\|\varphi\|_j^K = \sum_{|\alpha| \leqq j} \sup_{x \in K} |D^\alpha \varphi(x)|, \tag{1.3}$$

where α runs through Z^n. We denote by $\mathscr{D}(\Omega)$ the space of all functions in $C^\infty(\Omega)$ with compact support in Ω and equip $\mathscr{D}(\Omega)$ with the strict inductive limit topology. $\mathscr{E}(\Omega)$ is the space $C^\infty(\Omega)$ provided with the topology determined by all semi-norms of the form (1.3), where K runs through all compact sets in Ω and α runs through

* Received by the editors April 26, 1971.

† Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822.

Z^n. The spaces $\mathscr{D}'(\Omega)$ and $\mathscr{E}'(\Omega)$ are the respective topological duals of $\mathscr{D}(\Omega)$ and $\mathscr{E}(\Omega)$. $\mathscr{D}'(\Omega)$ is called the space of distributions on Ω, and $\mathscr{D}'_F(\Omega)$ are the distributions of finite order (see Schwartz [11]). Also note that $\mathscr{E}'(\Omega)$ is the space of all distributions with compact support (see Schwartz [11]).

The convolution of two distributions S and φ is always defined provided at least one of them has compact support (for the definition of support of a distribution see Schwartz [11]). Here we shall define convolution only where φ is in $\mathscr{D}(\Omega)$ and S is in $\mathscr{E}'(\Omega)$.

$$S * \varphi(x) = \langle S, \tau_x \check{\varphi} \rangle, \tag{1.4}$$

where $\check{\varphi}(y) = \varphi(-y)$. The trivial part of Lions' theorem [8] (Titchmarsh's theorem [12] in one dimension) shows that

$$\operatorname{supp} S * \varphi \subset \operatorname{supp} S + \operatorname{supp} \varphi, \tag{1.5}$$

and thus if (Ω_1, Ω_2) are a pair of open sets in R_n and

$$\Omega_1 + \operatorname{supp} S \subset \Omega_2, \tag{1.6}$$

then clearly $\operatorname{supp} S * \varphi \subset \Omega_2$ for all distributions φ with support in Ω_1.

Finally for all S in $\mathscr{E}'(R_n)$ we denote by $\hat{S}$ the Fourier transform of S which is defined by

$$\hat{S}(\xi) = \langle S, e^{-2\pi i \langle \cdot, \xi \rangle} \rangle. \tag{1.7}$$

2. Solution of the problem. We begin first by giving the basic definitions introduced by Hörmander and stating his result [7].

DEFINITION 2.1. Let $S \in \mathscr{E}'(R_n)$. The pair (Ω_1, Ω_2) of open sets in R_n is called *S-convex* if

$$\Omega_1 + \operatorname{supp} S \subset \Omega_2 \tag{2.1}$$

and given any compact set $K_2 \subset \Omega_2$ there exists a compact set $K_1 \subset \Omega_1$ such that $\varphi \in \mathscr{D}(\Omega_1)$ and $\operatorname{supp} S * \varphi \subset K_2$ imply that $\operatorname{supp} \varphi \subset K_1$.

DEFINITION 2.2. The distribution $S \in \mathscr{E}'(R_n)$ is said to be *invertible* if there exist constants A_1, A_2 and A_3 such that for every $\xi \in R_n$ one can find $\eta \in R_n$ such that

$$\begin{gathered} |\xi - \eta| \leqq A_1 \log(2 + |\xi|) \quad \text{and} \\ |\hat{S}(\eta)| \geqq (A_2 + |\xi|)^{-A_3}. \end{gathered} \tag{2.2}$$

Finally if we let T be the mapping from $\mathscr{D}(\Omega_1)$ into $\mathscr{D}(\Omega_2)$ defined by

$$T\varphi = S * \varphi, \qquad \varphi \in \mathscr{D}(\Omega_1),$$

and T^* the dual mapping from $\mathscr{D}'(\Omega_2)$ and $\mathscr{D}'(\Omega_1)$, we may state Hörmander's result.

THEOREM 2.1. *The following conditions are equivalent*:

(a) $T^* \mathscr{E}(\Omega_2) = \mathscr{E}(\Omega_1)$;

(b) $T^* \mathscr{D}'_F(\Omega_2) \supset \mathscr{E}(\Omega_1)$;

(c) $T^* \mathscr{D}'(\Omega_2) \supset \mathscr{E}(\Omega_1)$;

(d) T^{-1} *is a sequentially continuous operator from* $\mathscr{D}(\Omega_2)$ *to* $\mathscr{D}(\Omega_1)$;

(e) S *is invertible and* (Ω_1, Ω_2) *is an S-convex pair.*

Suppose we now have a finite family $\{S_i\}_{i=1}^N \subset \mathscr{E}'(R_n)$ and $\{\Omega_i\}_{i=0}^N$ a finite family of open sets on R_n satisfying

$$\Omega_0 + \operatorname{supp} S_i \subset \Omega_i, \qquad 1 \leqq i \leqq N. \tag{2.3}$$

Define the mapping T from $\mathscr{D}(\Omega_0)$ into $\prod_{i=1}^N \mathscr{D}(\Omega_i)$ by the relations

$$\mathrm{pr}_i \circ T(\varphi) = S_i * \varphi_i = T_i(\varphi), \qquad 1 \leqq i \leqq N, \tag{2.4}$$

where pr_i denotes the ith projection in the product space $\prod_{i=1}^N \mathscr{D}(\Omega_1)$. We equip $\prod_{i=1}^N \mathscr{D}(\Omega_i)$ with the product topology and since each T_i is a continuous mapping from $\mathscr{D}(\Omega_0)$ into $\mathscr{D}(\Omega_i)$ it follows that T is continuous. Clearly T is linear and thus there is a natural mapping T^* from $\bigoplus_{i=1}^N \mathscr{D}'(\Omega_i)$ into $\mathscr{D}(\Omega_0)$ defined by

$$\langle T^*(u_1, u_2, \cdots, u_N), \varphi\rangle = \sum_{i=1}^N \langle u_i, S_i * \varphi\rangle \tag{2.5}$$

for all φ in $\mathscr{D}(\Omega_0)$.

We now generalize Definitions 2.1 and 2.2 in the following way.

DEFINITION 2.3. The family $\{S_1, S_2, \cdots, S_N\} \subset \mathscr{E}'(R_n)$ is called *invertible* if there exist constants A_1, A_2 and A_3 such that for every $\xi \in R_n$ there exists $\eta \in R_n$ such that

$$|\xi - \eta| \leqq A_1 \log(2 + |\xi|) \quad \text{and}$$
$$\sum_{i=1}^N |\hat{S}_i(\eta)| \geqq (A_2 + |\xi|)^{-A_3}. \tag{2.6}$$

DEFINITION 2.4. Let $\{S_1, S_2, \cdots, S_N\} \subset \mathscr{E}'(R_n)$. We say that $(\Omega_0, \Omega_1, \cdots, \Omega_N)$ is an $\{S_i\}^N$*-convex* $(N + 1)$*-tuple* if and only if

$$\Omega_0 + \operatorname{supp} S_i \subset \Omega_i, \qquad 1 \leqq i \leqq N, \tag{2.7}$$

and given any compact set $K_i \subset \Omega_i$, $1 \leqq i \leqq N$, there exists compact $K_0 \subset \Omega_0$ such that $\varphi \in \mathscr{E}'(\Omega_0)$ and $\prod_{i=1}^N \operatorname{supp} S_i * \varphi \subset \prod_{i=1}^N K_i$ imply that $\operatorname{supp} \varphi \subset K_0$.

We wish to prove the following theorem.

THEOREM 2.2. *The following are equivalent*:

(a) $T^*[\bigoplus_{i=1}^N \mathscr{E}(\Omega_i)] = \mathscr{E}(\Omega_0)$;

(b) $T^*[\bigoplus_{i=1}^N \mathscr{D}'_F(\Omega_i)] \supset \mathscr{E}(\Omega_0)$;

(c) $T^*[\bigoplus_{i=1}^N \mathscr{D}'(\Omega_i)] \supset \mathscr{E}(\Omega_0)$;

(d) $T^{-1}: \prod_{i=1}^N \mathscr{D}(\Omega_i) \to \mathscr{D}(\Omega_0)$ *is sequentially continuous*;

(e) *the family* $\{S_1, S_2, \cdots, S_N\}$ *is invertible and* $(\Omega_0, \Omega_1, \cdots, \Omega_N)$ *is an* $\{S_i\}^N$*-convex* $(N + 1)$*-tuple.*

Clearly (a) → (b) → (c). We shall first show that (c) → (d).

LEMMA 2.1. *If* $T^*[\bigoplus_{i=1}^N \mathscr{D}'(\Omega_i)] \supset \mathscr{E}(\Omega_0)$, *then* T^{-1} *is a sequentially continuous mapping from* Image $T \subset \prod_{i=1}^N \mathscr{D}(\Omega_i)$ *into* $\mathscr{D}(\Omega_0)$.

Proof. For each $1 \leqq i \leqq N$, let $K_i \subset \Omega_i$ be a compact set. Define

$$\Phi = \left\{\varphi \in \mathscr{D}(\Omega_0) : \prod_{i=1}^{N} \operatorname{supp} S_i * \varphi \subset \prod_{i=1}^{N} K_i\right\} \tag{2.8}$$

and consider the mapping

$$(f, \varphi) \mapsto \int_{R_n} f\varphi \, dx \tag{2.9}$$

from $\mathscr{E}(\Omega_0) \times \Phi$ into C. The topology defined by the family of semi-norms

$$\| T\varphi \|_M = \sum_{i=1}^{N} \| S_i * \varphi \|_M$$

makes Φ into a metrizable space. Since $\mathscr{E}(\Omega_0)$ is a Fréchet space and (2.9) is a bilinear form, to show continuity of (2.9) it suffices to show that it is separately continuous. Continuity for fixed φ is trivial. Fix $f \in \mathscr{E}(\Omega_0)$. By hypothesis there exists $(u_1, u_2, \cdots, u_N)$ in $\bigoplus_{i=1}^{N} \mathscr{D}'(\Omega_i)$ such that $T^*(u_1, u_2, \cdots, u_N) = f$. Then we have

$$\begin{aligned} \left| \int_{R_n} f\varphi \, dx \right| &= \left| \int_{R_n} T^*(u_1, u_2, \cdots, u_N)\varphi \, dx \right| \\ &= \sum_{i=1}^{N} \left| \int_{R_n} T_i^* u_i \varphi \, dx \right| \\ &\leqq \sum_{i=1}^{N} |\langle u_i, S_i * \varphi \rangle|, \end{aligned} \tag{2.10}$$

and hence it is clear (2.9) is continuous for fixed $f \in \mathscr{E}(\Omega_0)$.

The remainder of the proof involves only functions on Ω_0 and thus is the same as in Hörmander's paper [7].

We now wish to show that (d) implies (e).

LEMMA 2.2. *If T^{-1} is sequentially continuous, then*

(i) $(\Omega_0, \Omega_1, \cdots, \Omega_N)$ *is an $\{S_i\}_{i=1}^{N}$-convex $(N + 1)$-tuple and*

(ii) *given any compact set $K_0 \subset \Omega_0$ there exist constants C and M such that*

$$\|\varphi\|_0 \leqq C \| T\varphi \|_M \tag{2.11}$$

for all $\varphi \in \mathscr{D}(R_n)$ with $\operatorname{supp} \varphi \subset K_0$.

Proof. The proof involves only minor changes of the proof in Hörmander's Theorem 3.2 [7], and thus will be omitted.

LEMMA 2.3. *If T^{-1} is a sequentially continuous mapping from its image in $\prod_{i=1}^{N} \mathscr{D}(\Omega_i)$ onto $\mathscr{D}(\Omega_0)$, then the family $\{S_1, S_2, \cdots, S_N\}$ is invertible.*

Proof. We may assume that $0 \in \Omega_0$. Let K_0 be a compact set in Ω_0 such that 0 is in the interior of K_0. From Lemma 2.2 we see that there exist constants C and M such that

$$\|\varphi\|_0 \leqq C \| T\varphi \|_M \tag{2.12}$$

for all $\varphi \in \mathscr{D}(\Omega_0)$ with $\operatorname{supp} \varphi \subset K_0$. Let $\psi \in \mathscr{D}(K_0)$ such that $\psi \geqq 0$ and $\int_{R_n} \psi \, dx = 1$. Define

$$\psi_k(x) = \mathop{*}_{i=1}^{k} k^n \psi_i(kx) \quad \text{for } k = 1, 2, \cdots,$$

where $\psi_i = \psi$ for $1 \leqq i \leqq k$. Then $\operatorname{supp} \psi_k \subset K_0$ and there exists a constant $a > 0$ such that

$$|\psi_k(x)| \leqq e^{-ak} \quad \text{for } |\xi| \geqq k.$$

Choose constants $A_1 > (\mu + M)/a$ and $A_3 > M$ where $\mu = \sum_{i=1}^{N}$ order S_i. Suppose that for no A_2 do we have (2.6) valid. It is clear that for any bounded sequence $\{\xi_j\} \subset R_n$ one can find an A_2 making (2.6) valid. Thus there exists a sequence $\{\xi_j\} \subset R_n$ such that $\xi_j \to \infty$ and

$$\sum_{i=1}^{N} |\hat{S}_i(\eta)| \leqq |\xi_j|^{-A_3}$$

for all

$$|\eta - \xi_j| \leqq A_1 \log(2 + |\xi_j|).$$

Let k_j = the integral part of $A_1 \log(2 + |\xi_j|)$ and

$$\varphi_j = e^{2\pi i \langle \xi_j, x \rangle} \psi_{k_j}.$$

Then $\hat{\varphi}_j(\xi) = \hat{\psi}_{k_j}(\xi - \xi_j)$ and $\varphi_j \in \mathscr{D}(K_0)$. As in Hörmander's Theorem 3.3 [7], $\|\varphi_j\|_0$ tends with j to ∞.

It remains to show that $\|T\varphi_j\|_M$ converges to 0. Since

$$\|T\varphi_j\|_M = \sum_{i=1}^{N} \|S_i * \varphi_j\|_M,$$

it suffices to show that $\|S_i * \varphi_j\|_M$ converges to 0 for each $1 \leqq i \leqq N$. But for each $1 \leqq i \leqq N$,

$$|S_i(\eta)| \leqq |\xi_j|^{-A_3}$$

for $|\eta - \xi_j| \leqq A_1 \log(2 + |\xi_j|)$. Thus by exactly the same argument as in Hörmander's Theorem 3.3, we have the required result.

We now turn to the important implication (e) → (a).

LEMMA 2.4. *Let $\varphi \in \mathscr{D}(\Omega_0)$, $\{S_1, S_2, \cdots, S_N\} \subset \mathscr{E}'(R_n)$ such that not all $S_i = 0$. Then for every $r > 0$ we have*

$$|\hat{\varphi}(\xi)| \leqq 2^N \sum_{i=1}^{N} \left[\sup_{|\xi - z| < 4r} |\hat{\psi}_i(z)| \sup_{|\xi - z| < 4r} |\hat{S}_i(z)| \right] \div \left[\sup_{|\xi - z| < r} \sum_{i=1}^{N} |S_i(z)| \right]^2 \tag{2.13}$$

for all $\xi \in C_n$.

Proof. By utilizing Harnack's inequality one can obtain

$$|\hat{\varphi}(\xi)| = |\hat{\psi}_i(\xi)/\hat{S}_i(\xi)| \leqq \sup_{|\xi - z| < 4r} |\hat{\psi}_i(z)| \sup_{|\xi - z| < 4r} |\hat{S}_i(z)| \Big/ \left[\sup_{|\xi - z| < r} |\hat{S}_i(z)| \right]^2$$

for all $\xi \in C_n$ and $1 \leqq i \leqq N$. Thus we have

$$|\hat{\varphi}(\xi)| \left[\sum_{i=1}^{N} \sup_{|\xi - z| < r} |\hat{S}_i(z)| \right]^2 \leqq 2^N \sum_{i=1}^{N} \sup_{|\xi - z| < 4r} |\hat{\psi}_i(z)| \sup_{|\xi - z| < 4r} |\hat{S}_i(z)|$$

for all $\xi \in C_n$. However, since

$$\sum_{i=1}^{N} \sup_{|\xi - z| < r} |\hat{S}_i(z)| \geqq \sup_{|\xi - z| < r} \sum_{i=1}^{N} |\hat{S}_i(z)| > 0$$

we have the result.

LEMMA 2.5. *If the family* $\{S_1, S_2, \cdots, S_N\}$ *is invertible and* $(\Omega_0, \Omega_1, \cdots, \Omega_N)$ *is an* $\{S_i\}^N$*-convex* $(N + 1)$*-tuple, then* T^* *is a surjection from* $\bigoplus_{i=1}^{N} \mathscr{E}(\Omega_i)$ *onto* $\mathscr{E}(\Omega_0)$.

Proof. The method of proof is due essentially to Malgrange [9]. Let $E = \bigoplus_{i=1}^{N} \mathscr{E}(\Omega_i)$ and $F = \mathscr{E}(\Omega_0)$. Since both E and F are reflexive Fréchet spaces we see that T^* is a surjection if and only if T is one-to-one and $T(F')$ is $\sigma(E', E)$-closed in E'. (See Bourbaki [1].)

Clearly T is one-to-one. It remains to show that $T[\mathscr{E}'(\Omega_0)]$ is weakly closed in $\prod_{i=1}^{N} \mathscr{E}'(\Omega_i)$. A standard theorem in topological vector spaces (see Bourbaki [1]) states that a linear subspace, V, of the dual of a Fréchet space, E, is closed if and only if $V \cap U^0$ is $\sigma(E', E)$-closed for all neighborhoods U of 0 in E.

Let U be an open neighborhood of 0 in $E = \bigoplus_{i=1}^{N} \mathscr{E}(\Omega_i)$. Then U^0 is $\sigma(E', E)$-compact. Since U is a neighborhood of 0, there exist fixed constants M and C and fixed compact sets $K_i \subset \Omega_i$, $1 \leqq i \leqq N$, such that

$$\left\{ (\varphi_i, \cdots, \varphi_N) : C \sum_{|\alpha| \leqq M} \operatorname{supp}_{x \in K_i} |D^\alpha \varphi_i(x)| \leqq 1, 1 \leqq i \leqq N \right\} \subset U.$$

Thus we have

$$\sum_{i=1}^{N} |\psi_i(\varphi_i)| \leqq C \sum_{i=1}^{N} \sum_{|\alpha| \leqq M} \operatorname{supp}_{x \in K_i} |D^\alpha \varphi_i(x)| \tag{2.14}$$

for all $(\varphi_1, \cdots, \varphi_N)$ in $\bigoplus_{i=1}^{N} \mathscr{E}(\Omega_i)$ and $(\psi_1, \cdots, \psi_N)$ in U^0.

We now easily see that if $(\psi_1, \cdots, \psi_N) \in U^0$, then

$$\prod_{i=1}^{N} \operatorname{supp} \psi_i \subset \prod_{i=1}^{N} K_i.$$

Choose compact $K_0 \subset \Omega_0$ such that $\varphi \in \mathscr{E}'(\Omega_0)$ and

$$\prod_{i=1}^{N} \operatorname{supp} S_i * \varphi \subset \prod_{i=1}^{N} K_i$$

imply that $\operatorname{supp} \varphi \subset K_0$.

Let $\{(\psi_1^\nu, \psi_2^\nu, \cdots, \psi_N^\nu)\}_\nu$ be a net in $U^0 \cap T[\mathscr{E}'(\Omega_0)]$ which converges weakly to $(\psi_1, \psi_2, \cdots, \psi_N)$ in E'. Clearly $(\psi_1, \psi_2, \cdots, \psi_N) \in U^0$, and thus

$$\prod_{i=1}^{N} \operatorname{supp} \psi_i \subset \prod_{i=1}^{N} K_i.$$

Since each $\mathscr{E}(\Omega)$ is semi-Montel (see Horvath [9]), so is $\bigoplus_{i=1}^{N} \mathscr{E}(\Omega_i)$, and thus $\beta|_L = \sigma|_L$ on every equicontinuous set L of $\prod_{i=1}^{N} \mathscr{E}'(\Omega_i)$. Hence $\{(\psi_1^\nu, \psi_2^\nu, \cdots, \psi_N^\nu)\}$ $\beta(E', E)$-converges to $(\psi_1, \psi_2, \cdots, \psi_N)$ in E'.

Let $\varphi_\nu \in \mathscr{E}'(\Omega_0)$ such that $\psi_i^\nu = S_i * \varphi_\nu$ for $1 \leqq i \leqq N$. Then $\operatorname{supp} \varphi_\nu \subset K_0$. Since $\{(\psi_1^\nu, \psi_2^\nu, \cdots, \psi_N^\nu)\} \subset U^0$ we have

$$
\begin{aligned}
\sum_{i=1}^{N} |\hat{\psi}_i^\nu(z)| &= \sum_{i=1}^{N} |\langle \psi_i^\nu, e^{-2\pi i \langle x,z\rangle}\rangle| \\
&\leqq C \sum_{i=1}^{N} \sum_{|\alpha| \leqq M} \sup_{x \in K_i} |D^\alpha e^{-2\pi i\langle x,z\rangle}| \qquad (2.15)\\
&\leqq C(1 + |z|)^M e^{2\pi b |\operatorname{Im} z|},
\end{aligned}
$$

where C, M and b are independent of ν. If A is a compact set in C_n, we have

$$\mathscr{A} = \{e^{-2\pi i\langle x,z\rangle} : z \in A\}$$

is a bounded set in $\mathscr{E}(\Omega_i)$ and thus $\psi_i^\nu \to \psi_i$ uniformly on $\mathscr{A}$ for $1 \leqq i \leqq N$. Hence we have

$$\hat{\psi}_i^\nu(z) = \langle \psi_i^\nu, e^{-2\pi i\langle x,z\rangle}\rangle$$

converges to

$$\hat{\psi}_i(z) = \langle \psi_i, e^{-2\pi i\langle x,z\rangle}\rangle$$

uniformly on compact sets of C_n, and consequently

$$\hat{\varphi}_\nu(z) = \hat{\psi}_1^\nu(z)/\hat{S}_1(z)$$

converges uniformly on compact subsets of C_n to an entire function $\hat{\varphi}(z)$. From Lemma 2.4 we obtain

$$|\hat{\varphi}_\nu(\xi)| \leqq \sum_{i=1}^{N} \sup_{|\xi - z| < 4r} |\hat{\psi}_i^\nu(z)| \sup_{|\xi - z| < 4r} |\hat{S}_\nu(z)| (A_2 + |\xi|)^{2A_3}$$

for $r = A_1 \log(2 + |\xi|)$. Thus we have

$$|\hat{\varphi}_\nu(z)| \leqq C(1 + |z|)^B e^{2\pi R |\operatorname{Im} z|}, \tag{2.16}$$

where the constants C, B and R are independent of ν, and thus (2.16) holds for $|\hat{\varphi}_\nu(z)|$. Hence $\hat{\varphi}_\nu \to \hat{\varphi}$ in $\mathscr{S}'$ and thus $\varphi_\nu \to \varphi$ in $\mathscr{S}'$. Now we have

$$S_i * \varphi = \lim S_i * \varphi_\nu = \lim \psi_i^\nu = \psi_i$$

for $1 \leqq i \leqq N$, and therefore $\operatorname{supp} \varphi \subset K_0$ and $(\psi_1, \psi_2, \cdots, \psi_N) \in U^0 \cap T(F')$.

We now attempt to provide a geometrical interpretation of the notion of $\{S_i\}^N$-convexity.

THEOREM 2.3. *If* $(\Omega_0, \Omega_1, \cdots, \Omega_N)$ *is an* $\{S_i\}^N$*-convex* $(N + 1)$*-tuple, then given any* $\varphi \in \mathscr{E}'(\Omega_0)$ *we have*

$$d(\operatorname{supp} \varphi, R_n \setminus \Omega_0) = \min_{1 \leqq i \leqq N} d(\operatorname{supp} S_i * \varphi, R_n \setminus \Omega_i). \tag{2.17}$$

Here $d(A, B)$ *denotes the distance from* A *to* B *for sets in* R_n.

Proof. Since $\Omega_0 + \operatorname{supp} S_i \subset \Omega_i$ we have

$$d(\operatorname{supp} \varphi, R_n \setminus \Omega_0) \subset d(\operatorname{supp} S_i * \varphi, R_n \setminus \Omega_i)$$

for $1 \leqq i \leqq N$ and thus

$$d(\operatorname{supp} \varphi, R_n \setminus \Omega_0) \leqq \min_{1 \leqq i \leqq N} d(\operatorname{supp} S_i * \varphi, R_n \setminus \Omega_i).$$

Let $\varphi \in C_0^\infty(\Omega_0)$ and $S = d(\operatorname{supp} S_i * \varphi, R_n \setminus \Omega_0)$. Thus $\varphi_a(x) = \varphi(x - a) \in C_0^\infty(\Omega_0)$ for all a such that $|a| < S$, but $\operatorname{supp} \varphi_a$ is not contained in any fixed compact subset of Ω_0 for all $|a| < S$. Suppose there exist compact sets $K_1, \cdots, K_N$ in $\Omega_1, \cdots, \Omega_N$ respectively such that $\operatorname{supp} S_i * \varphi_a \subset K_i$ for all i and $|a| < S$. Then by hypothesis there exists a compact set $K_0 \subset \Omega_0$ containing all $\operatorname{supp} \varphi_a$ such that $|a| < S$ (contradiction). Let $\{a_n\}$ be a sequence of vectors in R_n such that $|a_n| < S$ and $\operatorname{supp} \varphi_{a_n}$ is not contained in any compact set of Ω_0 for all n. Since there are only a finite number of S_i's, there exist $1 \leqq i_0 \leqq N$ and a subsequence of $\{a_n\}$, again call in $\{a_n\}$, such that $S_{i_0} * \varphi_{a_n}$ is not contained in any compact set of Ω_{i_0}. But $S_{i_0} * \varphi_{a_n} = S_{i_0} * \varphi(x - a_n)$. Thus

$$d(\operatorname{supp} S_{i_0} * \varphi, R_n \setminus \Omega_{i_0}) \leqq d(\operatorname{supp} \varphi, R_n \setminus \Omega_0).$$

It is interesting to note that Theorem 2.2 may be used to prove a result that has no analogue in the case of Hörmander's theorem.

THEOREM 2.4. *Let $\Omega_1, \Omega_2, \cdots, \Omega_N$ be a finite family of open sets in R_n having nonempty intersection Ω_0. If f is in $\mathscr{E}(\Omega_0)$, then there exists f_i in $\mathscr{E}(\Omega_i)$ for $1 \leqq i \leqq N$ such that*

$$f = \sum_{i=1}^{N} f_i \quad on\ \Omega_0. \tag{2.18}$$

Proof. Let $S_i = \delta$ for $1 \leqq i \leqq N$. Then $(\Omega_0, \Omega_1, \cdots, \Omega_N)$ is clearly an $\{S_i\}_{i=1}^N$-convex $(N + 1)$-tuple. Also every S_i is invertible and hence so is the family $\{S_i\}_{i=1}^N$. From the previous theorem we obtain

$$T^*\left[\bigoplus_{i=1}^{N} \mathscr{E}(\Omega_i)\right] = \mathscr{E}(\Omega_0).$$

That is, given f in $\mathscr{E}(\Omega_0)$ there exist f_i in $\mathscr{E}(\Omega_i)$ for $1 \leqq i \leqq N$ such that

$$f = T^*(f_1, \cdots, f_N) = \sum_{i=1}^{N} \check{S}_i * f_i|_{\Omega_0}$$
$$= \sum_{i=1}^{N} \delta * f_i|_{\Omega_0} = \left[\sum_{i=1}^{N} f_i\right]\Bigg|_{\Omega_0}.$$

REFERENCES

[1] N. BOURBAKI, *Espaces vectoriels topologiques*, Hermann et Cie, Paris, 1953–1955.
[2] L. EHRENPREIS, *Solutions of some problems of division. I*, Amer. J. Math., 76 (1954), pp. 883–903.
[3] ———, *Solutions of some problems of division. II*, Ibid., 77 (1955), pp. 286–292.
[4] ———, *Solutions of some problems of division, III*, Ibid., 78 (1956), pp. 685–715.
[5] ———, *Solutions of some problems of division. IV*, Ibid., 82 (1960), pp. 522–588.
[6] L. HÖRMANDER, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.
[7] ———, *On the range of convolution operators*, Ann. of Math., 76 (1962), pp. 148–169.
[8] J. LIONS, *Supports dans la transformation de Laplace*, J. Analyse Math., 2 (1952–1953), pp. 369–380.

[9] J. Horvath, *Topological Vector Spaces and Distributions*, Addison-Wesley, Reading, Mass., 1966.

[10] B. Malgrange, *Existence et approximation des solutions des équations aux derivées partielles et des équations de convolution*, Ann. Inst. Fourier (Grenoble), 6 (1955–1956), pp. 271–355.

[11] L. Schwartz, *Theorie des distributions*, Hermann, Paris, 1966.

[12] E. C. Titchmarsh, *The Theory of Functions*, Random House, Oxford, 1932.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

SPHERICAL MEANS OF CONJUGATE FOURIER INTEGRALS IN E_2 AT THE CRITICAL INDEX*

G. E. LIPPMAN†

Abstract. In this paper the following result is obtained: Let f be a function in $L_1(E_2)$ and let $K(r, \theta) = r^{-2} \sum (a_k \cos k\theta + b_k \sin k\theta)$ be a Calderón–Zygmund kernel satisfying $\sum k^4(|a_k| + |b_k|) < \infty$. Assuming that f satisfies a restricted Dini-type condition, at (x_0, y_0), the difference between the spherical Bochner–Riesz means of order 1/2 of the conjugate Fourier transform of f and the principal-valued Hilbert transform of f converges to 0. The proof depends upon facts concerning the Bessel functions of the first kind and on the one-dimensional Riemann–Lebesgue theorem for Fourier integrals.

1. Introduction. Let T_k be the k-dimensional torus and E_k be k-dimensional Euclidean space, $k \geqq 2$, let f be a function integrable on T_k and periodic in each coordinate of E_k, and x be a point in T_k. In 1936 Bochner [1] proved that the limit behavior at x of the Bochner–Riesz partial sums of order α of the Fourier series of f depends only on the values of f in any neighborhood (no matter how small) of x, as long as α remains greater than $(k - 1)/2$. In the same paper Bochner proved that an analogous result holds for Bochner–Riesz means of order α, for α greater than or equal to $(k - 1)/2$, of Fourier transforms of functions integrable on E_k, and that localization fails in the Fourier integral case for an index of summability less than the critical index $(k - 1)/2$. In the same paper Bochner also gave an ingenious proof of the existence of a function f periodic in E_k, integrable on the torus, identically zero in an open ball centered at the origin, with the property that the Bochner–Riesz sums of order $(k - 1)/2$ of its Fourier series diverge at $x = 0$, thus proving the failure of localization at the critical index of summability for Bochner–Riesz sums of Fourier series.

In 1954 Calderón and Zygmund [2], using their singular integral theory, defined the notion of conjugate multiple Fourier series. In 1961 Shapiro [8] proved localization theorems for Bochner–Riesz summability, of order greater than the critical index, for conjugate Fourier–Stieltjes series under very general conditions on the conjugate kernel. The following theorem in E_2 is a localization theorem at the critical index for Fourier integrals conjugate with respect to series of spherical harmonic Calderón–Zygmund kernels, and is similar to Theorem 1 [6, p. 43]. The function f is subjected to a "Dini-type" condition. The proof depends on fundamental lemmas involving Bessel functions, many of which are proved in [6, Chap. II].

2. Statement of Theorem.

THEOREM. *Let f be a function integrable on E_2, and let*

$$K(r, \theta) = r^{-2}\Omega(\theta) \tag{1}$$

be a Calderón–Zygmund kernel, where

$$\Omega(\theta) = \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta) \tag{2}$$

* Received by the editors February 19, 1971, and in revised form September 13, 1971.

† Department of Mathematics, California State College, Hayward, California 94542. This work is part of the author's doctoral dissertation prepared under the direction of Professor V. L. Shapiro at the University of California, Riverside.

satisfying

$$\sum_{k=1}^{\infty} k^4(|a_k| + |b_k|) < +\infty. \tag{3}$$

Assume that f satisfies the following restricted Dini-type condition at (x_0, y_0):

$$\int_0^\eta (dt/t) \int_0^{2\pi} |f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)|\, d\theta < +\infty \tag{4}$$

for some $\eta > 0$.

Then

$$\lim_{R\to\infty} \left\{ \int\int_{B(0,R)} \hat{K}(P)\hat{f}(P) \exp\{i(P, X_0)\}(1 - |P|^2/R^2)^{1/2}\, dP \right. \tag{5}$$
$$\left. -(2\pi)^{-2} \int_{E_2 - B(0,1/R)} f(X_0 - X)K(X)\, dX \right\} = 0.$$

In the above theorem $P = (p, q)$, $X = (x, y)$ and $X_0 = (x_0, y_0)$ are points in E_2. Notice that condition (4) implies the weaker hypothesis

$$\int_0^\eta (dt/t) \left| \int_0^{2\pi} f(x_0 + t\cos\theta, y_0 + t\sin\theta)\Omega(\theta)\, d\theta \right| < \infty \tag{6}$$

since $\Omega(\theta)$ is a trigonometric series which is uniformly absolutely convergent for $0 \leqq \theta < 2\pi$. In the proof of the theorem we shall use this hypothesis rather than (4) whenever possible.

3. Examples. In this section we cite some examples of spherical harmonic kernels satisfying (2) and (3).

The kernel $K(x, y) = x(x^2 + y^2)^{-3/2}$ arises in the study of Newtonian potential in E_3 (see, for example, [2]).

The wave kernel $K(x, y) = (x^2 - y^2)/(x^2 + y^2)^2$ is used in the study of sets of uniqueness for the vibrating string problem in [7] and is the real part of the kernel $K(x + iy) = (x + iy)^{-2}$ (see [3]).

4. Lemmas. In order to facilitate the proof, we shall prove several lemmas concerned with Bessel functions of the first kind:

$$J_n(t) = \sum_{j=0}^{\infty} (-1)^j (t/2)^{n+2j}/j!\Gamma(n + j + 1), \tag{7}$$

where t and n are real. From the well-known fact that

$$tJ_n(t) = 2(n - 1)J_{n-1}(t) - tJ_{n-2}(t)$$

it follows that

$$tJ_n(t) = 2\sum_{j=1}^{n-1} c_j J_j(t) + c_1' tJ_1(t) + c_0' tJ_0(t), \tag{8}$$

where

$$|c_j| = j \text{ or } 0, \quad |c_0'| = 1 \text{ or } 0, \quad \text{and} \quad |c_1'| = 1 \text{ or } 0 \tag{9}$$

(see [8, pp. 47–48, Lemma 3]).

LEMMA 1.

$$\int_0^t (1 - s^2/t^2)^{1/2} s J_1(s)\, ds = F(t) - \sin t,$$

where $|F(t) - B'| = O(t^{-1/2})$ *as* $t \to \infty$, B' *being a constant.*

Proof. Integrating by parts, and using the formula $\int J_1(s)\, ds = -J_0(s)$, we find that the above integral is equal to

$$\int_0^t (1 - s^2/t^2)^{1/2} J_0(s)\, ds - \int_0^t (1 - s^2/t^2)^{-1/2} (s/t)^2 J_0(s)\, ds = I_1(t) - I_2(t).$$

Define $P(t) = \int_0^t J_0(s)\, ds = \int_0^1 J_0(s)\, ds + \int_1^t J_0(s)\, ds$. The first integral is finite and the second one is equal to

$$\int_1^t s(J_0(s)/s)\, ds = J_1(t) - J_1(1) + \int_1^t (J_1(s)/s)\, ds$$

which is dominated by

$$|J_1(t)| + |J_1(1)| + \text{const.} \int_1^\infty s^{-3/2}\, ds.$$

Therefore, $P(\infty) = \lim_{t\to\infty} P(t)$ exists, and we notice that $P(\infty) - P(t) = O(t^{-1/2})$.

We consider $I_1(t)$ which equals

$$\int_0^t (1 - s^2/t^2)^{1/2} J_0(s)\, ds = \int_0^t (1 - s^2/t^2)^{-1/2} P(s)(s/t^2)\, ds.$$

Furthermore,

$$\int_0^t (1 - s^2/t^2)^{-1/2} (s/t^2)\, ds = 1;$$

hence,

$$I_1(t) - P(\infty) = \int_0^t (1 - s^2/t^2)^{-1/2} \{P(s) - P(\infty)\}(s/t^2)\, ds.$$

Next, let us consider

$$\left| t^{1/2} \int_0^t (1 - s^2/t^2)^{-1/2} \{P(s) - P(\infty)\}(s/t^2)\, ds \right|,$$

which is less than or equal to

$$\text{const.} \int_0^t (1 - s^2/t^2)^{-1/2} (s^{1/2}/t^{3/2})\, ds,$$

where the constant is independent of t. Let $u = s/t$. This last integral is then equal to

$$\int_0^1 (1 - u^2)^{-1/2} u^{1/2}\, du \leqq \int_0^1 (1 - u^2)^{-1/2}\, du = \pi/2.$$

Therefore $I_1(t) - P(\infty) = O(t^{-1/2})$. Since

$$I_2(t) = \int_0^t (1 - s^2/t^2)^{-1/2}(s/t)^2 J_0(s)\,ds$$

and

$$(s/t)^2 = (s/t) - (1 - s/t)(s/t),$$

we see that

$$\begin{aligned} I_2(t) &= \int_0^t (1 - s^2/t^2)^{-1/2}(s/t)J_0(s)\,ds - \int_0^t (1 - s/t)^{1/2}(1 + s/t)^{-1/2}(s/t)J_0(s)\,ds \\ &= I_3(t) - I_4(t). \end{aligned}$$

By Sonine's first integral and the fact that $J_{1/2}(t) = (2/\pi t)^{1/2} \sin t$, we have $I_3(t) = \sin t$.

Using the fact that $\int sJ_0(s)\,ds = sJ_1(s)$, and integrating $I_4(t)$ by parts, we obtain

$$\begin{aligned} I_4(t) = (1/2t^2) &\int_0^t (1 - s/t)^{1/2}(1 + s/t)^{-3/2} sJ_1(s)\,ds \\ &+ (1/2t^2)\int_0^t (1 - s^2/t^2)^{-1/2} sJ_1(s)\,ds, \end{aligned}$$

and this sum is dominated by a constant multiple of

$$t^{-1/2}\int_0^1 \{(1 - u)^{1/2} + (1 - u)^{-1/2}\}u^{1/2}\,du.$$

Since the integral is finite, we have $|I_4(t)| = O(t^{-1/2})$.

Hence, if we define $F(t) = I_1(t) + I_4(t)$, $B' = P(\infty)$, the conclusion of Lemma 1 holds.

LEMMA 2. *If n is a positive integer and $t > 0$, then*

$$\int_0^t (1 - r^2/t^2)^{1/2} rJ_n(r)\,dr = I_n(t) + I_n'(t) + I_n''(t), \tag{10}$$

where

$$I_n'(t) = -c_1'(n)A' \sin t, \tag{11}$$

$$I_n''(t) = c_0'(n)t^{1/2}J_{3/2}(t), \tag{12}$$

and there exist constants B_n and C_n, depending only on n, such that

$$|I_n(t) - B_n| < C_n/\sqrt{t}, \tag{13}$$

where

$$C_n = O(n^5). \tag{14}$$

The constant $c_0'(n)$ is such that

$$|c_0'(n)| = 0 \quad \textit{or} \quad 1$$

and

$$|c_1'(n)| = 0 \quad or \quad 1.$$

Proof.

$$\int_0^t (1 - r^2/t^2)^{1/2} J_n(r)\,dr = 2\sum_{j=1}^{n-1} c_j \int_0^t (1 - r^2/t^2)^{1/2} J_j(r)\,dr$$
$$+ c_1' \int_0^t (1 - r^2/t^2)^{1/2} r J_1(r)\,dr$$
$$+ c_0' \int_0^t (1 - r^2/t^2)^{1/2} r J_0(r)\,dr,$$

where $|c_j| = 0$ or j, and $|c_1'|$ and $|c_0'| = 1$ or 0. By Sonine's first integral [10, p. 373],

$$\int_0^t (1 - r^2/t^2)^{1/2} r J_0(r)\,dt = \sqrt{2}\Gamma(3/2) t^{1/2} J_{3/2}(t).$$

Therefore, define $I_n''(t) = c_0' \int_0^t (1 - r^2/t^2)^{1/2} r J_0(r)\,dr$. In Lemma 1 we have shown the existence of absolute constants B' and C' such that

$$\int_0^t (1 - r^2/t^2)^{1/2} r J_1(r)\,dr = F(t) - \sin t,$$

where $|F(t) - B'| \leqq C' t^{-1/2}$ for $t \geqq t_0 > 0$. We define $I_n'(t) = -c_1' \sin t$ and thus

$$I_n(t) = 2\sum_{j=1}^{n-1} c_j \int_0^t (1 - r^2/t^2)^{1/2} J_j(r)\,dr + c_1' F(t).$$

At this point we recall Lemma 4 from [8, p. 48]. Let $0 < \alpha \leqq 1$ and j be a nonnegative integer. Then there exists a constant b_j such that

$$\left| \int_0^t (1 - r^2/t^2)^\alpha J_j(r)\,dr - b_j \right| \leqq 30 C_\alpha (j+1)^3 t^{-1/2},$$

where b_j depends only on j and C_α only on α. Defining $B_n' = 2\sum_{j=1}^{n-1} c_j b_j$, we have

$$\left| 2\sum_{j=1}^{n-1} c_j \left\{ \int_0^t (1 - r^2/t^2)^{1/2} J_j(r)\,dr - b_j \right\} \right|$$
$$\leqq 2\sum_{j=1}^{n-1} |c_j| \left| \int_0^t (1 - r^2/t^2)^{1/2} J_j(r)\,dr - b_j \right|$$
$$\leqq 2\sum_{j=1}^{n-1} |c_j| \{30 C_{1/2} (j+1)^3 t^{-1/2}\}$$
$$\leqq 60 C_{1/2} t^{-1/2} \sum_{j=1}^{n-1} (j+1)^4$$
$$\leqq \text{const. } t^{-1/2} n^5.$$

Thus

$$|I_n(t) - B'_n - c'_1 B'| \leqq \text{const.}\, t^{-1/2} n^5 + c'_1 |F(t) - B'|$$
$$\leqq \text{const.}\, t^{-1/2} n^5 + C' t^{-1/2}.$$

By defining $B_n = B'_n - B'$ we have

$$|I_n(t) - B_n| \leqq \text{const.}\,(n^5 + 1)t^{-1/2}.$$

This proves Lemma 2.

LEMMA 3.

$$\int_{-\pi}^{\pi} \cos n\theta \exp\{-it\cos(\theta - \alpha)\}\, d\theta = \cos n\alpha\, \{2(-i)^n \pi J_n(t)\}, \tag{15}$$

$$\int_{-\pi}^{\pi} \sin n\theta \exp\{-it\cos(\theta - \alpha)\}\, d\theta = \sin n\alpha\, \{2\pi(-i)^n J_n(t)\} \tag{16}$$

for $t > 0$.

Proof of (15). (Equation (16) is proved similarly.)

$$\int_{-\pi}^{\pi} \cos n\theta \exp\{-it\cos(\theta - \alpha)\}\, d\theta$$
$$= \int_{-\pi}^{\pi} \cos n(\theta + \alpha) \exp\{-it\cos\theta\}\, d\theta$$
$$= \cos n\alpha \int_{-\pi}^{\pi} \cos n\theta \exp\{-it\cos\theta\}\, d\theta + \sin n\alpha \int_{-\pi}^{\pi} \text{odd function}$$
$$= \cos n\alpha \{2(-i)^n J_n(t)\pi\},$$

using the series expansion of $\exp\{-it\cos\theta\}$ in terms of Bessel functions and the orthogonality of the trigonometric functions.

5. Proof of Theorem. We assume without loss of generality that X_0 is the origin and $f(X_0) = 0$. Thus we need only show

$$\lim_{R\to\infty} \left\{ \int_{B(0,R)} \hat{K}(P)\hat{f}(P)(1 - |P|^2/R^2)^{1/2}\, dP - (2\pi)^{-2} \int_{E_2 - B(0,1/R)} f(-X)K(X)\, dX \right\} = 0. \tag{17}$$

Define

$$I_R = \int_{B(0,R)} \hat{K}(P)\hat{f}(P)(1 - |P|^2/R^2)^{1/2}\, dP. \tag{18}$$

By the definition of $\hat{f}$ and by Fubini's theorem,

$$I_R = (2\pi)^{-2} \int_{E_2} f(X)\, dX \int_{B(0,R)} \hat{K}(P)(1 - |P|^2/R^2)^{1/2} \exp\{-i(P, X)\}\, dP. \tag{19}$$

Expanding $\hat{K}$ by the formula given in [5, p. 14], we obtain

$$
\begin{aligned}
I_R = {} & (2\pi)^{-3} \int_{E_2} f(X)\, dX \int_0^R r(1 - r^2/R^2)^{1/2}\, dr \\
& \cdot \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (-i)^n n^{-1}(a_n \cos n\theta + b_n \sin n\theta) \exp\{-ir|X| \cos(\theta - \alpha)\}\, d\theta,
\end{aligned} \tag{20}
$$

where α is the polar angle of X. By Lemma 3,

$$
I_R = 1/(2\pi)^2 \int_{E_2} f(X)\, dX \sum_{n=1}^{\infty} A(n, \alpha) \int_0^R r(1 - r^2/R^2)^{1/2} J_n(r|X|)\, dr, \tag{21}
$$

where

$$
A(n, \alpha) = (-1)^n(a_n \cos n\alpha + b_n \sin n\alpha)/n. \tag{22}
$$

Since $|J_n| \leqq 1$ for every $n = 1, 2, \cdots$, $\sum_{n=1}^{\infty} A(n, \alpha) J_n(r|X|)$ is by hypothesis (3) uniformly absolutely convergent for all values of α, r and X. By a change of variables in the inner integral of (21),

$$
\begin{aligned}
I_R &= 1/(2\pi)^2 \int_{E_2} (f(X)/|X|^2) \sum_{n=1}^{\infty} A(n, \alpha) \int_0^{R|X|} r(1 - r^2/R^2|X|^2)^{1/2} J_n(r)\, dr \\
&= 1/(2\pi)^2 \left\{ \int_{E_2 - B(0,1/R)} + \int_{B(0,1/R)} \right\} (f(X)/|X|^2) \sum_{n=1}^{\infty} A(n, \alpha) \\
&\quad \cdot \int_0^{R|X|} r(1 - r^2/R^2|X|^2)^{1/2} J_n(r)\, dr.
\end{aligned} \tag{23}
$$

We first notice that

$$
\begin{aligned}
& \left| \int_{B(0,1/R)} (f(X)/|X|^2) \sum_{n=1}^{\infty} A(n, \alpha) \int_0^{R|X|} r(1 - r^2/R^2|X|^2)^{1/2} J_n(r)\, dr \right| \\
& \leqq \left| \int_0^{1/R} t^{-1} \sum_{n=1}^{\infty} A(n, \alpha) \int_0^{Rt} (1 - r^2/R^2t^2)^{1/2} r J_n(r)\, dr \right. \\
& \qquad \left. \cdot \int_0^{2\pi} f(t \cos \theta, t \sin \theta)\, d\theta\, dt \right|.
\end{aligned} \tag{24}
$$

Since $(1 - r^2/R^2t^2)^{1/2}|J_n(r)| \leqq 1$ for all r and t such that $0 < r \leqq Rt$, the right-hand side of (24) is dominated by

$$
\int_0^{1/R} R^2 t \sum_{n=1}^{\infty} |A(n, \alpha)| \left| \int_0^{2\pi} f(t \cos \theta, t \sin \theta)\, d\theta \right| dt,
$$

and since $\sum_{n=1}^{\infty} |A(n, \alpha)|$ is uniformly bounded and $t \leqq 1/R$, then this last integral is dominated by

$$
\int_0^{1/R} (1/t) \left| \int_0^{2\pi} f(t \cos \theta, t \sin \theta)\, d\theta \right| dt,
$$

which is $o(1)$ as $R \to \infty$ by hypothesis (4).

We now consider the integral in (23) over $E_2 - B(0, 1/R)$. By Lemma 2, the inner integral in (23) is equal to

$$-c_1'(n)\sin R|X| + c_0'(n)(R|X|)^{1/2}J_{3/2}(R|X|) + I_n(R|X|) \tag{25}$$

for each positive integer n, where there exist constants B_n and C_n depending only on N such that $C_n = O(n^5)$ and

$$|I_n(R|X|) - B_n| \leqq C_n/\sqrt{R|X|} \tag{26}$$

for $|X| \neq 0$. Putting (23) and (25) together we obtain

$$\begin{aligned} I_R = &-(2\pi)^{-2}\int_{E_2 - B(0,1/R)} f(X)|X|^{-2}\sum_{n=1}^{\infty} A(n,\alpha)c_1'(n)\sin R|X|\,dX \\ &+(2\pi)^{-2}\int_{E_2 - B(0,1/R)} f(X)|X|^{-2}\sum_{n=1}^{\infty} A(n,\alpha)c_0'(n)(R|X|)^{1/2}J_{3/2}(R|X|)\,dX \\ &+(2\pi)^{-2}\int_{E_2 - B(0,1/R)} f(X)|X|^{-2}\sum_{n=1}^{\infty} A(n,\alpha)I_n(R|X|)\,dX + o(1) \\ &\text{as} \quad R \to \infty. \end{aligned} \tag{27}$$

We consider each of these three integrals separately.

The first integral in (27) can be written as the sum of two integrals, the integral over $E_2 - B(0, \delta)$ and the integral over $B(0, \delta) - B(0, 1/R)$ for some $\delta > 0$ independent of R (δ will be specified later). Consider the integral

$$\begin{aligned} &\int_{E_2 - B(0,\delta)} f(X)|X|^{-2}\sum_{n=1}^{\infty} A(n,\alpha)c_1'(n)\sin R|X|\,dX \\ &= \int_{-\infty}^{\infty} \chi_{(\delta,\infty)}(r)r^{-1}\sin Rr\int_0^{2\pi}\sum_{n=1}^{\infty} c_1'(n)A(n,\alpha)f(r\cos\alpha, r\sin\alpha)\,d\alpha\,dr. \end{aligned} \tag{28}$$

Since the function

$$\chi_{(\delta,\infty)}(r)r^{-1}\int_0^{2\pi}\sum_{n=1}^{\infty} c_1'(n)A(n,\alpha)f(r\cos\alpha, r\sin\alpha)\,d\alpha$$

is integrable over the real line, by the Riemann–Lebesgue theorem for functions integrable on E_1, the integral on the left side of (28) goes to 0 as $R \to \infty$.

Given $\varepsilon > 0$, by hypothesis (4) we can pick $\delta > 0$ so that the integral of the absolute value of the integrand over $B(0, \delta)$ is bounded by ε. Thus the first integral in (27) goes to 0 as $R \to \infty$.

The second integral is equal to

$$\begin{aligned} &\int_{E_2 - B(0,1/R)} f(X)|X|^{-2}\sum_{n=1}^{\infty} c_0'(n)A(n,\alpha) \\ &\cdot\{\text{const.}\sin R|X|(R|X|)^{-1} - \text{const.}\cos R|X|\}\,dX, \end{aligned} \tag{29}$$

where the constants are independent of n and R.

Following the same reasoning as we did for the first integral in (27) we see that the term with $\cos R|X|$ as a factor approaches zero as $R \to \infty$. For the other term we break the integral into two terms:

$$\int_{B(0,\delta)-B(0,1/R)} + \int_{E_2-B(0,\delta)},$$

where, as before, $\delta > 0$ is chosen by hypothesis (4) so that for $\varepsilon > 0$ given,

$$\int_{B(0,\delta)} \left| f(X)|X|^{-2} \sum_{n=1}^{\infty} c_0'(n)A(n,\alpha) \right| dX < \varepsilon. \tag{30}$$

Since $|\sin R|X||/(R|X|) \leqq 1$, this implies that the integral over $B(0,\delta) - B(0,1/R)$ of the integrand in the first term of (29) is bounded by ε.

$$\int_{E_2-B(0,\delta)} f(X)|X|^{-2} \sum_{n=1}^{\infty} c_0'(n)A(n,\alpha) \sin R|X|/(R|X|)\, dX$$

goes to 0 as $R \to \infty$ by the one-dimensional Riemann–Lebesgue theorem, since δ was chosen independent of R. Thus the second integral in (27) goes to 0 as $R \to \infty$.

We have remaining the third integral in (27):

$$\int_{E_2-B(0,1/R)} f(X)|X|^{-2} \sum_{n=1}^{\infty} A(n,\alpha)I_n(R|X|)\, dX. \tag{31}$$

Since the other integrals in (27) go to 0 as $R \to \infty$, in order to complete the proof of the theorem we must show that, as $R \to \infty$, the difference between the integral (31) and

$$\int_{E_2-B(0,1/R)} f(-X)K(X)\, dX \tag{32}$$

approaches 0.

Expanding the integrand in (31) we have (31) equal to

$$\int_{E_2-B(0,1/R)} f(X)|X|^{-2} \sum_{n=1}^{\infty} (1/n)(a_n \cos n\alpha + b_n \sin n\alpha)I_n(R|X|)\, dX$$

while (32) is equal to

$$\int_{E_2-B(0,1/R)} f(-X)|X|^{-2}\left(\sum_{n=1}^{\infty} (a_n \cos n\alpha + b_n \sin n\alpha)\right) dX$$
$$= \int_{E_2-B(0,1/R)} f(X)|X|^{-2}\left(\sum_{n=1}^{\infty} (a_n \cos n\alpha + b_n \sin n\alpha)(-1)^n\right) dX$$

since $\cos n(\alpha + \pi) = (-1)^n \cos n\alpha$, for $n = 1, 2, \cdots$. We therefore need only to show that, as $R \to \infty$,

$$\int_{E_2-B(0,1/R)} f(X)|X|^{-2} \sum_{n=1}^{\infty} (a_n \cos n\alpha + b_n \sin n\alpha)(-1)^n(I_n(R|X|)/n - 1)\, dX \to 0. \tag{33}$$

Without loss of generality we shall assume that $b_n = 0$ for $n = 1, 2, \cdots$. The effect of this will be to show that (33) holds for the cosine series only; a similar argument will also work for the sine series, thus establishing (33). Therefore, we shall prove

$$\int_{E_2 - B(0,1/R)} f(X)|X|^{-2} \sum_{n=1}^{\infty} (-1)^n a_n \cos n\alpha (I_n(R|X|)/n - 1)\, dX \tag{34}$$

goes to 0 as $R \to \infty$.

The fact that will allow us to show that (34) goes to 0 is that for each positive integer n the constant B_n in (13) in Lemma 2 is actually equal to n. The proof of this fact is as follows:

Assume that $a_n = 0$ for $n \geqq 1$, $n \neq m$, where m is a fixed positive integer. We shall show that

$$\lim_{R\to\infty} \int_{E_2 - B(0,1/R)} f(X)|X|^{-2} a_m \cos m\alpha (I_m(R|X|)/m - B_m/m)\, dX = 0. \tag{35}$$

This implies that

$$\begin{aligned} &\int_{B(0,R)} \hat{K}(P)\hat{f}(P)(1 - |P|^2/R^2)^{1/2}\, dP \\ &\qquad -(-1)^m B_m/((2\pi)^2 m) \int_{E_2 - B(0,1/R)} f(-X)K(X)\, dX \end{aligned} \tag{36}$$

goes to 0 as $R \to \infty$, where

$$K(X) = a_m |X|^{-2} \cos m\alpha, \tag{37}$$

α being the polar angle of X. However, for $f \in C_0^\infty(E_2)$ and with (37) defining $K(X)$, for $\varepsilon > 0$,

$$\begin{aligned} \lim_{R\to\infty} &\int_{B(0,R)} \hat{K}(P)\hat{f}(P)(1 - |P|^2/R^2)^{1/2}\, dP \\ &= \int_{E_2} \hat{K}(P)\hat{f}(P)\, dP \\ &= \lim_{R\to\infty} \int_{B(0,R)} \hat{K}(P)\hat{f}(P)(1 - |P|^2/R^2)^{1/2+\varepsilon}\, dP \\ &= (2\pi)^{-2} \int_{E_2} f(-X)K(X)\, dX \\ &= \lim_{R\to\infty} (2\pi)^{-2} \int_{E_2 - B(0,1/R)} f(-X)K(X)\, dX, \end{aligned} \tag{38}$$

since all of the above integrals and limits exist, and by virtue of Theorem 1 [6, p. 43].

Therefore $B_m/m = 1$.

We now prove (35) in the following way.

By (13) and (14) in Lemma 2 the difference

$$I_m(R|X|) - B_m$$

is dominated by $m^5/(R|X|)^{1/2}$. Therefore there is a constant C, independent of

m, R and $|X|$, such that

$$|I_{.n}(R|X|)/m - B_m/m| \leqq Cm^4/(R|X|)^{1/2}. \tag{39}$$

Inequality (39) implies that the integral in (35) is dominated by

$$\left\{\int_\delta^\infty + \int_{1/R}^\delta\right\} R^{-1/2}t^{-3/2}m^4 \left|\int_0^{2\pi} f(t\cos\alpha, t\sin\alpha)\cos m\alpha\, d\alpha\right| dt, \tag{40}$$

where $\delta > 0$ is independent of R. The integral over (δ, ∞) goes to 0 as $R \to \infty$ since it is $O(R^{-1/2})$.

Let $\varepsilon > 0$ and choose $\delta > 0$ so that the integral over $[0, \delta]$ in hypothesis (6) is dominated by ε. In the integral over $(1/R, \delta)$ in (40), $1/\sqrt{Rt} \leqq 1$, so this integral is dominated by $\varepsilon|a_m|m^4$. This proves (35).

We now have the fact that $B_n = n$ for every positive integer n, and we need only show (34) holds in order to complete the theorem.

The integrand in (34) is dominated by a constant multiple of

$$|f(X)||X|^{-2} \sum_{n=1}^\infty |a_n|n^4(R|X|)^{-1/2}.$$

Since by hypothesis (3),

$$\sum_{n=1}^\infty |a_n|n^4 \text{ is finite},$$

we have

$$\begin{aligned} &\lim_{R\to\infty}\left|\int_{E_2 - B(0,1/R)} f(X)|X|^{-2} \sum_{n=1}^\infty (-1)^n a_n \cos n\alpha (I_n(R|X|)/n - 1)\, dX\right| \\ &\leqq \text{const.} \lim_{R\to\infty} \int_{1/R}^\infty \sum_{n=1}^\infty |a_n|n^4 R^{-1/2}t^{-3/2} \int_0^{2\pi} |f(t\cos\theta, t\sin\theta)|\, d\theta\, dt. \end{aligned} \tag{41}$$

Applying the same reasoning as used in proving (35) and using hypothesis (4) in place of (6) we see that the right-hand side of (41) is equal to 0.

This completes the proof of the theorem.

In a forthcoming article [4] we shall present localization theorems in E_k, $k \geqq 2$, for Bochner–Riesz means of order $(k - 1)/2$ for Fourier series and integrals conjugate with respect to the spherical harmonic kernels which, in E_2, are special cases of the conjugate kernels discussed here.

REFERENCES

[1] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc., 40 (1936), pp. 175–207.

[2] A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math., 88 (1952), pp. 25–139.

[3] ———, *Singular integrals and periodic functions*, Studia Math., 14 (1954), pp. 249–271.

[4] C. P. Chang and G. E. Lippman, *Exponential sums arising in conjugate multiple Fourier series*; *spherical summability of conjugate multiple Fourier series and integrals at the critical index* (in preparation).

[5] V. L. Shapiro, *The conjugate Fourier–Stieltjes integral in the plane*, Bull. Amer. Math. Soc., 65 (1959), pp. 12–15.

[6] ———, *Fourier series in several variables*, Ibid., 70 (1964), pp. 48–93.

[7] ———, *Sets of uniqueness for the vibrating string problem*, Trans. Amer. Math. Soc., 141 (1969), pp. 127–146.

[8] ———, *Topics in Fourier and geometric analysis*, Mem. Amer. Math. Soc., 39 (1961), pp. 1–100.

[9] E. C. TITCHMARSH, *Theory of Fourier Integrals*, Oxford University Press, Oxford, 1948.

[10] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1966.

[11] A. ZYGMUND, *Trigonometric Series*, vol. I, Cambridge University Press, Cambridge, 1959.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

BERGMAN OPERATORS FOR ELLIPTIC EQUATIONS IN FOUR INDEPENDENT VARIABLES*

DAVID COLTON†

Abstract. Integral operators are obtained which map analytic functions of three complex variables onto solutions of linear elliptic partial differential equations in four independent variables. An inversion formula is given and used to construct a complete family of solutions for the elliptic equation under investigation.

1. Introduction. The theory of integral operators for elliptic partial differential equations was initiated by S. Bergman [1] and I. N. Vekua [20], both of whom constructed operators which map analytic functions of a single complex variable onto twice continuously differentiable (class C^2) solutions of the elliptic equation

$$\Delta_2 u + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \tag{1.1}$$

These operators were then used to construct complete families of solutions and to investigate the analytic properties of solutions to (1.1). Recently, Colton [3], [4], [5] was able to extend the results of Bergman and Vekua to the case of three independent variables, that is, the equation

$$\Delta_3 u + a(x, y, z)u_x + b(x, y, z)u_y + c(x, y, z)u_z + d(x, y, z)u = 0. \tag{1.2}$$

More specifically, integral operators were obtained in [3], [4] and [5] which map analytic functions of *two* complex variables onto C^2-solutions of (1.2), and were then used for purposes of analytic continuation and to construct a complete family of solutions to (1.2). This work was the culmination of the efforts of several mathematicians, among them Bergman [1], Tjong [18], [19], Colton and Gilbert [6] and Gilbert and Lo [14]. In this paper we indicate how the approach used to treat equation (1.2) can be extended to treat elliptic equations in four independent variables, that is, the equation

$$\begin{aligned}\Delta_4 u + a(x_1, x_2, x_3, x_4)u_{x_1} + b(x_1, x_2, x_3, x_4)u_{x_2} + c(x_1, x_2, x_3, x_4)u_{x_3} \\ + d(x_1, x_2, x_3, x_4)u_{x_4} + f(x_1, x_2, x_3, x_4)u = 0.\end{aligned} \tag{1.3}$$

Our methods unfortunately do not appear applicable to elliptic equations in more than four variables, and so at present it seems that the use of integral operators in investigating the analytic theory of elliptic equations is restricted to equations in two, three and four variables.

Until a few years ago integral operators for elliptic equations in four independent variables were available only for the harmonic equation and certain classes of equations with spherically symmetric coefficients [11], [12], [13], [16]. Recently, however, Colton and Gilbert obtained an integral operator which mapped analytic functions of *three* complex variables onto an unspecified *subspace* of solutions to (1.3) in the special case when $a = b = c = d = 0$ (see [6]).

---

* Received by the editors April 26, 1971, and in revised form June 29, 1971.

† Department of Mathematics, Indiana University, Bloomington, Indiana 47401. This work was supported in part by the National Science Foundation under Grant GP-27232.

If in addition the coefficient $f(x_1, x_2, x_3, x_4)$ was independent of x_1, then Colton and Gilbert were able to construct an operator which mapped ordered pairs of analytic functions of three complex variables onto the space of C^2-solutions of (1.3). This last result was then used to investigate Cauchy's problem for certain classes of elliptic equations in four independent variables and hyperbolic equations in three space variables and one time variable [6], [7]. In the present paper we overcome the problem of showing that our operator maps analytic functions onto the whole space of real-valued C^2-solutions of (1.3) by carefully choosing new independent variables, reducing the question of invertibility to the problem of showing that a Goursat problem for an ultrahyperbolic equation in the space of four complex variables is well-posed, and then solving an integral equation associated with this Goursat problem. We shall furthermore give an explicit formula for constructing the analytic function associated with a given real-valued C^2-solution of (1.3) by our integral operator. A special case of this last result is a new inversion formula for the operator Re $\mathbf{G}_4$, where $\mathbf{G}_4$ is Gilbert's generalization of the Bergman–Whittaker operator [11, pp. 75–82] and "Re" denotes "take the real part." As an application of our main theorem we shall construct a complete family of solutions for (1.3) in a bounded, simply connected domain in Euclidean four-space R^4.

For the sake of brevity we only consider the special case of (1.3) when $a = b = c = d = 0$. The extension to the more general case can easily be made by combining the results of this paper with the approach used in [4] for the case of three independent variables. We furthermore assume that the coefficient $f(x_1, x_2, x_3, x_4)$ is an entire function of x_1, x_2, x_3 and x_4 (considered as complex variables), although with slight modification our results remain valid when $f(x_1, x_2, x_3, x_4)$ is only assumed to be analytic inside some polydisc in the space of four complex variables. It will also always be assumed that $f(x_1, x_2, x_3, x_4)$ is real-valued for x_1, x_2, x_3 and x_4 real. Since much of our analysis is based on the ideas of [3], it might be helpful if the reader had access to this paper.

2. The operator $\mathbf{P}_4$. In this section we consider the partial differential equation

$$\Delta_4 u + f(x_1, x_2, x_3, x_4)u = 0, \tag{2.1}$$

where $f(x_1, x_2, x_3, x_4)$ is a real-valued (for x_1, x_2, x_3, x_4 real) entire function of its independent (complex) variables. Our first result is the following theorem which is central to the analysis which follows.

THEOREM 2.1. *Let* $Y = \frac{1}{2}(x_1 + ix_2)$, $Y^* = \frac{1}{2}(x_1 - ix_2)$, $Z = \frac{1}{2}(x_3 + ix_4)$, $Z^* = -\frac{1}{2}(x_3 - ix_4)$, *and let* $u(x_1, x_2, x_3, x_4)$ *be a real-valued* C^2*-solution of* (2.1) *in a neighborhood of the origin. Then* $U(Y, Y^*, Z, Z^*) = u(x_1, x_2, x_3, x_4)$ *is an analytic function of* Y, Y^*, Z, Z^* *in some neighborhood of the origin in* $\mathbb{C}^4$, *in the space of four complex variables, and is uniquely determined by the function* $U(Y, 0, Z, Z^*)$.

Remark. Note that $Y = \overline{Y^*}$, $Z = \overline{-Z^*}$ if and only if x_1, x_2, x_3, x_4 are real.

Proof of Theorem 2.1. The fact that $U(Y, Y^*, Z, Z^*)$ is analytic follows from the fact that C^2-solutions of second order linear elliptic equations with analytic coefficients are analytic functions of their independent variables (cf. [10, p. 164]).

Hence, locally we can write

$$U(Y, Y^*, Z, Z^*) = \sum_{l,m,n,p=0}^{\infty} c_{lmnp} Y^l Y^{*m} Z^n Z^{*p}, \tag{2.2}$$

$$U(Y, 0, Z, Z^*) = \sum_{l,n,p=0}^{\infty} c_{l0np} Y^l Z^n Z^{*p}, \tag{2.3}$$

$$U(0, Y^*, Z, Z^*) = \sum_{m,n,p=0}^{\infty} c_{0mnp} Y^{*m} Z^n Z^{*p}. \tag{2.4}$$

Since $u(x_1, x_2, x_3, x_4)$ is real-valued, we have that for x_1, x_2, x_3, x_4 real,

$$U(Y, Y^*, Z, Z^*) = \overline{U(Y, Y^*, Z, Z^*)}, \tag{2.5}$$

where the bar denotes complex conjugation. This implies that for x_1, x_2, x_3, x_4 real,

$$\sum_{l,m,n,p=0}^{\infty} c_{lmnp} Y^l Y^{*m} Z^n Z^{*p} = \sum_{l,m,n,p=0}^{\infty} \overline{c_{lmnp}} Y^{*l} Y^m (-Z^*)^n (-Z)^p \tag{2.6}$$

or

$$c_{lmnp} = (-1)^{n+p} \overline{c_{mlpn}}. \tag{2.7}$$

Equations (2.3), (2.4) and (2.7) now show that $U(0, Y^*, Z, Z^*)$ is uniquely determined from $U(Y, 0, Z, Z^*)$. However in the Y, Y^*, Z, Z^* variables, (2.1) becomes an equation of ultrahyperbolic type, viz.

$$U_{YY^*} - U_{ZZ^*} + F(Y, Y^*, Z, Z^*)U = 0, \tag{2.8}$$

where

$$F(Y, Y^*, Z, Z^*) \equiv f(x_1, x_2, x_3, x_4). \tag{2.9}$$

From Hormander's generalized Cauchy–Kowalewski theorem [15, pp. 116–119], [2], we have that $U(Y, Y^*, Z, Z^*)$ is uniquely determined from the Goursat data $U(0, Y^*, Z, Z^*)$ and $U(Y, 0, Z, Z^*)$, which we have already seen are determined from $U(Y, 0, Z, Z^*)$ alone. The theorem is now proved.

We now begin to construct an integral operator which maps $U(Y, 0, Z, Z^*)$ onto $U(Y, Y^*, Z, Z^*)$. We first introduce the following notation:

$$\begin{aligned} \xi_1 &= \eta^{-1}\zeta^{-1}Y^*, \\ \xi_2 &= \eta^{-1}\zeta^{-1}Y^* + \eta^{-1}Z^*, \\ \xi_3 &= \eta^{-1}Z^* + Y, \\ \xi_4 &= \zeta^{-1}Z + Y, \end{aligned} \tag{2.10}$$

$$\mu = \xi_2 + \xi_4 = Y + \zeta^{-1}Z + \eta^{-1}Z^* + \eta^{-1}\zeta^{-1}Y^*, \tag{2.11}$$

where ζ, η are complex variables such that $1 - \varepsilon < |\zeta| < 1 + \varepsilon$, $1 - \varepsilon < |\eta| < 1 + \varepsilon$, $0 < \varepsilon < \frac{1}{2}$. Noting that the Jacobian of the transformation (2.10) is equal to $-(\eta\zeta)^{-2} \neq 0$, one can prove the following theorem by straightforward differentiation and integration by parts (cf. [6, Theorem 4.1]).

THEOREM 2.2. *Let D be a neighborhood of the origin in the μ-plane, $B = \{(\zeta, \eta): 1 - \varepsilon < |\zeta| < 1 + \varepsilon,\ 1 - \varepsilon < |\eta| < 1 + \varepsilon\}$, G a neighborhood of the origin in $(\xi_1, \xi_2, \xi_3, \xi_4)$-space and $T = \{t: |t| \leqq 1\}$. Let $f(\mu, \zeta, \eta)$ be an analytic function of three complex variables in the product domain $D \times B$ and $E^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) \equiv E(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ be a regular solution of the partial differential equation*

$$2\mu t(E^*_{13} + E^*_{14} + E^*_{23} - E^*_{34} + \eta\zeta F^* E^*) + (1 - t^2)E^*_{1t} - \frac{1}{t}E^*_1 = 0 \tag{2.12}$$

in $G \times B \times T$, where $F^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \equiv F(Y, Y^*, Z, Z^*)$, and

$$E^*_i = \frac{\partial E^*}{\partial \xi_i}, \quad E^*_{ij} = \frac{\partial^2 E^*}{\partial \xi_i \partial \xi_j}, \quad E^*_{1t} = \frac{\partial^2 E^*}{\partial \xi_i \partial t}, \qquad i, j = 1, 2, 3, 4.$$

Then,

$$U(Y, Y^*, Z, Z^*) = \mathbf{P}_4\{f\}$$

$$= -\frac{1}{4\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1}\int_{\gamma} E(Y, Y^*, Z, Z^*, \zeta, \eta, t) f(\mu(1 - t^2), \zeta, \eta) \cdot \frac{dt}{\sqrt{1 - t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}, \tag{2.13}$$

where γ is a path in T joining $t = -1$ and $t = +1$, is a (complex-valued) solution of (2.1) *which is regular in a neighborhood of the origin in (Y, Y^*, Z, Z^*)-space.*

We must now show that the integral operator $\mathbf{P}_4$ exists; that is, we must show the existence of a function $E(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ satisfying the conditions of Theorem 2.2.

THEOREM 2.3. *Let $D_r = \{(\xi_1, \xi_2, \xi_3, \xi_4): |\xi_i| < r, i = 1, 2, 3, 4\}$, where r is an arbitrary positive number, and let $B_{2\varepsilon} = \{(\zeta, \eta): |\zeta - \zeta_0| < 2\varepsilon, |\eta - \eta_0| < 2\varepsilon\}, 0 < \varepsilon < \frac{1}{2}$, where ζ_0, η_0 are arbitrary with $|\zeta_0| = |\eta_0| = 1$. Then, for each n, $n = 0, 1, 2, \cdots$, there exists a unique function $p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ which is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$ (the bar denoting closure) and which satisfies*

$$p^{(n+1)}_1 = -\frac{1}{2n + 1}\{2p^{(n)}_{13} + 2p^{(n)}_{14} + 2p^{(n)}_{13} - 2p^{(n)}_{34} + \eta\zeta F^* p^{(n)}\}, \tag{2.14}$$

$$\begin{aligned} &p^{(0)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) = 1, \\ &p^{(n+1)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta) = 0, \qquad n = 0, 1, 2, \cdots, \end{aligned} \tag{2.15}$$

where

$$p^{(n)}_i = \frac{\partial p^{(n)}}{\partial \xi_i}, \qquad p^{(n)}_{ij} = \frac{\partial^2 p^{(n)}}{\partial \xi_i \partial \xi_j}, \qquad i, j = 1, 2, 3, 4.$$

The function

$$E^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1 + \sum_{n=1}^{\infty} t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \tag{2.16}$$

is a solution of (2.12) *which is regular in the product domain $G_R \times B \times T$, where R*

is an arbitrary positive number, and

$$G_R = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi| < R, i = 1, 2, 3, 4\},$$

$$(2.17) \qquad B = \{(\zeta, \eta) : 1 - \varepsilon < |\zeta| < 1 + \varepsilon, 1 - \varepsilon < |\eta| < 1 + \varepsilon\}, \; 0 < \varepsilon < \tfrac{1}{2},$$

$$T = \{t : |t| \leqq 1\}.$$

The function defined in (2.16) *satisfies*

$$(2.18) \qquad E^*(0, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1.$$

Proof. It is easily verified from (2.14) and (2.15) that $p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ exists, is uniquely determined and is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$ for $n = 0, 1, 2, \cdots$. Straightforward differentiation and collection of terms shows that the series (2.16) formally satisfies (2.12). It remains to be shown that this series converges absolutely and uniformly in $G_R \times B \times T$. To this end, note that since $\bar{B}$ is a compact subset of the (ζ, η)-space, there are finitely many points (ζ_j, η_j) with $|\zeta_j| = |\eta_j| = 1$, $j = 1, 2, \cdots, N$, such that B is covered by the union of sets

$$(2.19) \qquad N_j = \{(\zeta, \eta) : |\zeta - \zeta_j| < \tfrac{3}{2}\varepsilon, |n - \eta_j| < \tfrac{3}{2}\varepsilon\}, \quad j = 1, 2, \cdots, N.$$

Hence it is sufficient to show that the series converges absolutely and uniformly in $\bar{G}_R \times \bar{N}_j \times T$. To this end we majorize the $p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ in $D_R \times B_{2\varepsilon}$. Since $F(Y, Y^*, Z, Z^*)$ is an entire function, it follows that $F^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ is regular in $\bar{D}_r \times \bar{B}_{2\varepsilon}$, and hence we have

$$(2.20) \qquad \begin{aligned} F^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \ll C\left(1 - \frac{\xi_1}{r}\right)^{-1}\left(1 - \frac{\xi_2}{r}\right)^{-1}\left(1 - \frac{\xi_3}{r}\right)^{-1} \\ \cdot \left(1 - \frac{\xi_4}{r}\right)^{-1}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-1}\left(1 - \frac{\eta - \eta_0}{2\varepsilon}\right)^{-1} \end{aligned}$$

for some $C > 0$ and $(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$ in $D_r \times B_{2\varepsilon}$. In (2.20) the symbol "$\ll$" means "is dominated by." The use of dominants is a standard tool in the analytic theory of partial differential equations, and the reader unfamiliar with their use is referred to [1] or [11] for further details. From (2.14), (2.15) and (2.10) it is a somewhat lengthy but straightforward procedure to show by induction that in $D_r \times B_{2\varepsilon}$ we have

$$(2.21) \qquad \begin{aligned} p_1^{(n)} \ll M(8 + \delta)^n(2n - 1)\left(1 - \frac{\xi_1}{r}\right)^{-(2n-1)}\left(1 - \frac{\xi_2}{r}\right)^{-(2n-1)}\left(1 - \frac{\xi_3}{r}\right)^{-(2n-1)} \\ \cdot \left(1 - \frac{\xi_4}{r}\right)^{-(2n-1)}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-n}\left(1 - \frac{\eta - \eta_0}{2\varepsilon}\right)^{-n} r^{-n}, \end{aligned}$$

where M and δ are positive constants independent of n. (For details of the proof of closely related results the reader is referred to [3], [6] and [18].) Equation (2.21) now implies (after some slight manipulation) that in $D_r \times B_{2\varepsilon}$ we have

$$(2.22) \qquad \begin{aligned} p^{(n)} \ll M(8 + \delta)^n(2n)^{-1}(2n - 1)^{-1}\left(1 - \frac{\xi_1}{r}\right)^{-2n}\left(1 - \frac{\xi_2}{r}\right)^{-(2n-1)} \\ \cdot \left(1 - \frac{\xi_3}{r}\right)^{-(2n-1)}\left(1 - \frac{\xi_4}{r}\right)^{-(2n-1)}\left(1 - \frac{\zeta - \zeta_0}{2\varepsilon}\right)^{-n}\left(1 - \frac{\eta - \eta_0}{2\varepsilon}\right)^{-n} r^{-n} \end{aligned}$$

which implies that in $\bar{D}_r \times \bar{N}_j$ we have

$$
\begin{aligned}
|p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)| \leqq M(8 + \delta)^n (2n)^{-1}(2n - 1)^{-1}\left(1 - \frac{|\xi_1|}{r}\right)^{-2n} \\
\cdot \left(1 - \frac{|\xi_2|}{r}\right)^{-(2n-1)}\left(1 - \frac{|\xi_3|}{r}\right)^{-(2n-1)} \\
\cdot \left(1 - \frac{|\xi_4|}{r}\right)^{-(2n-1)}\left(1 - \frac{|\zeta - \zeta_j|}{2\varepsilon}\right)^{-n} \\
\cdot \left(1 - \frac{|\eta - \eta_j|}{2\varepsilon}\right)^{-n} r^{-n}.
\end{aligned} \tag{2.23}
$$

Now consider $|t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)|$ in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$, where

$$D_{\alpha r} = \{(\xi_1, \xi_2, \xi_3, \xi_4) : |\xi_i| < r/\alpha ; \alpha > 1, i = 1, 2, 3, 4\}.$$

In $\bar{D}_{\alpha r} \times \bar{N}_j \times T$ we have

$$
\begin{aligned}
&1 - \frac{|\xi_i|}{r} \geqq \frac{\alpha - 1}{\alpha}, \qquad i = 1, 2, 3, 4, \\
&1 - \frac{|\zeta - \zeta_j|}{2\varepsilon} \geqq \frac{1}{4}, \qquad 1 - \frac{|\eta - \eta_j|}{2\varepsilon} \geqq \frac{1}{4}, \\
&|\mu| = |\xi_2 + \xi_4| \leqq \frac{2r}{\alpha}, \qquad |t| \leqq 1.
\end{aligned} \tag{2.24}
$$

Thus, from (2.23) and (2.24) we have that in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$,

$$
\begin{aligned}
|t^{2n}\mu^n p^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)| \leqq Mr(\alpha - 1)^3(2n - 1)^{-1}(2n)^{-1}\alpha^{-3} \\
\cdot (32\alpha^7(8 + \delta)(\alpha - 1)^{-8})^n.
\end{aligned} \tag{2.25}
$$

If we choose α such that

$$32\alpha^7(8 + \delta)(\alpha - 1)^{-8} < 1, \tag{2.26}$$

then the series (2.16) converges absolutely and uniformly in $\bar{D}_{\alpha r} \times \bar{N}_j \times T$. By taking $r = \alpha R$ we can now conclude that $E^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t)$ is regular in $\bar{G}_R \times \bar{N}_j \times T$ for each $j = 1, 2, \cdots, N$, and hence in $G_R \times B \times T$. Equation (2.18) follows from (2.15).

We now want to show that *every* real-valued C^2-solution $u(x_1, x_2, x_3, x_4)$ of (2.1) which is defined in some neighborhood of the origin in R^4 can be represented locally in the form

$$u(x_1, x_2, x_3, x_4) = \operatorname{Re} \mathbf{P}_4\{f\}. \tag{2.27}$$

We shall furthermore show that the associated analytic function $f(\mu, \zeta, \eta)$ has a simple representation in terms of the Goursat data $U(Y, 0, Z, Z^*)$ for $U(Y, Y^*, Z, Z^*) \equiv u(x_1, x_2, x_3, x_4)$. These results will then enable us to construct a complete family of solutions (in the L^∞-norm) for (2.1).

THEOREM 2.4. *Let $u(x_1, x_2, x_3, x_4)$ be a real-valued C^2-solution of* (2.1) *in some neighborhood of the origin in R^4. Then there exists an analytic function of three complex variables $f(\mu, \zeta, \eta)$ which is regular for μ in some neighborhood of the origin and $|\zeta| < 1 + \varepsilon, |\eta| < 1 + \varepsilon, \varepsilon > 0$, such that locally $u(x_1, x_2, x_3, x_4) = \operatorname{Re} \mathbf{P}_4\{f\}$. In particular, denote by $U(Y, Y^*, Z, Z^*) \equiv u(x_1, x_2, x_3, x_4)$ the extension of $u(x_1, x_2, x_3, x_4)$ to the (Y, Y^*, Z, Z^*)-space, and let*

$$g(\mu, \zeta, \eta) = \frac{\partial^2}{\partial \mu^2}\left\{\int_0^1 \int_0^1 \mu^2(1 - t)[2U(\mu t, 0, \xi(1 - t)\mu\zeta, (1 - t)(1 - \xi)\mu\eta) - U(0, 0, \xi(1 - t)\mu\zeta, (1 - t)(1 - \xi)\mu\eta)]\, dt\, d\xi\right\}. \tag{2.28}$$

Then,

$$f(\mu, \zeta, \eta) = -\frac{1}{2\pi}\int_{\gamma'} g(\mu(1 - t^2)\zeta, \eta)\frac{dt}{t^2}, \tag{2.29}$$

where γ' is a rectifiable arc joining the points $t = -1$ and $t = +1$ and not passing through the origin.

Remark. Equation (2.29) can be inverted by the formula (cf. [11, p. 114])

$$g(\mu, \zeta, \eta) = \int_{\gamma} f(\mu(1 - t^2), \zeta, \eta)\frac{dt}{\sqrt{1 - t^2}}, \tag{2.30}$$

where the path γ is defined in Theorem 2.2.

Proof of Theorem 2.4. The fact that $u(x_1, x_2, x_3, x_4)$ is a strong solution of (2.1) implies that $u(x_1, x_2, x_3, x_4)$ is an analytic function of its independent variables in some neighborhood of the origin. Furthermore, since $F(Y, Y^*, Z, Z^*)$ is real-valued (for $Y = \overline{Y^*}, Z = -\overline{Z^*}$), $\operatorname{Re} \mathbf{P}_4\{f\}$ is a real-valued solution of (2.1) for any function $f(\mu, \zeta, \eta)$ which is analytic in the product domain $D \times B$ (see Theorem 2.2). Now suppose that locally $g(\mu, \zeta, \eta)$, $f(\mu, \zeta, \eta)$ and $F(Y, Y^*, Z, Z^*)$ have the expansions

$$\begin{aligned}
g(\mu, \zeta, \eta) &= \sum_{n=0}^{\infty} \sum_{\substack{k=0 \\ k+l \leq n}}^{n} \sum_{l=0}^{n} a_{nkl}\mu^n\eta^k\zeta^l, \\
f(\mu, \zeta, \eta) &= -\frac{1}{2\pi}\int_{\gamma'} g(\mu(1 - t^2), \zeta, \eta)\frac{dt}{t^2} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{k=0 \\ k+l \leq n}}^{n} \sum_{l=0}^{n} a_{nkl}\frac{\Gamma(n + 1)}{\Gamma(n + \frac{1}{2})\Gamma(\frac{1}{2})}\mu^n\eta^k\zeta^l \\
F(Y, Y^*, Z, Z^*) &= \sum_{l,m,n,p=0}^{\infty} b_{lmnp}Y^l Y^{*m} Z^n Z^{*p},
\end{aligned} \tag{2.31}$$

and let the analytic functions $\bar{g}(\mu, \zeta, \eta)$, $\bar{f}(\mu, \zeta, \eta)$, $\bar{F}(Y, Y^*, Z, Z^*)$ be defined by replacing a_{nkl} and b_{lmnp} by $\overline{a_{nkl}}$ and $\overline{b_{lmnp}}$, respectively, in (2.31). Let $\bar{E}(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ be the generating function corresponding to the differential equation

$U_{YY^*} - U_{ZZ^*} + \bar{F}(Y, Y^*, Z, Z^*)U = 0$. Then for x_1, x_2, x_3, x_4 real we can write

$$\operatorname{Re} \mathbf{P}_4\{f\} = -\frac{1}{8\pi^2} \int_{|\zeta|=1} \int_{|\eta|=1} \int_\gamma E(Y, Y^*, Z, Z^*, \zeta, \eta, t) \cdot f(\mu(1-t^2), \zeta, \eta) \frac{dt}{\sqrt{1-t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta}$$
$$-\frac{1}{8\pi^2} \int_{|\zeta|=1} \int_{|\eta|=1} \int_\gamma \bar{E}(Y^*, Y, -Z^*, -Z, \zeta, \eta, t) \cdot \bar{f}(\bar{\mu}(1-t^2), \zeta, \eta) \frac{dt}{\sqrt{1-t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta}, \tag{2.32}$$

where $\bar{\mu} = Y^* - \zeta^{-1}Z^* - \eta^{-1}Z + \eta^{-1}\zeta^{-1}Y$. Now from Theorem 2.1 we know that $U(Y, Y^*, Z, Z^*)$ is uniquely determined by the function $U(Y, 0, Z, Z^*)$, and hence, using (2.18) and (2.32) we try to determine $f(\mu, \zeta, \eta)$ from the equation

$$U(Y, 0, Z, Z^*) = -\frac{1}{8\pi^2} \int_{|\zeta|=1} \int_{|\eta|=1} \int_\gamma f(\mu_1(1-t^2)\zeta, \eta) \frac{dt}{\sqrt{1-t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta}$$
$$-\frac{1}{8\pi^2} \int_{|\zeta|=1} \int_{|\eta|=1} \int_\gamma \bar{E}(0, Y, -Z^*, -Z, \zeta, \eta) \cdot \bar{f}(\mu_2(1-t^2), \zeta, \eta) \frac{dt}{\sqrt{1-t^2}} \frac{d\eta}{\eta} \frac{d\zeta}{\zeta}, \tag{2.33}$$

where $\mu_1 = Y + \zeta^{-1}Z + \eta^{-1}Z^*$ and $\mu_2 = \eta^{-1}\zeta^{-1}Y - \zeta^{-1}Z^* - \eta^{-1}Z$. To this end we first write $\bar{E}^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) \equiv \bar{E}(Y, Y^*, Z, Z^*, \zeta, \eta, t)$ in its series expansion

$$\bar{E}^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta, t) = 1 + \sum_{n=1}^{\infty} t^{2n}\mu^n \bar{p}^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta), \tag{2.34}$$

where, from Theorem 2.3, we have

$$\bar{p}^{(1)} = -\eta\zeta \int_0^{\xi_1} \bar{F}^*(\xi_1', \xi_2, \xi_3, \xi_4, \zeta, \eta)\, d\xi_1', \tag{2.35}$$

$$\bar{p}_1^{(n+1)} = -\frac{1}{2n+1}\left\{2\bar{p}_{13}^{(n)} + 2\bar{p}_{14}^{(n)} + 2\bar{p}_{23}^{(n)} - 2\bar{p}_{34}^{(n)} + \eta\zeta\bar{F}^*\bar{p}^{(n)}\right\},$$
$$\bar{p}^{(n+1)}(0, \xi_2, \xi_3, \xi_4, \zeta, \eta) = 0, \tag{2.36}$$

with $\bar{F}^*(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta) \equiv \bar{F}(Y, Y^*, Z, Z^*)$. From (2.10) and (2.35) we have

$$\bar{p}^{(1)} = -\eta\zeta \int_0^{\xi_1} \bar{F}(\xi_3 + \xi_1' - \xi_2, \eta\zeta\xi_1', \zeta(\xi_4 - \xi_3 + \xi_2 - \xi_1'), \eta(\xi_2 - \xi_1'))\, d\xi_1'$$
$$= -\int_0^{Y^*} \bar{F}(Y + \eta^{-1}\zeta^{-1}\tau - \eta^{-1}\zeta^{-1}Y^*, \tau,$$
$$\zeta(\zeta^{-1}Z - \eta^{-1}\zeta^{-1}Y^* - \eta^{-1}\zeta^{-1}\tau),$$
$$\eta(\eta^{-1}\zeta^{-1}Y^* + \eta^{-1}Z^* - \eta^{-1}\zeta^{-1}\tau))\, d\tau. \tag{2.37}$$

The fact that $\bar{F}(Y, Y^*, Z, Z^*)$ is an analytic function of Y, Y^*, Z and Z^* now implies that

$$(2.38)\quad \begin{aligned} -\frac{1}{8\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1}\int_{\gamma} & t^2\mu_2\tilde{p}^{(1)}(0, Y, -Z^*, -Z, \zeta, \eta) \\ & \cdot \bar{f}(\mu_2(1-t^2), \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} = 0, \end{aligned}$$

$$\tilde{p}^{(1)}(Y, Y^*, Z, Z^*, \zeta, \eta) \equiv \bar{p}^{(1)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$$

since the Laurent series of the integrand has no terms involving $\zeta^l\eta^m$ for both $l > -2$ and $m > -2$. A similar calculation using (2.36) and induction shows that

$$(2.39)\quad \begin{aligned} -\frac{1}{8\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1}\int_{\gamma} & t^{2n}\mu_2^n\tilde{p}^{(n)}(0, Y, -Z, Z^*, \zeta, \eta) \\ & . \bar{f}(\mu_2(1-t^2), \zeta, \eta)\frac{dt}{\sqrt{1-t^2}}\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} = 0, \end{aligned}$$

$$\tilde{p}^{(n)}(Y, Y^*, Z, Z^*, \zeta, \eta) \equiv \bar{p}^{(n)}(\xi_1, \xi_2, \xi_3, \xi_4, \zeta, \eta)$$

for $n = 1, 2, 3, \cdots$. Because of the uniform convergence of the series in (2.34), we can substitute this series into (2.33) and integrate termwise to conclude that

$$(2.40)\quad \begin{aligned} U(Y, 0, Z, Z^*) = & -\frac{1}{8\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1} g(\mu_1, \zeta, \eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta} \\ & -\frac{1}{8\pi^2}\int_{|\zeta|=1}\int_{|\eta|=1} \bar{g}(\mu_2, \zeta, \eta)\frac{d\eta}{\eta}\frac{d\zeta}{\zeta}, \end{aligned}$$

where we have made use of (2.30). To complete the proof of the theorem it now suffices to show that (2.28) gives the solution of the integral equation (2.40). In order to show this we let

$$(2.41)\quad U(Y, 0, Z, Z^*) = \sum_{n,k,l=0}^{\infty} \gamma_{nkl} Y^n Z^{*k} Z^l$$

and equate coefficients of $Y^n Z^{*k} Z^l$ on both sides of (2.40). This gives

$$(2.42)\quad \begin{aligned} 2n!k!l!\gamma_{nkl} &= (n+k+l)!a_{n+k+l,k,l}, \qquad n > 0, \\ 2k!l!\gamma_{0kl} &= (k+l)!a_{k+l,k,l} + (k+l)!(-1)^{k+l}\overline{a_{k+l,l,k}}. \end{aligned}$$

Since $U(0, 0, Z, Z^*)$ is real-valued for x_3 and x_4 real, we have from (2.41) that $\overline{\gamma_{0kl}} = (-1)^{k+l}\gamma_{0lk}$, and hence, we can assume without loss of generality that

$(-1)^{k+l}\overline{a_{k+l,l,k}} = a_{k+l,k,l}$. Equations (2.41) and (2.42) now give

$$
\begin{aligned}
U(Y,0,Z,Z^*) &= \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\Gamma(n+k+l+1)}{\Gamma(n+1)\Gamma(k+1)\Gamma(l+1)}\,a_{n+k+l,k,l}Y^nZ^{*k}Z^l \\
&\quad + \tfrac{1}{2}U(0,0,Z,Z^*) \\
&= \tfrac{1}{2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=k+l}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-k-l+1)\Gamma(k+1)\Gamma(l+1)} \\
&\quad \cdot a_{nkl}Y^{n-k-l}Z^{*k}Z^l + \tfrac{1}{2}U(0,0,Z,Z^*) \\
&= \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k+l\leq n}}^{n}\sum_{l=0}^{n}\frac{\Gamma(n+1)}{\Gamma(n-k-l+1)\Gamma(k+1)\Gamma(l+1)} \\
&\quad \cdot a_{nkl}Y^{n-k-l}Z^{*k}Z^l + \tfrac{1}{2}U(0,0,Z,Z^*).
\end{aligned} \tag{2.43}
$$

From the definition of the beta function (cf. [8, p. 9]) we can now write

$$
\begin{aligned}
&\int_0^1 (1-t)[U(tY,0,(1-t)Z,(1-t)Z^*) - \tfrac{1}{2}U(0,0,(1-t)Z,(1-t)Z^*)]\,dt \\
&\quad = \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k+l\leq n}}^{n}\sum_{l=0}^{n}\frac{\Gamma(k+l+2)}{(n+2)(n+1)\Gamma(l+1)\Gamma(k+1)}a_{nkl}Y^{n-k-l}Z^{*k}Z^l,
\end{aligned} \tag{2.44}
$$

and hence,

$$
\begin{aligned}
&\int_0^1\int_0^1 (1-t)[U(tY,0,\xi(1-t)Z,(1-t)(1-\xi)Z^*) \\
&\qquad - \tfrac{1}{2}\,U(0,0,\xi(1-t)Z,(1-t)(1-\xi)Z^*)]\,dt\,d\xi \\
&\quad = \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k+l\leq n}}^{n}\sum_{l=0}^{n}\frac{a_{nkl}}{(n+2)(n+1)}Y^{n-k-l}Z^{*k}Z^l,
\end{aligned} \tag{2.45}
$$

which implies that

$$
\begin{aligned}
&\frac{\partial^2}{\partial\mu^2}\left\{\int_0^1\int_0^1 \mu^2(1-t)[U(\mu t,0,\xi(1-t)\mu\zeta,(1-t)(1-\xi)\mu\eta)\right. \\
&\qquad \left. - \tfrac{1}{2}U(0,0,\xi(1-t)\mu\zeta,(1-t)(1-\xi)\mu\eta)]\,dt\,d\xi\right\} \\
&\quad = \tfrac{1}{2}\sum_{n=0}^{\infty}\sum_{\substack{k=0\\k+l\leq n}}^{n}\sum_{l=0}^{n}a_{nkl}\mu^n\eta^k\zeta^l = \tfrac{1}{2}g(\mu,\zeta,\eta).
\end{aligned} \tag{2.46}
$$

Equation (2.28) follows immediately from (2.46), and this proves the theorem.

Note. When $F(Y,Y^*,Z,Z^*) \equiv 0$, our operator $\mathbf{P}_4$ reduces to Gilbert's operator $\mathbf{G}_4$ (see [11, pp. 75–82]), and (2.28) gives a new inversion formula for the operator $\operatorname{Re}\mathbf{G}_4$. It is of interest to compare (2.28) with the inversion formula given by Kreyszig for complex-valued harmonic functions in four independent variables [16], [11, p. 78].

Theorems 2.2, 2.3 and 2.4 can now be used to construct a complete family of solutions in the L^∞-norm for (2.1). The proof of the following theorem exactly parallels that for the case of three independent variables, and the reader is referred to Theorem 3.2 of [3] for further details. Briefly, the proof proceeds as follows: Since (2.1) is elliptic and has analytic coefficients, it possesses the unique continuation property and hence the Runge approximation property (cf. [17]). Hence it suffices to find a complete family of (real-valued) solutions defined in an arbitrarily large sphere S in R^4. From Garabedian's work on Cauchy's problem for analytic systems [10, pp. 614–619] it is possible to conclude that the Cauchy data for solutions of (2.1) defined in S must be regular in some convex region B in $\mathbb{C}^3$, the space of three complex variables. Since convex domains are Runge domains of the first kind [9, p. 229] and solutions of (2.1) defined in S depend continuously on their Cauchy data in B, we can approximate solutions in S by solutions having polynomial Cauchy data, that is, entire solutions of (2.1). Such (real-valued) entire solutions can then be approximated by (real-valued) solutions having polynomial Goursat data in the (Y, Y^*, Z, Z^*)-space. But by Theorem 2.4, real-valued solutions $u(x_1, x_2, x_3, x_4)$ of (2.1) with polynomial Goursat data can be represented in the form

$$u(x_1, x_2, x_3, x_4) = \operatorname{Re} \mathbf{P}_4\{h_N\}, \tag{2.47}$$

where

$$h_N(\mu, \zeta, \eta) = \sum_{n=0}^{N} \sum_{\substack{k=0 \\ k+l \leqq n}}^{n} \sum_{l=0}^{n} a_{nkl}\mu^n\eta^k\zeta^l, \tag{2.48}$$

from which follows the theorem below. In the statement of the theorem "Im" denotes "take the imaginary part."

THEOREM 2.5. *Let G be a bounded, simply connected domain in R^4, and define*

$$\begin{aligned} u_{2n,k,l}(x_1, x_2, x_3, x_4) &= \operatorname{Re} \mathbf{P}_4\{\mu^n\eta^k\zeta^l\}, \\ u_{2n+1,k,l}(x_1, x_2, x_3, x_4) &= \operatorname{Im} \mathbf{P}_4\{\mu^n\eta^k\zeta^l\}, \end{aligned} \tag{2.49}$$

where $0 \leqq n < \infty$, $l = 0, 1, \cdots, n$, $k = 0, 1, \cdots, n$, $k + l \leqq n$. Then the set $\{u_{nkl}\}$ is a complete family of solutions in the L^∞-norm for (2.1) *in the space of real-valued C^2-solutions of* (2.1) *defined in G.*

REFERENCES

[1] S. BERGMAN, *Integral Operators in the Theory of Linear Partial Differential Equations*, Springer-Verlag, Berlin, 1961.

[2] L. CESARI, *Functional analysis and boundary value problems*, Analytic Theory of Differential Equations, P. F. Hsieh and A. W. J. Stoddart, eds., Springer-Verlag Lecture Note Series, vol. 183, Springer-Verlag, Berlin, 1971.

[3] D. COLTON, *Integral operators for elliptic equations in three independent variables. I*, Applicable Analysis, to appear.

[4] ———, *Integral operators for elliptic equations in three independent variables. II*, Ibid., to appear.

[5] ———, *Bergman operators for elliptic equations in three independent variables*, Bull. Amer. Math. Soc., 77 (1971), pp. 752–756.

[6] D. COLTON AND R. P. GILBERT, *An integral operator approach to Cauchy's problem for $\Delta_{p+2}u(\mathbf{x}) + F(\mathbf{x})u(\mathbf{x}) = 0$*, this Journal, 2 (1971), pp. 113–132.

[7] ———, *New results on the approximation of solutions to partial differential equations: The method of particular solutions*, Analytic Theory of Differential Equations, P. F. Hsieh and A. W. J. Stoddart, eds., Springer-Verlag Lecture Note Series, vol. 183, Springer-Verlag, Berlin, 1971.

[8] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. TRICOMI, *Higher Transcendental Functions*, vol. I, McGraw-Hill, New York, 1953.

[9] B. A. FUKS, *Special Chapters in the Theory of Analytic Functions of Several Complex Variables*, American Mathematical Society, Providence, 1965.

[10] P. R. GARABEDIAN, *Partial Differential Equations*, John Wiley, New York, 1964.

[11] R. P. GILBERT, *Function Theoretic Methods in Partial Differential Equations*, Academic Press, New York, 1969.

[12] ———, *The construction of solutions for boundary value problems by function theoretic methods*, this Journal, 1 (1970), pp. 96–114.

[13] R. P. GILBERT AND H. C. HOWARD, *On a class of elliptic partial differential equations*, Portugal. Math., 26 (1967), pp. 353–373.

[14] R. P. GILBERT AND C. Y. LO, *On the approximation of solutions of elliptic partial differential equations in two and three dimensions*, this Journal, 2 (1971), pp. 17–30.

[15] L. HORMANDER, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1964.

[16] E. KREYSZIG, *Kanonische Integraloperatoren zur Erzeugung harmonischer Funktionen von der Veranderlichen*, Arch. Math., 14 (1963), pp. 193–203.

[17] P. D. LAX, *A stability theory of abstract differential equations and its application to the study of local behavior of solutions of elliptic equations*, Comm. Pure Appl. Math., 8 (1956), pp. 747–766.

[18] B. L. TJONG, *Operators generating solutions of certain partial differential equations and their properties*, Thesis, University of Kentucky, Lexington, 1968.

[19] ———, *Operators generating solutions of* $\Delta_3\psi(x, y, z) + F(x, y, z)\psi(x, y, z) = 0$ *and their properties*, Analytic Methods in Mathematical Physics, R. P. Gilbert and R. G. Newton, eds., Gordon and Breach, New York, 1970.

[20] I. N. VEKUA, *New Methods for Solving Elliptic Equations*, John Wiley, New York, 1967.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

DUAL INTEGRAL EQUATIONS WITH TRIGONOMETRIC KERNELS AND TEMPERED DISTRIBUTIONS*

R. P. SRIVASTAV†

Abstract. Of concern here are two theorems pertaining to the questions of existence and uniqueness of solutions of the dual integral equations

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\xi|^{\alpha}\psi(\xi)\,e^{i\xi x}\,d\xi = f(x), \qquad 0 \leqq |x| < 1, \quad \alpha = \pm 1,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(\xi)\,e^{i\xi x}\,d\xi = 0, \qquad |x| > 1.$$

The integrals are interpreted as the classical Abel limits. This furnishes a computational device for the evaluation of Fourier transforms of tempered distributions. Using the theories of Fourier transforms and singular integral equations, an explicit solution is constructed, which is shown to be unique.

1. Introduction. In this paper we examine, for existence and uniqueness of solutions, dual integral equations

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\xi|^{\alpha}\psi(\xi)\,e^{i\xi x}\,d\xi = f(x), \qquad 0 \leqq |x| < 1, \tag{1.1}$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(\xi)\,e^{i\xi x}\,d\xi = 0, \qquad |x| > 1, \tag{1.2}$$

where $\alpha = \pm 1$. The integrals are to be interpreted as Abel limits given by the relation

$$\int_{-\infty}^{\infty} p(\xi)\,d\xi \equiv \lim_{c\to 0+}\int_{-\infty}^{\infty} e^{-c|\xi|}p(\xi)\,d\xi, \tag{1.3}$$

where the limit is taken in the sense of S', the space of tempered distributions. The equality in the above equations is also to be considered in the distributional sense in S'. ψ, being the Fourier transform of a tempered distribution with a compact support, must be a tempered C^{∞}-function. Hence, we consider the existence and uniqueness of the solutions in this class of functions. In notation and terminology, we follow Bremermann [1]. The object of this paper is to determine (if possible, uniquely) a tempered C^{∞}-function ψ, whose Fourier transform is a distribution derivative of a prescribed order of a continuous function with its support contained in $[-1, 1]$ and for which the Fourier transform of $|\xi|^{\alpha}\psi(\xi)$ is equal to f on $(-1, 1)$, where $\alpha = \pm 1$ and f is a distribution derivative of prescribed order of a continuous function.

* Received by the editors March 5, 1970, and in revised form September 10, 1971.

† Department of Applied Mathematics and Statistics, Division of Mathematical Sciences, State University of New York at Stony Brook, Stony Brook, Long Island, New York 11790. This research was supported in part by the Atomic Energy Commission under Contract AT(30-1)3869.

The pair of equations (1.1) and (1.2) arise in a natural way when the Fourier transform is used for the solution of mixed boundary value problems of elasticity. Formal solutions of these equations are known (see [6, pp. 96–106]). Not much, it seems, has been done to determine the conditions which will give a unique solution. A paper due to I. W. Busbridge [2] may very well be the only one to consider dual integral equations of the type (1.1) and (1.2) with any attention to mathematical rigor. The equations considered here are not included in Busbridge's discussion and our techniques are also very different. There is one feature in common between this paper and [2]. We too consider the equations on open intervals. This is so because the equality of distributions is defined for open sets. We believe that a mathematically precise formulation of the uniqueness problem, which can be used in applications, will be of interest. The classical theory of functions and the related theory of integral transforms seem to be inadequate for this purpose. For, it is known that the physical quantities represented by (1.1) and (1.2) are not in $L_2(-\infty, \infty)$ but in $L_1(-\infty, \infty)$, and their Fourier transforms in general may not be in $L_1(-\infty, \infty)$. Mathematical techniques of classical analysis required to deal with L_1-functions tend to be abstruse. In contrast, considerable simplification can be achieved by assuming that the right-hand sides of equations (1.1) and (1.2) are tempered distributions.

Section 2 contains an example which motivated this investigation. In § 3 we introduce a nonstandard definition of the Fourier transform for tempered distributions using the classical concept of Abel summability instead of Parseval's relation which is commonly used. We also show that our definition is equivalent to the standard one. The definition given here has the advantage that it facilitates computation. Section 4 is devoted to a discussion of the case $\alpha = 1$, while the case $\alpha = -1$ is the subject of § 5.

2. An example. Let us consider the problem of finding the stress distribution in an infinite isotropic elastic medium containing a Griffith crack, which is subject to an internal pressure varying along the length of the crack. It is well known [7, p. 26] that the elastic field can be expressed in terms of an auxiliary function $\psi(\xi)$, which satisfies the dual integral equations

$$\frac{d}{dx}\int_0^\infty \psi(\xi)\sin \xi x\, d\xi = f(x), \qquad 0 < x < 1, \tag{2.1}$$

$$\int_0^\infty \psi(\xi)\cos \xi x\, d\xi = 0, \qquad x > 1. \tag{2.2}$$

If the constraints of the physical problem are disregarded, it is easy to see that $\psi(\xi) = J_0(\xi)$ satisfies the homogeneous pair obtained by putting $f(x) \equiv 0$ (see Watson [9, p. 405]). Another solution of the homogeneous system can be obtained by putting

$$\psi(\xi) = \int_0^1 g(t) J_0(\xi t)\, dt,$$

where $g(t)$ is an auxiliary function. We note that (2.2) is identically satisfied and

(2.1) is found to be equivalent to the integral equation

$$\frac{d}{dx}\int_0^x \frac{g(t)\,dt}{\sqrt{(x^2 - t^2)}} = f(x), \qquad 0 < x < 1. \tag{2.3}$$

Equation (2.3) has a nontrivial solution for $f \equiv 0$, viz., $g(t) = \text{const}$. For $g(t) = 1$, we find that

$$\psi(\xi) = \int_0^1 J_0(t\xi)\,dt.$$

Moreover,

$$\int_0^\infty \psi(\xi)\cos(\xi x)\,d\xi = \begin{cases} \log\left\{\dfrac{1 + \sqrt{1 - x^2}}{x}\right\}, & 0 < x < 1, \\ 0, & x > 1, \end{cases} \tag{2.4}$$

and

$$\int_0^\infty \psi(\xi)\sin \xi x\,d\xi = \begin{cases} \pi/2, & 0 < x \leqq 1, \\ \sin^{-1}(1/x), & x \geqq 1. \end{cases} \tag{2.5}$$

Obviously, (2.4) and (2.5) satisfy the requirements of a formal solution. This shows that a more precise formulation is called for.

3. Preliminary mathematics. Let T be a tempered distribution. Then, there exists a tempered function ϕ and a positive integer k such that $\phi^{(k)} = T$ (see [1, p. 121]). We define two operators of Fourier transformation on the space S' by the relations

$$\mathscr{F}_+ T = (-ix)^k \lim_{c \to 0+} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-c|\xi|}\phi(\xi)\,e^{i\xi x}\,d\xi, \tag{3.1}$$

$$\mathscr{F}_- T = (ix)^k \lim_{c \to 0+} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-c|\xi|}\phi(\xi)\,e^{-i\xi x}\,d\xi, \tag{3.2}$$

where the limits are to be taken in S'. It may be observed that $(ix)^k$ and $(-ix)^k$ both are multipliers of S'. Since $\phi(\xi)$ is a tempered function,

$$\Phi_c(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-c|\xi|}\phi(\xi)\,e^{i\xi x}\,d\xi \tag{3.3}$$

is a tempered C^∞-function for $c > 0$. As $c \to 0$, $\Phi_c(x)$ as well as $\Phi_c(-x)$ converge to functionals in S'. Let $\theta(x)$ belong to S, the space of testing functions of rapidly decreasing C^∞-functions. Then

$$\langle \Phi_c(x), \theta(x) \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \theta(x) \int_{-\infty}^\infty e^{-c|\xi|}\phi(\xi)\,e^{i\xi x}\,d\xi.$$

Changing the order of integration, which is obviously justified, we get the equation

$$\langle \Phi_c(x), \theta(x) \rangle = \int_{-\infty}^\infty e^{-c|\xi|}\phi(\xi)\hat{\theta}(\xi)\,d\xi,$$

where

$$\hat{\theta}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} \theta(x)\, dx = \mathscr{F}_+ \theta$$

and $\hat{\theta}$ is also a rapidly decreasing C^∞-function. Hence

$$\lim_{c \to 0} \langle \Phi_c(x), \theta(x) \rangle = \langle \phi, \hat{\theta} \rangle. \tag{3.4}$$

The Fourier transform of ϕ, whether it is a tempered function or a tempered distribution, is usually defined by the functional relation

$$\langle \mathscr{F}_+ \phi, \theta \rangle = \langle \phi, \mathscr{F}_+ \theta \rangle. \tag{3.5}$$

For tempered functions our definition is easily seen to be equivalent from the preceding computation. For tempered distributions the equivalence follows from Theorem 2 of [1, p. 88]. We observe that for every testing function θ in S,

$$\langle \mathscr{F}_+ T, \theta \rangle = \langle T, \mathscr{F}_+ \theta \rangle, \tag{3.6a}$$

$$\langle \mathscr{F}_- T, \theta \rangle = \langle T, \mathscr{F}_- \theta \rangle, \tag{3.6b}$$

$$\langle \mathscr{F}_+ \mathscr{F}_- T, \theta \rangle = \langle \mathscr{F}_- T, \mathscr{F}_+ \theta \rangle = \langle T, \mathscr{F}_- \mathscr{F}_+ \theta \rangle = \langle T, \theta \rangle. \tag{3.7}$$

Likewise,

$$\langle \mathscr{F}_- \mathscr{F}_+ T, \theta \rangle = \langle T, \theta \rangle.$$

Hence

$$\mathscr{F}_+ \mathscr{F}_- T = \mathscr{F}_- \mathscr{F}_+ T = T. \tag{3.8}$$

For $T = \phi^{(k)}$,

$$\begin{aligned} \langle \mathscr{F}_+ \phi^{(k)}, \theta \rangle &= \langle \phi^{(k)}, \mathscr{F}_+ \theta \rangle \\ &= (-1)^k \langle \phi(x), \mathscr{F}_+ (i\xi)^k \theta(\xi) \rangle. \end{aligned}$$

Hence

$$\langle \mathscr{F}_+ \phi^{(k)}, \theta \rangle = (-1)^k \langle (i\xi)^k \mathscr{F}_+ \phi, \theta \rangle. \tag{3.9}$$

Similarly,

$$\begin{aligned} \langle \mathscr{F}_- \phi^{(k)}, \theta \rangle &= \langle \phi^{(k)}, \mathscr{F}_- \theta \rangle \\ &= (-1)^k \langle \phi, \mathscr{F}_- (-i\xi)^k \theta(\xi) \rangle, \end{aligned}$$

in other words,

$$\langle \mathscr{F}_- \phi^{(k)}, \theta \rangle = \langle (i\xi)^k \mathscr{F}_- \phi, \theta \rangle. \tag{3.10}$$

In general, $|\xi|$ is not a multiplier for S', but if $\psi(\xi)$ is a tempered C^∞-function, so is $|\xi|\psi(\xi)$ and it belongs to S'.

4. Dual integral equations of a crack problem. For $\alpha = 1$, we have the following theorem.

THEOREM 1. *Let $f(x)$ be continuous on $(-1, 1)$ and right and left continuous respectively at -1 and 1. Then, there is a unique tempered C^∞-function ψ possessing* *the following properties*:

(i) *The Fourier transform of* ψ *is a continuous function whose support is contained in* $[-1, 1]$.

(ii) *The Fourier transform of* $|\xi|\psi(\xi)$ *is a tempered distribution which is equal to* $f(x)$ *on* $(-1, 1)$, *and there exists a tempered function whose distribution derivative is equal to this tempered distribution.*

Proof. We prove this theorem by explicitly constructing a solution, which we show to be unique. Let

$$\mathscr{F}_+\psi = q(x), \qquad -1 \leqq x \leqq 1, \tag{4.1}$$

where $q(x)$ is a continuous function such that $q(-1) = q(1) = 0$, a condition which is required by the continuity of the Fourier transform of ψ. We set $q(x) = 0$ for $|x| \geqq 1$. Obviously,

$$\psi = \mathscr{F}_- q = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} q(x)\, e^{-i\xi x}\, dx. \tag{4.2}$$

Further,

$$\begin{aligned} \mathscr{F}_+\{|\xi|\psi(\xi)\} &= \lim_{c \to 0+} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-c|\xi|}\, e^{i\xi x}|\xi|\, d\xi \int_{-1}^{1} q(y)\, e^{-iy\xi}\, dy \\ &= \lim_{c \to 0+} \frac{1}{2\pi} \int_{-1}^{1} q(y)\, dy \int_{-\infty}^{\infty} e^{-c|\xi|}\, e^{i\xi(x-y)}|\xi|\, d\xi. \end{aligned}$$

However,

$$\int_{-\infty}^{\infty} e^{-c|\xi|}\, e^{i\xi(x-y)}|\xi|\, d\xi = 2\frac{d}{dx}\left\{\frac{x-y}{c^2 + (x-y)^2}\right\}. \tag{4.3}$$

Hence

$$\mathscr{F}_+\{|\xi|\psi(\xi)\} = \lim_{c \to 0+} \frac{1}{\pi} \frac{d}{dx} \int_{-1}^{1} \frac{q(y)(x-y)}{c^2 + (x-y)^2}\, dy, \tag{4.4}$$

where the limit, we may recall, is to be taken in the sense of S'. Equivalently, for every testing function θ belonging to S,

$$\langle \mathscr{F}_+\{|\xi|\psi(\xi)\}, \theta(x)\rangle = -\left\langle \frac{1}{\pi} \int_{-1}^{1} \frac{q(y)\, dy}{x-y}, \theta'(x) \right\rangle, \tag{4.5}$$

where the prime denotes differentiation and the singular integrals here and henceforth are to be interpreted as Cauchy principal values. Since on $(-1, 1)$,

$$\mathscr{F}_+\{|\xi|\psi(\xi)\} = f, \tag{4.6}$$

for every testing function θ whose support is contained in $(-1, 1)$, we have

$$-\left\langle \frac{1}{\pi} \int_{-1}^{1} \frac{q(y)\, dy}{x-y}, \theta'(x) \right\rangle = \langle f, \theta \rangle. \tag{4.7}$$

Let

$$F(x) = \int_{-1}^{x} f(t)\, dt, \qquad -1 \leqq x \leqq 1.$$

$F(x)$ is absolutely continuous and satisfies a Lipschitz condition on $[-1, 1]$. In fact, it is differentiable on $(-1, 1)$ with right and left derivatives at -1 and 1 respectively. Obviously,

$$\langle f, \theta \rangle = \langle F', \theta \rangle = -\langle F, \theta' \rangle .$$

Equation (4.7) may therefore be rewritten as

$$\left\langle \frac{1}{\pi} \int_{-1}^{1} \frac{q(y)\,dy}{x-y} - F, \theta' \right\rangle = 0. \tag{4.8}$$

Thus (4.7) is equivalent to the singular integral equation

$$\frac{1}{\pi} \int_{-1}^{1} \frac{q(y)\,dy}{x-y} = F(x) + C_0, \qquad -1 \leqq x \leqq 1, \tag{4.9}$$

where C_0 is an arbitrary constant. The solution of this integral equation is well known (see, for example, Tricomi [8, pp. 173–179]). All solutions of (4.9) which belong to L_p for $1 < p < 2$ are given by

$$q(y) = \frac{1}{\pi} \int_{-1}^{1} \sqrt{\left(\frac{1-x^2}{1-y^2}\right)} \frac{F(x) + C_0}{x-y} dx + \frac{k}{\sqrt{(1-y^2)}}, \tag{4.10}$$

where k is an arbitrary constant. Since $F(x)$ satisfies a Lipschitz condition, by the Plemelj–Privalov theorem (see Muskhelishvili [4] or Pogorzelski [5]), the function

$$q_0(y) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{(1-x^2)}[F(x) + C_0]}{x-y} dx$$

is Hölder continuous. Hence $q(y)$ is continuous at $y = \pm 1$ if and only if

$$k = \frac{1}{\pi} \int_{-1}^{1} \frac{xF(x)\,dx}{\sqrt{(1-x^2)}} = \frac{1}{\pi} \int_{-1}^{1} f(t)\sqrt{(1-t^2)}\,dt \tag{4.11}$$

and

$$C_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{F(x)\,dx}{\sqrt{(1-x^2)}} = \frac{1}{\pi} \int_{-1}^{1} f(t) \cos^{-1}(t)\,dt. \tag{4.12}$$

Substituting these values in (4.10) we find that

$$q(y) = \frac{1}{\pi}\sqrt{(1-y^2)} \int_{-1}^{1} \frac{F(x)\,dx}{(x-y)\sqrt{(1-x^2)}} \tag{4.13}$$

is the only continuous solution of (4.9) and hence (4.2) and (4.13) give us the unique tempered C^∞-function with the desired properties.

In particular, if $f(x) = 1$, then $F(x) = x + 1$, and $q(y) = 1 - y^2$. Hence

$$\psi(\xi) = \sqrt{\frac{\pi}{2}} \frac{J_1(\xi)}{\xi}$$

and $\mathscr{F}_+\{|\xi|\psi(\xi)\}$ in the sense of D' is equal to 1 on $(-1, 1)$ and to $|x|(x^2 - 1)^{-1/2} - 1$ for $|x| > 1$. The above analysis serves to show that in the case of constant internal pressure, the normal component of the stress tensor across the line of crack has a square root singularity.

We believe that the conditions imposed on $f(x)$ in the theorem proved above are unduly restrictive and that the result will hold for a larger class of functions. However, these conditions are generally satisfied by functions encountered in applications and our goal in this paper is to strive for simplicity rather than generality.

5. Dual integral equations for $\alpha = -1$. A theory analogous to the one of the previous section can be developed for the pair of equations

$$\mathscr{F}_+\{|\xi|^{-1}\psi(\xi)\} = f(x), \qquad |x| < 1, \tag{5.1}$$

$$\mathscr{F}_+\{\psi(\xi)\} = 0, \qquad |x| > 1. \tag{5.2}$$

We have the following theorem.

THEOREM 2. *Let* $f(x)$ *be Hölder continuous of order* $\alpha > 1/2$ *on* $(-1, 1)$ *and right and left continuous at* -1 *and* 1 *respectively. Further, suppose that*

$$\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{1 - x^2}} = 0. \tag{5.3}$$

Then, there exists a uniquely determined tempered C^∞*-function with the following properties*:

(i) *The Fourier transform of* $|\xi|^{-1}\psi(\xi)$ *is a continuous function which is equal to* $f(x)$ *for* $|x| \leqq 1$.

(ii) *The Fourier transform of* $\psi(\xi)$ *is a tempered distribution which is the distribution derivative of a tempered function whose support is contained in* $[-1, 1]$.

Remark. The condition (5.3) was noted by Lowengrub [3, p. 71] and Sneddon [6, p. 101] when $f(x)$ is known to be an even function. In physical problems, this condition is merely a restatement of some physical law. For example, in the case of an appropriate indentation problem it is an equilibrium condition.

Proof. As in the case of Theorem 1, we construct a solution whose uniqueness follows from the theory of singular integral equations. Let

$$\mathscr{F}_+\{\psi(\xi)\} = \frac{d}{dx}q(x),$$

where $q(x)$ vanishes outside $[-1, 1]$. q is to be determined so that $\psi(\xi)$ will satisfy the equations (5.1) and (5.2). From (3.8) it follows that

$$\psi(\xi) = \mathscr{F}_-\left\{\frac{dq}{dx}\right\}, \tag{5.4}$$

and using (3.10) we get the relation

$$\psi(\xi) = \frac{i\xi}{\sqrt{2\pi}} \int_{-1}^{1} q(y)\, e^{-iy\xi}\, dy. \tag{5.5}$$

On substituting this expression for $\psi(\xi)$ in (5.1) and computing the Fourier transform, we obtain the equation

$$-\frac{1}{\pi} \int_{-1}^{1} \frac{q(y)\,dy}{x - y} = f(x), \qquad -1 < x < 1. \tag{5.6}$$

All solutions of (5.6) belonging to $L_p(-1, 1)$ for $1 < p < 2$ are given by

$$q(y) = -\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x^2}{1-y^2}}\frac{f(x)\,dx}{x-y} + \frac{k}{\sqrt{1-y^2}}, \qquad -1 < y < 1, \tag{5.7}$$

where k is an arbitrary constant. In contrast with the previous case, the constant available now is one shy and by choosing it appropriately we can only ensure continuity on the left or on the right. Condition (5.3), however, enables us to get a continuous solution. We may put

$$k = -\frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}f(x)\,dx. \tag{5.8}$$

This ensures continuity at $y = 1$. For q to be continuous at -1 it is necessary and sufficient that

$$k = \frac{1}{\pi}\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}f(x)\,dx, \tag{5.9}$$

which is true if $f(x)$ satisfies (5.3). After some elementary computation we find that

$$q(y) = \begin{cases} -\dfrac{1}{\pi}\sqrt{1-y^2}\displaystyle\int_{-1}^{1}\frac{[f(x)-f(y)]\,dx}{(x-y)\sqrt{1-x^2}}, & |y| \leqq 1, \\ 0, & |y| \geqq 1. \end{cases} \tag{5.10}$$

Equation (5.5) in conjunction with (5.7) gives a solution of (5.1) and (5.2).

This solution is unique; for, if there were more than one solution in the class of admissible functions, then the homogeneous pair obtained by putting $f = 0$ would have a nontrivial solution, which could be expressed by the relation (5.5). It is clear from (5.7) that the only possible solutions belonging to $L_p(-1, 1)$ tend to infinity as y tends to ± 1, unless of course the constant k is chosen to be zero.

We wish to add that the theory presented here can be extended in a straightforward way to deal with the pair of dual integral equations

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\xi|^{\alpha}(1 + \lambda e^{-2\xi\delta})\psi(\xi)\,e^{i\xi x}\,d\xi = f(x), \qquad -1 < x < 1,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(\xi)\,e^{i\xi x}\,d\xi = 0, \qquad |x| > 1,$$

where λ is any complex number and δ is a positive real number.

REFERENCES

[1] H. BREMERMANN, *Distributions, Complex Variables, and Fourier Transforms*, Addison-Wesley, Reading, Mass., 1965.

[2] I. W. BUSBRIDGE, *Dual integral equations*, Proc. London Math. Soc., 44 (1938), pp. 115–129.

[3] M. LOWENGRUB, *Some dual trigonometric equations with an application to elasticity*, Internat. J. Engrg. Sci., 4 (1966), pp. 69–79.

[4] N. I. MUSKHELISHVILI, *Singular Integral Equations*, P. Noordhoff, Groningen, 1953.

[5] W. POGORZELSKI, *Integral Equations and their Applications*, Pergamon Press, New York, 1966.

[6] I. N. SNEDDON, *Mixed Boundary Value Problems in Potential Theory*, North-Holland, Amsterdam, 1966.

[7] I. N. SNEDDON AND M. LOWENGRUB, *Crack Problems in the Mathematical Theory of Elasticity*, John Wiley, New York, 1969.

[8] F. G. TRICOMI, *Integral Equations*, Interscience, New York, 1957.

[9] G. N. WATSON, *Theory of Bessel Functions*, Cambridge University Press, London, 1962.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

HYPERGEOMETRIC INTEGRAL EQUATIONS OF A GENERAL KIND AND FRACTIONAL INTEGRATION*

TILAK RAJ PRABHAKAR†

Abstract. Some integral equations with the kernels containing a confluent hypergeometric function in two variables are studied by the use of fractional integration. These equations include as special cases the hypergeometric integral equations as well as the confluent hypergeometric integral equations discussed by earlier authors.

1. Introduction. In this note we consider the integral equations

$$\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1\left(\alpha, \beta, \gamma; \lambda(x-t), 1 - \frac{x}{t}\right) f(t)\,dt \overset{\circ}{=} g(x), \tag{1.1}$$

$$\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1\left(\alpha, \beta, \gamma; \lambda(x-t), 1 - \frac{t}{x}\right) f(t)\,dt \overset{\circ}{=} g(x), \tag{1.2}$$

where Φ_1 is a confluent hypergeometric function in two variables and $\overset{\circ}{=}$ means "equals almost everywhere." For $\lambda = 0$, these equations reduce to hypergeometric integral equations studied in the case $a = 0$ by Love [2] and for $\beta = 0$ to confluent hypergeometric integral equations discussed by the author [3], [4]. We show that (1.1) and (1.2) can be discussed by a systematic use of fractional integration in the same way as in [2]–[4]. Section 3 is the central part of the paper and theorems on solutions of (1.1) and (1.2) follow as easy consequences thereof.

2. Preliminaries. As usual, the function $\Phi_1(\alpha, \beta, \gamma; x, y)$ is defined by the double series [1, 5.7.1 (20)]

$$\Phi_1(\alpha, \beta, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_n x^m y^n}{(\gamma)_{m+n} m!\,n!} \tag{2.1}$$

for $|y| < 1$ and by its analytic continuation elsewhere. For $\operatorname{Re}\gamma > \operatorname{Re}\alpha > 0$, the function has the integral representation

$$\Phi_1(\alpha, \beta, \gamma; x, y) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}(1-uy)^{-\beta} e^{ux}\,du \tag{2.2}$$

which, on putting $u = 1 - t$, yields the transformation

$$\Phi_1(\alpha, \beta, \gamma; x, y) = e^x(1-y)^{-\beta}\Phi_1\left(\gamma-\alpha, \beta, \gamma; -x, \frac{-y}{1-y}\right). \tag{2.3}$$

The fractional integration operator. We denote by L the linear space of (equivalence classes) of complex-valued functions f which are Lebesgue integrable on $[a, b]$, with the norm $\|f\| = \int_a^b |f(x)|\,dx$. Throughout it is assumed that $0 < a < b < \infty$. For complex μ with $\operatorname{Re}\mu > 0$, the linear operator $I^\mu: L \to L$ is

* Received by the editors June 2, 1970, and in final revised form October 13, 1971.
† Department of Mathematics, Ramjas College, University of Delhi, Delhi-7, India.

defined by

$$(I^\mu f)(x) = \int_a^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)} f(t)\,dt \tag{2.4}$$

for almost all $x \in [a, b]$. Standard properties of I^μ including its definition when $\operatorname{Re}\mu \leqq 0$ are well known ([2]–[4]).

The operator $A(\alpha, \beta, \gamma, \lambda)$. If α, β, γ and λ are complex numbers with $\operatorname{Re}\gamma > 0$, then setting

$$(A(\alpha, \beta, \gamma, \lambda)f)(x) = \int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1\left(\alpha, \beta, \gamma; \lambda(x-t), 1 - \frac{x}{t}\right) f(t)\,dt$$

for almost all $x \in [a, b]$, it is easily verified that $A(\alpha, \beta, \gamma, \lambda)$ is a bounded linear operator on L into itself.

3. The main result. To prove Theorem 1, which may be regarded as the main result, we require the following lemma.

LEMMA 1. *If* $\operatorname{Re}\mu > 0$ *and* $\operatorname{Re}\gamma > 0$, *then for* $0 < t < x$,

$$\begin{aligned}\int_t^x \frac{(x-v)^{\mu-1}}{\Gamma(\mu)} \frac{(v-t)^{\gamma-1}}{\Gamma(\gamma)} \Phi_1\left(\alpha, \beta, \gamma; \lambda(v-t), 1 - \frac{v}{t}\right) dv \\ = \frac{(x-t)^{\gamma+\mu-1}}{\Gamma(\gamma+\mu)} \Phi_1\left(\alpha, \beta, \gamma+\mu; \lambda(x-t), 1 - \frac{x}{t}\right).\end{aligned} \tag{3.1}$$

Proof. If $\operatorname{Re}\mu > 0$, $\operatorname{Re}\gamma > 0$, z_1 and z_2 are complex, the z_2-plane supposed cut along $z_2 \geqq 1$, then using (2.2) it is easily seen that

$$\begin{aligned}\int_0^1 \frac{(1-s)^{\mu-1}}{\Gamma(\mu)} \frac{s^{\gamma-1}}{\Gamma(\gamma)} \Phi_1(\alpha, \beta, \gamma; z_1 s, z_2 s)\,ds \\ = \frac{1}{\Gamma(\gamma+\mu)} \Phi_1(\alpha, \beta, \gamma+\mu; z_1, z_2),\end{aligned} \tag{3.2}$$

which on putting $s = (v-t)/(x-t)$, $z_1 = \lambda(x-t)$, $z_2 = -(x-t)/t$ gives (3.1).

THEOREM 1. *If* $\operatorname{Re}\gamma > 0$ *and* $\operatorname{Re}(\gamma+\mu) > 0$, *then operating on L, we have*

$$I^\mu A(\alpha, \beta, \gamma, \lambda) = A(\alpha, \beta, \gamma+\mu, \lambda). \tag{3.3}$$

Proof. (i) Suppose that $\operatorname{Re}\mu > 0$. If $f \in L$, then $A(\alpha, \beta, \gamma, \lambda)f$ and therefore $I^\mu A(\alpha, \beta, \gamma, \lambda)f$, also, exists in L so that for almost all $x \in [a, b]$,

$$\begin{aligned}I^\mu A(\alpha, \beta, \gamma, \lambda)f(x) = \int_a^x \frac{(x-v)^{\mu-1}}{\Gamma(\mu)} dv \int_a^v \frac{(v-t)^{\gamma-1}}{\Gamma(\gamma)} \\ \cdot \Phi_1\left(\alpha, \beta, \gamma; \lambda(v-t), 1 - \frac{v}{t}\right) f(t)\,dt.\end{aligned} \tag{3.4}$$

By an application of Fubini's theorem to reverse the order of integration and using (3.1), we find that the repeated integral (3.4) a.e. in $[a, b]$ equals

$$\int_a^x \frac{(x-t)^{\gamma+\mu-1}}{\Gamma(\gamma+\mu)} \Phi_1\left(\alpha, \beta, \gamma+\mu; \lambda(x-t), 1 - \frac{x}{t}\right) f(t)\,dt,$$

which gives (3.3) in the case $\operatorname{Re}\mu > 0$.

(ii) Suppose that $\operatorname{Re}\mu < 0$. For $\operatorname{Re}(\gamma + \mu) > 0$ and $f \in L$, we have, by case (i),

$$I^{-\mu}A(\alpha, \beta, \gamma + \mu, \lambda)f = A(\alpha, \beta, \gamma, \lambda)f,$$

which leads to (3.3) in this case also.

(iii) Suppose finally that $\operatorname{Re}\mu = 0$. By (i),

$$I^{1+\mu}A(\alpha, \beta, \gamma, \lambda)f = A(\alpha, \beta, \gamma + \mu + 1, \lambda)f$$

and also

$$I^{1}A(\alpha, \beta, \gamma + \mu, \lambda)f = A(\alpha, \beta, \gamma + \mu + 1, \lambda)f.$$

Equating the left members we have

$$A(\alpha, \beta, \gamma + \mu, \lambda)f = I^{-1}I^{1+\mu}A(\alpha, \beta, \gamma, \lambda)f,$$

in which the right side, by the definition [3, (2.4)] of I^{μ} for $\operatorname{Re}\mu = 0$, is $I^{\mu}A(\alpha, \beta, \gamma, \lambda)f$.

THEOREM 2. *If* $\operatorname{Re}\gamma > 0$, $\operatorname{Re}\alpha > 0$ *and* $f \in L$, *then for almost all* x *in* $[a, b]$,

$$A(\alpha, \beta, \gamma, \lambda)f(x) = I^{\gamma-\alpha}x^{-\beta}\, e^{\lambda x}I^{\alpha}x^{\beta}\, e^{-\lambda x}f(x). \tag{3.5}$$

Proof. By Theorem 1, applied twice,

$$I^{\alpha}A(\alpha, \beta, \gamma, \lambda)f = I^{\gamma}A(\alpha, \beta, \alpha, \lambda)f$$

which, as can be easily verified, leads to

$$A(\alpha, \beta, \gamma, \lambda)f = I^{\gamma-\alpha}A(\alpha, \beta, \alpha, \lambda)f. \tag{3.6}$$

Using the reduction formula

$$\Phi_1(\alpha, \beta, \alpha; x, y) = e^{x}(1 - y)^{-\beta},$$

we find that

$$A(\alpha, \beta, \alpha, \lambda)f(x) = x^{-\beta}\, e^{\lambda x}I^{\alpha}x^{\beta}\, e^{-\lambda x}f(x),$$

which combines with (3.6) to give (3.5).

COROLLARY. *If the integral equation*

$$\int_a^x \frac{(x - t)^{\gamma-1}}{\Gamma(\gamma)}\, \Phi_1\left(\alpha, \beta, \gamma; \lambda(x - t), 1 - \frac{x}{t}\right) f(t)\, dt = g(x), \tag{3.7}$$

for $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\gamma > 0$ *and* $a \leqq x \leqq b$, *has a solution* $f \in L$, *then the solution is unique.*

Proof. From Theorem 2 together with the theorem on the uniqueness of fractional integrals, it easily follows that

$$A(\alpha, \beta, \gamma, \lambda)f = 0 \quad \text{implies that } f = 0.$$

4. The integral equations. We first consider two lemmas.

LEMMA 2. *If* $\operatorname{Re}\alpha > 0$, *then the integral equation*

$$I^{\alpha}\, e^{-\lambda x}f(x) = e^{-\lambda x}I^{\alpha}\phi(x), \qquad a \leqq x \leqq b, \tag{4.1}$$

has, for each $f \in L$, *a solution* ϕ *in* L, *and conversely for each* $\phi \in L$, *a solution* f *in* L.

Proof. This proceeds on the same lines as in the particular cases $\lambda = \pm 1$ [3, Thms. 7 and 8].

LEMMA 3. *If* $\operatorname{Re}\alpha > 0$, *then the equation*

$$I^{\alpha}x^{\beta}f(x) \overset{\circ}{=} x^{\beta}I^{\alpha}\phi(x), \qquad a \leqq x \leqq b, \tag{4.2}$$

has, for each $f \in L$, *a solution* ϕ *in* L, *and conversely.*

Proof. This is a simple adaptation of [2, Thm. 5].

The following results can be easily derived from Theorem 2 and Lemmas 2 and 3.

THEOREM 3. *If* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\gamma > 0$, *then the integral equation*

$$\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1\left(\alpha,\beta,\gamma;\lambda(x-t),1-\frac{x}{t}\right)f(t)\,dt \overset{\circ}{=} g(x), \qquad 0 < a \leqq x \leqq b, \tag{4.3}$$

has a solution f *in* L *if and only if* $I^{-\gamma}g$ *exists and* $I^{-\gamma}g \in L$.

Also, if $I^{-\gamma}g$ exists, then the solution $f \in L$ is given by

$$f(x) = e^{\lambda x}x^{-\beta}I^{-\alpha}\,e^{-\lambda x}x^{\beta}I^{\alpha-\gamma}g(x). \tag{4.4}$$

THEOREM 4. *If* $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\gamma > 0$, *then the integral equation*

$$\int_a^x \frac{(x-t)^{\gamma-1}}{\Gamma(\gamma)}\Phi_1\left(\alpha,\beta,\gamma;\lambda(x-t),1-\frac{t}{x}\right)f(t)\,dt \overset{\circ}{=} g(x), \qquad a \leqq x \leqq b, \tag{4.5}$$

possesses a solution f *in* L *if and only if* $I^{-\gamma}g$ *exists and belongs to* L; *also such a solution is given by*

$$f(x) = I^{\alpha-\gamma}\,e^{\lambda x}x^{\beta}I^{-\alpha}\,e^{-\lambda x}x^{-\beta}g(x).$$

Proof. Using (2.3), we can convert (4.5) into the form of (4.3) which is solved by Theorem 3.

REFERENCES

[1] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vol. I, McGraw-Hill, New York, 1953.

[2] E. R. LOVE, *Some integral equations involving hypergeometric functions*, Proc. Edinburgh Math. Soc., 15 (1967), pp. 169–198.

[3] TILAK RAJ PRABHAKAR, *Two singular integral equations involving confluent hypergeometric functions*, Proc. Cambridge Philos. Soc., 66 (1969), pp. 71–89.

[4] ———, *Some integral equations with Kummer's functions in the kernels*, Canad. Math. Bull., 14 (1971), pp. 391–404.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

A PROOF OF A THEOREM BY DOLEZAL*

REUVEN MEIDAN†

Abstract. Recently, Dolezal [1] stated a theorem: *Suppose* $\{\varphi_n\}$ *is a sequence in* $\mathscr{D}$ *converging to zero with respect to the weak topology generated on* $\mathscr{D}$ *by the subspace of the distributions of finite order in* $\mathscr{D}'$. *Then,* $\{\varphi_n\} \to 0$ *for the initial topology of* $\mathscr{D}$. In this paper a proof of the theorem based on the theory of topological vector spaces is presented.

Recently, Dolezal [1] stated the following theorem.

THEOREM. *Let* $\{\varphi_n\}$ *be a sequence in* $\mathscr{D}$ *which converges to zero with respect to the weak topology generated on* $\mathscr{D}$ *by the subspace of the distributions of finite order in* $\mathscr{D}'$. *Then* $\{\varphi_n\} \to 0$ *with respect to the initial topology on* $\mathscr{D}$.

The proof given by Dolezal is based on elementary considerations. It is the purpose of this note to propose a different proof based on the theory of topological vector spaces.

Notation. Let $\mathscr{D}$ denote the space of infinitely differentiable testing functions on R, and $\mathscr{D}'$ the space of distributions on R'. $\{\varphi_n\}$ is, by assumption, a sequence of functions in $\mathscr{D}$ which converges to zero with respect to the weak topology generated on $\mathscr{D}$ by the distributions of finite order in $\mathscr{D}'$. Namely, $\{\langle f, \varphi_n\rangle\} \to 0$ for every finite order distribution f, where $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathscr{D}$. Let m be a nonnegative integer, $m \geqq 0$. $\mathscr{D}^m$ denotes the space of m-times continuously differentiable functions on R^n with compact support. K is a compact subset of R^n. $\mathscr{D}^m(K)$ denotes the subset of $\mathscr{D}^m$ consisting of functions whose supports are contained in K. Similarly, $\mathscr{D}(K)$ is used to denote the space of infinitely differentiable functions with supports contained in K.

Proof of the theorem. In order to establish convergence in $\mathscr{D}$ with respect to the initial topology we first verify that all φ_n have their supports contained in some compact subset K of R^n. Let m be a nonnegative integer, $m \geqq 0$, and consider the space $\mathscr{D}^m$. The φ_n are members of $\mathscr{D}^m$. Since the dual of $\mathscr{D}^m$ consists of distributions of the order m, hence of finite order, $\{\varphi_n\} \to 0$ weakly in $\mathscr{D}^m$. Consequently, $\{\varphi_n\}$ is weakly bounded in $\mathscr{D}^m$. By Mackey's theorem (Treves [2, p. 371]) it follows that $\{\varphi_n\}$ is also bounded with respect to the initial topology of $\mathscr{D}^m$. But this implies (Treves [2, p. 139]) that all φ_n are contained in some $\mathscr{D}^m(K)$, where K is a compact subset of R^n. Consequently, the supports of all φ_n are contained in K.

To complete the proof we show that $\{\varphi_n\}$ converges to zero in $\mathscr{D}(K)$. Indeed, the dual of $\mathscr{D}(K)$ is contained in the subspace of finite order distributions in $\mathscr{D}'$ (Treves [2, p. 258]). Hence, $\{\varphi_n\} \to 0$ weakly in $\mathscr{D}(K)$. Again, by Mackey's theorem, $\{\varphi_n\}$ is bounded in $\mathscr{D}(K)$ for the initial topology. We use the fact that $\mathscr{D}(K)$ is a Montel space and that, on bounded sets of Montel spaces, the initial and the weak topologies coincide (Treves [2, p. 376]). Consequently $\{\varphi_n\} \to 0$ for the initial topologies of $\mathscr{D}(K)$, which completes the proof.

It is instructive to point out that the complication of the proof is due to the fact that the convergence of $\{\varphi_n\}$ in $\mathscr{D}$ is assumed to hold only for the weak topology

* Received by the editors April 19, 1971, and in revised form September 13, 1971.

† Haifa, Israel.

generated by *finite order* distributions. If, for instance, the stronger assumption of weak convergence with respect to *all* distributions were assumed, then the proof would be simpler and would only consist of the second part of the above.

REFERENCES

[1] V. DOLEZAL, *A representation of linear continuous operators on testing functions and distributions*, this Journal, 1 (1970), pp. 491–506.

[2] F. TREVES, *Topological Vectors Spaces, Distributions and Kernels*, Academic Press, New York, 1967.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

UNIFORM ASYMPTOTIC STABILITY OF EVOLUTIONARY PROCESSES IN A BANACH SPACE*

RICHARD DATKO†

Abstract. This paper contains two major results. The first one is to obtain necessary and sufficient conditions for the uniform asymptotic stability of linear evolutionary processes which are defined in a general Banach space and whose norms can increase no faster than an exponential. This is the substance of Theorem 1 and its corollaries. The second is to extend the bounded-input, bounded-output criteria of O. Perron and R. Bellman to evolutionary processes in a Banach space. This is done in Theorems 6 through 8.

In the special case of Hilbert spaces it is shown that uniform asymptotic stability of the evolutionary processes considered is equivalent to the existence of positive Hermitian bilinear functionals whose derivatives along trajectories give rise to negative bilinear Hermitian functionals. This is the analogue of the usual Lyapunov theory for linear differential equations defined in a finite Euclidean space.

Introduction. The purpose of this paper is to extend certain results on the uniform asymptotic stability of homogeneous linear differential equations in R^n to a class of evolutionary processes in a general Banach space. This extension has been partially accomplished in [3] and [4] for the case where the process either eventually became a compact semi-group of operators on a Banach space or where the process was a semi-group of operators on a Hilbert space. Thus, this paper is in a sense a sequel to [3] and [4].

By the term evolutionary process, as it is used in this paper, is meant a two-parameter family of endomorphisms $S(t, s)$, with $0 \leqq s \leqq t < \infty$, defined on a Banach space X which has the transitive property $S(t, s)S(s, t_0) = S(t, t_0)$ if $0 \leqq t_0 \leqq s \leqq t$. Thus $S(t, s)$ is the infinite-dimensional analogue of a fundamental matrix associated with the solutions of a linear differential equation in R^n (see, e.g., [5, p. 286]). Examples of the type of evolutionary process considered in this paper are those generated by solutions of linear differential-difference equations, solutions of some classes of parabolic partial differential equations and any semi-group of operators of class C_0 defined on a Banach space. Existence, uniqueness and other properties of evolutionary process have been extensively investigated (see, e.g., [14], [16] and [18]). However, to the best of our knowledge their stability properties in the sense of Lyapunov have not been systematically investigated in an abstract setting, although processes associated with differential-difference equations have been well studied (see, e.g., [12], [13] and [15]). This paper is an attempt to develop a general theory for the uniform asymptotic stability of a large class of evolutionary processes.

The paper consists of five sections and an Appendix. Section 1 introduces some notation and the basic definitions which will be used throughout the paper. Section 2 contains the main results of the paper. In this section, necessary and sufficient conditions are found for the uniform asymptotic stability of a general class of evolutionary processes defined on a Banach space. The main restrictions

* Received by the editors April 6, 1971, and in revised form July 29, 1971.

† Department of Mathematics, Georgetown University, Washington, D.C. 20007. This work was supported in part by the National Science Foundation under Grant GP-20143.

on these processes are that their norms can increase no faster than an exponential function. The main theorem in this section is Theorem 1. Following Theorem 1 are several corollaries which give alternate criteria for uniform asymptotic stability. Theorem 2 gives a sufficient condition for asymptotic stability. Example 1 shows that in general, asymptotic stability does not imply uniform asymptotic stability even for semi-groups of class C_0. However, it is proved in Theorem 3 that if for some positive t_0 a semi-group of class C_0 is compact, then asymptotic stability implies uniform asymptotic stability.

Section 3 considers the special case of evolutionary operators defined on a Hilbert space. In this case, the results of § 2 can be phrased in terms of positive Hermitian mappings of the space into itself. The stability criteria then become the exact analogues of the well-known Lyapunov theory for finite-dimensional spaces (see, e.g., [5] or [6]). An application of the results of this section can be found in [11] (the first corollary to Theorem 4).

Section 4 is a perturbation result which indicates how the usual stability theory of ordinary differential equations can be extended to evolutionary processes in a Banach space. Section 5 extends a bounded-input, bounded-output criteria of O. Perron for determining the uniform asymptotic stability of a linear system of differential equations in R^n to evolutionary processes in a Banach space. The extension of the Perron condition is in the spirit of [1] and not in the direction taken in the work of Massera and Schäffer [17]. An interesting problem would be to attempt to extend their work to the types of processes considered in this paper.

1. Notation, conventions and basic definitions. X will denote a real or complex Banach space with a norm $|\cdot|$. The zero vector in X will be denoted by 0. H will denote a real or complex Hilbert space with an inner product $(\cdot, \cdot)$. $\mathscr{L}(X, X)$ will stand for the space of continuous linear mappings from X into itself. In the case of a Hilbert space, a mapping $B \in \mathscr{L}(H, H)$ will be called positive Hermitian and denoted by $B > 0$ if $B = B^*$ and $(Bx, x) > 0$ for $x \neq 0$. The symbol Δ will denote the set defined by

$$\Delta = \{(t, t_0): 0 \leqq t_0 \leqq t < \infty\}. \tag{1}$$

DEFINITION 1. A family of mappings $\{S(t, t_0)\} \subset \mathscr{L}(X, X)$ with $(t, t_0) \in \Delta$ will be called a *strongly continuous evolutionary process with exponential growth* if for all (s, t_0) and (t, s) in Δ and x in X:

(2)
(a) $$S(t, s)S(s, t_0)x = S(t, t_0)x;$$
(b) there exist constants $M_1 \geqq 1$ and $\omega > 0$ such that

$$|S(t, t_0)| \leqq M_1 e^{\omega(t - t_0)}; \tag{3}$$

(4)
(c) $S(\cdot, t_0)$ is strongly continuous for $t \geqq t_0$; and
(d) $$\lim_{t \to t_0^+} S(t, t_0)x = x$$
for all x in X.

A family of mappings satisfying (a) through (d) will be called an evolutionary process of class $C(0, e)$. Thus, in this paper, statements of the form, "Let $S(t, t_0)$ be an evolutionary process of class $C(0, e) \cdots$," will be met with frequently. Unless otherwise explicitly mentioned, an evolutionary process of class $C(0, e)$ will always be denoted by $S(t, t_0)$.

DEFINITION 2. If the evolutionary process $S(t, t_0)$ satisfies (a), (c) and (d) of Definition 1, and in addition

$$S(t, t_0)x = S(t - t_0, 0)x \tag{5}$$

for all $(t, t_0) \in \Delta$ and $x \in X$, then it is called a *semi-group of class* C_0. For semi-groups of class C_0 condition (b) is automatically satisfied (see, e.g., [7]).

Remark 1. If $S(t, t_0)$ is a semi-group of class C_0, then there exists a dense set $D \subset X$ and a linear operator $A: D \to X$ such that if $x \in D$,

$$\lim_{t \to 0^+} \frac{S(t, 0)x}{t} = Ax. \tag{6}$$

The mapping A is in general unbounded and is called the infinitesimal generator of the semi-group.

DEFINITION 3. An evolutionary process of class $C(0, e)$ will be called *uniformly stable* if there exists an $M \geqq 1$ such that $|S(t, t_0)x| \leqq M|x|$ for all $(t, t_0) \in \Delta$ and $x \in X$.

DEFINITION 4. An evolutionary process of class $C(0, e)$ will be called *asymptotically stable* if $\lim_{t \to \infty} S(t, t_0)x = 0$ for all $x \in X$.

DEFINITION 5. An evolutionary process of class $C(0, e)$ will be called *uniformly asymptotically stable* if given any $\varepsilon > 0$ there exists a $T(\varepsilon) > 0$ such that $|S(t, t_0)| < \varepsilon$ wherever $t \geqq T(\varepsilon) + t_0$.

DEFINITION 6. An evolutionary process of class $C(0, e)$ will be called *uniformly exponentially stable* if there exist constants $M \geqq 1$ and $\alpha > 0$ such that $|S(t, t_0)| \leqq Me^{-\alpha(t - t_0)}$ for all $(t, t_0) \in \Delta$.

Remark 2. It is easy to see for evolutionary processes of class $C(0, e)$ that uniform exponential stability implies uniform asymptotic stability which implies asymptotic stability.

In the sequel, it will be shown that uniform asymptotic stability implies uniform exponential stability, a well-known result for X a finite-dimensional space. However, it will also be shown by means of an example that in the case of semi-groups of class C_0, asymptotic stability does not in general imply uniform asymptotic stability, a property which is true if X is finite-dimensional.

2. Some stability properties of evolutionary processes.

LEMMA 1. *Let $S(t, t_0)$ be an evolutionary process of class $C(0, e)$. Then, if $S(t, t_0)$ is uniformly asymptotically stable, it is uniformly exponentially stable.*

Proof. Let $S(t, t_0)$ be uniformly asymptotically stable. Choose $\varepsilon_0 = 1/2$. Then there exists $T(\varepsilon_0) > 0$ such that $|S(t, t_0)| \leqq \varepsilon_0$ for all $t \geqq T(\varepsilon_0) + t_0$. Hence if $(t, t_0) \in \Delta$, then

$$t - t_0 = nT(\varepsilon_0) + \tau,$$

where n is a natural number and $0 \leqq \tau \leqq T(\varepsilon_0)$. By properties (a) and (b) of Definition 1,

$$
\begin{aligned}
|S(t, t_0)| &= S[nT(\varepsilon_0) + t_0 + \tau, nT(\varepsilon_0) + t_0] \prod_{k=0}^{n-1} S[(n-k)T(\varepsilon_0) + t_0, \\
&\qquad\qquad (n-k-1)T(\varepsilon_0) + t_0] \\
&\leqq M_1 e^{\omega\tau}(1/2)^n = M_1 \exp[\omega t] \exp[-n \ln 2] \\
&= M_1 \exp\left[\omega\tau - \frac{nT(\varepsilon_0)\ln 2}{T(\varepsilon_0)}\right] \\
&= M_1 \exp\left[\omega\tau - \frac{nT(\varepsilon_0) + \tau}{T(\varepsilon_0)} + \frac{\tau}{T(\varepsilon_0)}\right] \\
&= M_1 \exp[\omega\tau + \tau/T(\varepsilon_0)] \exp[-(t - t_0)/T(\varepsilon_0)] \\
&\leqq M_1 \exp[\omega T(\varepsilon_0) + 1] \exp[-(t - t_0)/T(\varepsilon_0)].
\end{aligned}
$$

Hence if $M = M_1 \exp[\omega T(\varepsilon_0) + 1]$ and $-\alpha = -1/T(\varepsilon_0)$, then $|S(t, t_0)| \leqq Me^{-\alpha(t-t_0)}$. This completes the proof of the lemma.

THEOREM 1. *An evolutionary process $S(t, t_0)$ of class $C(0, e)$ will be uniformly exponentially stable if and only if for each $x \in X$, there exists a finite constant $M(x)$, depending only on x, such that for all $t_0 \geqq 0$,*

$$\int_{t_0}^{\infty} |S(t, t_0)x|^2 \, dt \leqq M(x).$$

Proof. Proof of necessity. Assume $|S(t, t_0)| \leqq Me^{-\alpha(t-t_0)}$ fo r all $(t, t_0) \in \Delta$, where $M \geqq 1$ and $\alpha > 0$. Then if $x \in X$ and $(t, t_0) \in \Delta$,

$$\int_{t_0}^{t} |S(s, t_0)x|^2 \, ds \leqq M^2 \int_{t_0}^{t} e^{-2\alpha(s-t_0)}|x|^2 \, ds \leqq \frac{M^2}{2\alpha}|x|^2 = M(x).$$

This completes the proof of necessity.

Proof of sufficiency. We first prove that there exists a finite positive constant M_2 such that for all $x \in X$ and $t_0 \geqq 0$,

$$\int_{t_0}^{\infty} |S(t, t_0)x|^2 \, dt \leqq M_2|x|^2. \tag{7}$$

To see this let us define for all $(t, t_0) \in \Delta$ the family of semi-norms $\{\phi(t, t_0)\}$ given by

$$\phi(t, t_0)(x) = \left(\int_{t_0}^{t} |S(s, t_0)x|^2 \, ds\right)^{1/2}. \tag{8}$$

Since for each $x \in X$ and pair $(t, t_0) \in \Delta$, $\phi(t, t_0)(x) \leqq M(x)$, it follows from the principle of uniform boundedness (see, e.g., [10, p. 68]) that $\lim_{|x| \to 0} \phi(t, t_0)(x) = 0$ uniformly in X; hence, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for all $(t, t_0) \in \Delta$,

$$\phi(t, t_0)(x) \leqq \varepsilon \quad \text{if } |x| < \delta(\varepsilon).$$

Since $\phi(t, t_0)(\cdot)$ is a semi-norm for each $(t, t_0) \in \Delta$, this is equivalent to the relation

$$\phi(t, t_0)(x) \leqq \varepsilon|x|/\delta(\varepsilon)$$

for all $x \in X$. Thus if we set $M_2 = \varepsilon^2/\delta^2(\varepsilon)$ it follows that (7) holds for all $x \in X$.

We shall next prove the existence of a finite constant M_3 such that for all $(t_0, t) \in \Delta$ and $x \in X$,

$$(9) \qquad |S(t, t_0)x| \leqq M_3|x|.$$

The proof of (9) is by means of contradiction. Thus, assume there exists a sequence $\{(\bar{t}_n, t_n)\} \subset \Delta$ and a sequence $\{x_n\} \subset X$ with $|x_n| = 1$ for each n such that the inequalities

$$(10) \qquad M_1 n \leqq |S(\bar{t}_n, t_n)x_n| \leqq M_1\, e^{\omega(\bar{t}_n - t_n)}$$

are satisfied.

Equations (2) and (3) imply that for n fixed and $s \in [t_n, \bar{t}_n]$,

$$M_1 n \leqq |S(\bar{t}_n, s)S(s, t_n)x_n| \leqq M_1\, e^{\omega(\bar{t}_n - s)}|S(s, t_n)x_n|.$$

Hence,

$$(11) \qquad n\, e^{-\omega(\bar{t}_n - s)} \leqq |S(s, t_n)x_n|$$

for all $s \in [t_n, \bar{t}_n]$. Squaring both sides of (11), integrating between t_n and $\bar{t}_n$ and applying inequality (7), we obtain the inequality

$$(12) \qquad \frac{n^2}{2\omega}[1 - e^{-2\omega(\bar{t}_n - t_n)}] \leqq M_2.$$

First we observe that (10) guarantees that $\bar{t}_n - t_n \geqq (1/\omega)\ln n$ for each n. Hence, there exists n_0 such that if $n \geqq n_0$, the term in the brackets on the left-hand side of (12) is greater than 1/2. Thus, for $n \geqq n_0$,

$$(13) \qquad \frac{n^2}{2\omega}\frac{1}{2} \leqq M_2.$$

But this is impossible. This contradiction establishes the inequality (9).

We now prove that for any $t_0 \geqq 0$, $x \in X$ and $\varepsilon > 0$, where $\sqrt{M_2}\varepsilon < 1$, there exists some point $\bar{t}$ in the interval $[t_0, t_0 + 1/\varepsilon^2]$ for which

$$(14) \qquad |S(\bar{t}, t_0)x| = \sqrt{M_2}\varepsilon|x|.$$

Since $S(\cdot, t_0)$ is strongly continuous of class $C(0, e)$ on the interval $[t_0, \infty)$ and (7) has been shown to hold, we can find for each $x \neq 0$ in X a first time $T(x, t_0, \varepsilon) > 0$ such that

$$|S(t_0 + T(x, t_0, \varepsilon), t_0)x| = \sqrt{M_2}\varepsilon|x|$$

and

$$\sqrt{M_2}\varepsilon|x| < |S(t, t_0)x|$$

for

$$t \in [t_0, T(x, t_0, \varepsilon) + t_0).$$

Thus, because of (7), the inequality

$$M_2\varepsilon^2 T(x, t_0, \varepsilon) \leqq \int_{t_0}^{T(x,t_0,\varepsilon)+t_0} |S(t, t_0)x|^2\, dt \leqq M_2|x|^2$$

applies. This means that

$$T(x, t_0, \varepsilon) \leqq 1/\varepsilon^2. \tag{15}$$

Hence, $T(x, t_0, \varepsilon)$ depends only on ε. This establishes (14).

Inequality (9) and equality (14) imply that if $t_0 \geqq 0$ and $x \in X$, then

$$|S(t, t_0)x| \leqq M_3\sqrt{M_2\varepsilon}|x| \tag{16}$$

if $0 \leqq \sqrt{M_2\varepsilon} < 1$ and $t \geqq t_0 + 1/\varepsilon^2$. Thus, (16) shows that $S(t, t_0)$ is uniformly asymptotically stable, and by Lemma 1, this implies that $S(t, t_0)$ is uniformly exponentially stable.

Remark 3. Theorem 1 remains valid if we replace the condition

$$\int_{t_0}^{\infty} |S(t, t_0)x|^2 \, dt \leqq M(x) \tag{17}$$

by

$$\int_{t_0}^{\infty} |S(t, t_0)x|^p \, dt \leqq M(x), \tag{18}$$

where $1 \leqq p < \infty$. The proof of necessity is obvious. The proof of sufficiency is almost verbatim that given in Theorem 1. The reason for not establishing Theorem 1 for condition (18) is that condition (17) is a natural generalization of known results when X is finite-dimensional and to avoid unnecessary computation in proving Theorem 1.

If X is a finite m-dimensional vector space over the real numbers, the evolutionary processes we have been discussing are most often solutions to ordinary differential equations of the form $\dot{x}(t) = A(t)x(t)$, where $A(t)$ is a continuous $m \times m$ matrix. For such systems there exist necessary and sufficient conditions for uniform asymptotic stability which are stated in terms of Lyapunov functions (see, e.g., [5] or [6]). The following corollaries to Theorem 1 extend these results to evolutionary processes of class $C(0, e)$ defined on infinite-dimensional Banach spaces.

COROLLARY 1. *Let $S(t, t_0)$ be an evolutionary process of class $C(0, e)$ which is uniformly asymptotically stable. Then there exists a unique continuous mapping $V: X \times [0, \infty) \to [0, \infty)$ such that for each x in X and $t_0 \geqq 0$:*

(i) *the mapping*

$$t \to V(S(t, t_0)x, t) = \tilde{V}(t, x, t_0) \tag{19}$$

from $[t_0, \infty) \to [0, \infty)$ has the property that $\lim_{t\to\infty} \tilde{V}(t, x, t_0) = 0$;

(ii)

$$d\tilde{V}(t)/dt = -|S(t, t_0)x|^2; \tag{20}$$

(iii) *there exists a finite positive constant M_4 such that for all $x \in X$ and $t \in [0, \infty)$ the inequality*

$$V(x, t) \leqq M_4|x|^2 \tag{21}$$

is satisfied.

Conversely, if such a mapping V exists satisfying (19), (20) *and* (21), *then the process is uniformly asymptotically stable.*

Proof. If $S(t, t_0)$ is uniformly asymptotically stable, then it is exponentially stable by Lemma 1, and hence the mapping V defined by

$$V(x, t) = \int_t^\infty |S(s, t)x|^2 \, dt \tag{22}$$

is well-defined and by (7) satisfies an inequality of the form (21). Moreover, since $S(t, t_0)$ is exponentially stable and satisfies (21), it follows that

$$V(S(t, t_0)x, t_0) \leqq M_4|S(t, t_0)x|^2 \leqq M_4 M^2 e^{-2\alpha(t-t_0)}|x|. \tag{23}$$

Hence $\tilde{V}(t, x, t_0) \to 0$ as $t \to \infty$.

From (22) and (2) we see that if $t \geqq t_0$,

$$V(S(t, t_0)x, t) = \int_t^\infty |S(s, t)S(t, t_0)x|^2 \, ds = \int_t^\infty |S(s, t_0)x|^2 \, ds, \tag{24}$$

and hence,

$$d\tilde{V}(t)/dt = -|S(t, t_0)x|^2,$$

which establishes (20).

Uniqueness is a consequence of the equality

$$\begin{aligned} \tilde{V}(t, x, t_0) &= V(x, t_0) + \int_{t_0}^t \frac{d\tilde{V}(s, x, t_0)\, ds}{ds} \\ &= \tilde{V}(t_0, x, t_0) - \int_{t_0}^t |S(s, t_0)x|^2 \, ds \end{aligned} \tag{25}$$

and the fact that $V(t, x, t_0) \to 0$ as $t \to \infty$.

The converse is due to (21) and (25). For, from (25) and (21),

$$\int_{t_0}^t |S(s, t_0)x|^2 ds = \tilde{V}(t_0, x, t_0) = V(x, t_0) \leqq M_4|x|^2, \tag{26}$$

and by Theorem 1, (26) is sufficient for uniform exponential stability.

COROLLARY 2. *A necessary and sufficient condition that an evolutionary process of class $C(0, e)$ be uniformly asymptotically stable is the existence of a mapping $V: X \times [0, \infty) \to [0, \infty)$ such that $V(0, t) \equiv 0$ for all $t \in [0, \infty)$, $V(x, t) \leqq M_4|x|^2$ for some finite constant M_4 and all $x \in X$ and that*

$$\frac{dV}{dt}(S(t, t_0)x, t) \leqq -|S(t, t_0)x|^2 \tag{27}$$

for all x in X.

Proof. If $S(t, t_0)$ is uniformly asymptotically stable, then by Corollary 1, a V satisfying the hypotheses of the corollary exists and is given by (22).

If, on the other hand, a V satisfying the hypotheses of the corollary exists, then for each $x \in X$ and $t_0 \geqq 0$,

$$0 \leqq V(S(t, t_0)x, t) = V(x, t) + \int_{t_0}^{t} \frac{dV}{dt}(S(s, t_0)x, s)\, ds$$

$$\leqq V(x, t) - \int_{t_0}^{t} |S(s, t_0)x|^2\, ds \leqq M_4|x|^2 - \int_{t_0}^{t} |S(s, t_0)x|^2\, ds.$$

Hence,

$$\int_{t_0}^{\infty} |S(s, t_0)x|^2\, ds \leqq M_4|x|^2. \tag{28}$$

By Theorem 1, condition (28) implies uniform asymptotic stability.

The following corollary is a minor weakening of Theorem 1 for semi-groups of class C_0.

COROLLARY 3. *If $S(t, t_0)$ is a semi-group of class C_0 and $\int_0^\infty |S(t, 0)x|^2\, dt < \infty$ for all x in X, then $S(t, t_0)$ is uniformly asymptotically stable.*

Proof. The proof is a consequence of (5), since if $x \in X$ and $t_0 \geqq 0$,

$$\int_{t_0}^{\infty} |S(t, t_0)x|^2\, dt = \int_{t_0}^{\infty} |S(t - t_0, 0)x|^2\, dt = \int_{0}^{\infty} |S(t, 0)x|^2\, dt.$$

Thus an estimate of the form $\int_{t_0}^\infty |S(t, t_0)x|^2\, dt \leqq M(x)$ holds for all $t_0 \geqq 0$ if it holds for $t_0 = 0$, and hence Theorem 1 applies.

DEFINITION 7. An evolutionary process of class $C(0, e)$ is said to have *periodic behavior of period* $\beta > 0$ if

$$S(t + \beta, \beta + t_0) = S(t, t_0) \tag{29}$$

for all $(t, t_0) \in \Delta$.

Remark 4. If X is finite-dimensional, then an evolutionary process with periodic behavior most frequently corresponds to the set of solutions of an ordinary differential equation of the form

$$\dot{x}(t) = A(t)x(t), \qquad A(t + \beta) = A(t).$$

The following corollary to Theorem 1 is the analogue for evolutionary processes with periodic behavior of Corollary 3.

COROLLARY 4. *An evolutionary process of class $C(0, e)$ with periodic behavior $\beta > 0$ is uniformly asymptotically stable if and only if for each x in X,*

$$\int_{\beta}^{\infty} |S(t, \beta)x|^2\, dt < \infty. \tag{30}$$

Proof. (i) Observe that if $\int_{t_0}^\infty |S(t, t_0)x|^2\, dt < \infty$ for each $x \in X$ and $t_1 \leqq t_0$, then

$$\int_{t_1}^{\infty} |S(t, t_1)x|^2\, dt = \int_{t_1}^{t_0} |S(t, t_1)x|^2\, dt + \int_{t_0}^{\infty} |S(t, t_0)S(t_0, t_1)x|^2\, dt$$

$$= \frac{M_1^2}{2\omega}[e^{2\omega(t_0, t_1)} - 1]|x|^2 + \int_{t_0}^{\infty} |S(t, t_0)S(t_0, t_1)x|^2\, dt. \tag{31}$$

By application of the principle of uniform boundedness in the same manner as was done in the proof of Theorem 1, it is easy to show that there exists a finite constant $M(t_0)$ such that

$$\int_{t_0}^{\infty} |S(t, t_0)x|^2\, dt \leqq M(t_0)|x|^2. \tag{32}$$

Using (32) we can write (31) as

$$\begin{aligned} \int_{t_1}^{\infty} |S(t, t_1)x|^2\, dt &\leqq \frac{M_1^2}{2\omega}[e^{2\omega(t_0 - t_1)} - 1]|x|^2 + M(t_0)\, e^{2\omega(t_0 - t_1)}|x|^2 \\ &\leqq \left[\frac{M_1^2}{2\omega} + M(t_0)\right] e^{2\omega t_0}|x|^2. \end{aligned} \tag{33}$$

Clearly the right side of (33) depends only on t_0 and x.

Now let $S(t, t_0)$ have periodic behavior with period $\beta > 0$. Then if $t_0 \geqq \beta$, we can write $t_0 = n\beta + \tau$, where $0 \leqq \tau < \beta$ and n is an integer. Hence,

$$\begin{aligned} S(t, t_0)x &= S(t_0 + (t - t_0), t_0)x \\ &= S[(t - t_0) + n\beta + \tau, n + \tau]x = S[(t - t_0) + \tau, \tau]x. \end{aligned} \tag{34}$$

Thus using (33), we have

$$\begin{aligned} \int_{t_0}^{\infty} |S(t, t_0)x|^2\, dt &= \int_{t_0}^{\infty} |S[(t - t_0) + \tau, \tau]x|^2\, dt \\ &= \int_{\tau}^{\infty} |S(t, \tau)x|^2\, dt \leqq \left[\frac{M_1^2}{2\omega} + M(\beta)\right] e^{2\omega\beta}|x|^2. \end{aligned} \tag{35}$$

The relation (35) is equivalent to (7), and hence by Theorem 1, $S(t, t_0)$ is uniformly asymptotically stable.

(ii) If $S(t, t_0)$ is uniformly asymptotically stable, then (30) is satisfied.

This completes the proof of the corollary.

COROLLARY 5. (i) *If $S(t, t_0)$ is a semi-group of class C_0 which is uniformly asymptotically stable, then the V function of Corollary* 1 *is independent of t.*

(ii) *If $S(t, t_0)$ is an evolutionary process of class $C(0, e)$ with periodic behavior $\beta > 0$, then the V function of Corollary* 1 *is periodic of period β.*

Proof. Proof of (i).

$$\begin{aligned} V(x, t) &= \int_{t}^{\infty} |S(s, t)x|^2\, ds = \int_{t}^{\infty} |S(s - t, 0)x|^2\, ds \\ &= \int_{0}^{\infty} |S(s, 0)x|^2\, ds = V(x, 0). \end{aligned} \tag{36}$$

Proof of (ii).

$$\begin{aligned} V(x, t + \beta) &= \int_{t+\beta}^{\infty} |S(s, t + \beta)x|^2\, ds = \int_{t+\beta}^{\infty} |S(s - \beta, t)x|^2\, ds \\ &= \int_{t}^{\infty} |S(s, t)x|^2\, ds = V(x, t). \end{aligned} \tag{37}$$

The next theorem gives a sufficient condition for asymptotic stability of evolutionary processes of class $C(0, e)$. We shall show by means of an example that the condition is not necessary.

THEOREM 2. *Let $S(t, t_0)$ be an evolutionary process of class $C(0, e)$ such that for each $x \in X$ and $t_0 \geqq 0$, the improper integral $\int_{t_0}^{\infty} |S(t, t_0)x|^2\, dt$ is convergent. Then $S(t, t_0)$ is asymptotically stable.*

Proof. By means of the principle of uniform boundedness, it can be shown as was done in proving Theorem 1 that for each $t_0 \geqq 0$ there exists a finite constant $M_2(t_0)$ such that for all $x \in X$,

$$\int_{t_0}^{\infty} |S(t, t_0)x|^2\, dt \leqq M_2(t_0)|x|^2. \tag{38}$$

Similarly we can find $M_3(t_0)$ such that

$$|S(t, t_0)x| \leqq M_3(t_0)|x| \tag{39}$$

for all $x \in X$.

And finally, it can be shown, as was done in Theorem 1, that if $\varepsilon > 0$ is such that $\sqrt{M_2(t_0)}\varepsilon < 1$, then for each $x \in X$ there is a $\bar{t} \in [t_0, t_0 + 1/\varepsilon^2]$ such that

$$|S(\bar{t}, t_0)x| = \varepsilon|x|. \tag{40}$$

Thus (39) and (40) imply that if $t \geqq t_0 + 1/\varepsilon^2$, then

$$|S(t, t_0)x| \leqq M_3(t_0)\varepsilon|x|, \tag{41}$$

which proves asymptotic stability.

COROLLARY. *Let $S(t, t_0)$ be an evolutionary process of class $C(0, e)$ and suppose there exists a mapping $V\colon X \times [0, \infty) \to [0, \infty)$ such that:*

(i) *$V(0, t) = 0$ for all $t \in [0, \infty)$ and for all x in X;*

(ii)
$$\frac{dV}{dt}(S(t, t_0)x, t) \leqq -|S(t, t_0)x|^2.$$

Then $S(t, t_0)$ is asymptotically stable.

Proof. If x is in X and $t_0 \geqq 0$, then

$$0 \leqq V(S(t, t_0)x, t) \leqq V(x, t_0) - \int_{t_0}^{t} |S(s, t_0)x|^2\, ds.$$

Thus

$$\int_{t_0}^{\infty} |S(s, t_0)x|^2\, ds \leqq V(x, t_0) \tag{42}$$

for all x in X, and hence, by Theorem 2, $S(t, t_0)$ is asymptotically stable.

Remark 5. If $S(t, t_0)$ is a semi-group of class C_0 which satisfies the hypothesis of Theorem 2, then $S(t, t_0)$ is uniformly exponentially stable. This follows from the fact that for semi-groups,

$$\int_{t_0}^{\infty} |S(t, t_0)x|^2\, dt = \int_{0}^{\infty} |S(t, 0)x|^2\, dt$$

for all $x \in X$ and $t_0 \geqq 0$, and hence Theorem 1 applies.

The following example shows that, in contrast to the case where X is finite-dimensional, asymptotic stability does not imply uniform asymptotic stability for semi-groups of operators of class C_0. Thus the hypothesis of Theorem 2 is not necessary for asymptotic stability of evolutionary processes of class $C(0, e)$. Since if it were, then by Remark 5, all asymptotically stable semi-groups of class C_0 would be uniformly stable, contradicting the example.

Example 1. Let ℓ_2 be the Hilbert space of all real sequences $a = (a_1, \cdots, a_n, \cdots)$ such that $\sum_{i=1}^{\infty} a_i^2 < +\infty$. On ℓ_2 we define the bounded linear operator A by the equation

$$Aa = (-a_1 + a_2, -a_2 + a_3, \cdots, -a_n + a_{n+1}, \cdots) \tag{43}$$

and the infinite system of differential equations

$$\dot{x}_i = -x_i + x_{i+1}, \qquad i = 1, 2, \cdots, \tag{44}$$

where x_i is the ith coordinate of a vector x in ℓ_2.

It is not difficult to verify that the solution of system (44) with initial vector $x(0) = a$ is given by the vector-valued function $e^{At}a$ and that the ith coordinate of $e^{At}a$ has the expression

$$x_i(t) = e^{-t} \sum_{n=0}^{\infty} a_{i+n} \frac{t^n}{n!}. \tag{45}$$

By means of a few elementary estimates (see the Appendix) it can be shown that $|e^{At}a| \to 0$ as $t \to \infty$ for every $a \in \ell_2$. Thus e^{At} is an asymptotically stable semi-group. However, if e^{At} were exponentially stable, then by Remark 3, the integral $\int_0^\infty |e^{At}a| \, dt$ would be convergent for each $a \in \ell_2$ and this would imply that each coordinate $x_i(t)$ of $e^{At}a$ would have a convergent integral $\int_0^\infty x_i(t) \, dt$. Since $(1/n!) \int_0^\infty t^n e^{-t} \, dt = 1$ for every natural number, it follows from (45) that $\lim_{t \to \infty} \int_0^t x_i(s) \, ds = \sum_{n=0}^{\infty} a_{i+n}$ which is divergent if, for example, a is the sequence with coordinates $a_n = 1/n$. Hence, e^{At} is asymptotically stable, but not uniformly asymptotically stable.

There is, however, an important category of semi-groups, which includes solutions of autonomous differential-difference equations, for which asymptotic stability implies uniform asymptotic stability. This result is described in the next theorem.

THEOREM 3. *Let $S(t)$ be a semi-group of class C_0 such that for some t_0, where $t_0 > 0$, $S(t_0)$ is a compact operator. Then if $S(t)$ is asymptotically stable, it is uniformly asymptotically stable.*

Proof. Since $S(t_0)$ is compact, its spectrum consists only of isolated points in the point spectrum and possibly the origin (see, e.g., [7]). Thus, the spectral radius of $S(t_0)$ is determined solely by its point spectrum. But if τ is in the point spectrum of $S(t_0)$, this implies that there exists a λ in the point spectrum of the infinitesimal generator A of $S(t)$ such that $\tau = e^{\lambda t_0}$ (see, e.g., [7]). Because $S(t)$ is asymptotically stable, it follows that $|\tau| < 1$ for all τ in the point spectrum of $S(t_0)$. Moreover, since the spectrum of $S(t_0)$ can accumulate only at the origin, it follows that there exists α, with $0 < \alpha < 1$, such that the spectrum of $S(t_0)$ lies in the interior of the circle $|Z| \leqq \alpha$. Hence the spectrum of the infinitesimal generator A of $S(t)$ lies in the half-plane $\operatorname{Re} Z \leqq (\ln \alpha)/t_0 < 0$ (see, e.g., [7]). This means that given $(\ln \alpha)/2t_0$ there exists a constant $K \geqq 1$ such that $|S(t)| \leqq K \exp[(\ln \alpha)/2t_0]t$ (see, e.g., [7]). This implies that $S(t)$ is uniformly exponentially stable and hence uniformly asymptotically stable.

3. The special case of X a Hilbert space.

THEOREM 4. *Let $S(t, t_0)$ be an evolutionary process of class $C(0, e)$ defined on a Hilbert space H. Then $S(t, t_0)$ will be uniformly asymptotically stable if and only if there exists a uniformly bounded mapping $B: [0, \infty) \to \mathscr{L}(H, H)$ such that for each $t \in [0, \infty)$, $B(t) > 0$ and for all $x \in H$ and $(t, t_0) \in \Delta$,*

$$\frac{d}{dt}(B(t)S(t, t_0)x, S(t, t_0)x) \leqq -|S(t, t_0)x|^2. \tag{46}$$

Proof. Proof of necessity. If $S(t, t_0)$ is uniformly asymptotically stable, by Lemma 1 it is also uniformly exponentially stable. Hence $|S(t, t_0)| \leqq M\,e^{-\alpha(t-t_0)}$ for all $(t, t_0) \in \Delta$. Let $S^*(t, t_0)$ denote the adjoint of $S(t, t_0)$. Clearly $|S^*(t, t_0)| \leqq M\,e^{-\alpha(t-t_0)}$ for all $(t, t_0) \in \Delta$. Thus the mapping

$$B(t) = \int_t^\infty S^*(s, t)S(s, t)\,ds \tag{47}$$

in $\mathscr{L}(H, H)$ is well-defined for each $t \in [0, \infty)$. Moreover, for each x in H and $(t, t_0) \in \Delta$,

$$(B(t)S(t, t_0)x, S(t, t_0)x) = \int_t^\infty |S(s, t_0)x|^2\,ds. \tag{48}$$

Consequently,

$$\frac{d}{dt}(B(t)S(t, t_0)x, S(t, t_0)x) = -|S(t, t_0)x|^2. \tag{49}$$

The uniform boundedness of the mappings $B(t)$ follows from (7); that is,

$$(B(t)x, x) \leqq M_2|x|^2 \tag{50}$$

for all $t \in [0, \infty)$ and $x \in H$, and hence $B(t)$ is uniformly bounded.

Proof of sufficiency. Assume $B(t)$ satisfies the hypotheses of the theorem. Let $x \in H$ and $t_0 \geqq 0$. Then,

$$\begin{aligned} 0 &\leqq (B(t)S(t, t_0)x, S(t, t_0)x) \\ &= (B(t_0)x_0, x_0) + \int_{t_0}^t \frac{d}{ds}(B(s)S(s, t_0)x, S(s, t_0)x)\,ds \\ &\leqq (B(t_0)x_0, x_0) - \int_{t_0}^t |S(s, t_0)x|^2\,ds. \end{aligned}$$

Thus,

$$\int_{t_0}^\infty |S(s, t_0)x|^2\,ds \leqq (B(t_0)x, x) \leqq \sup_t |B(t)|\,|x|^2. \tag{51}$$

Since $\sup_t |B(t)|$ is by assumption finite, Theorem 1 applies and $S(t, t_0)$ is uniformly asymptotically stable.

COROLLARY. (i) *If $S(t, t_0)$ is a semi-group of class C_0 defined on H which is uniformly asymptotically stable, then the mapping $t \to B(t)$ of Theorem 3 will be constant if equality holds in* (46).

(ii) *If $S(t, t_0)$ has periodic behavior of period $\beta > 0$ and is uniformly asymptotically stable, then $B(t)$ will be periodic of period β if equality holds in* (46).

Proof. The proof is similar to the proof of Corollary 5 to Theorem 1, and is omitted.

Example 2. As in Example 1, let $H = \ell_2$ and this time, e^{At} will be the semigroup defined by the infinitesimal generator

$$Aa = (-2a_1 + a_2, \cdots, -2a_n + a_{n+1}, \cdots). \tag{52}$$

The ith coordinate of $e^{At}a$ has the expression

$$x_i(t) = e^{-2t} \sum_{n=0}^{\infty} a_{i+n} \frac{t^n}{n!}. \tag{53}$$

Let $B > 0$ in $\mathscr{L}(\ell_2, \ell_2)$ be given by

$$B = I/2 \quad (I \text{ is the identity mapping in } \ell_2). \tag{54}$$

For each $a \in \ell_2$ and $t \in [0, \infty)$, we have after a bit of computation,

$$\begin{aligned} \frac{d}{dt}(B\, e^{At}a, e^{At}a) &= -2 \sum_{i=1}^{\infty} x_i^2(t) + \sum_{i=1}^{\infty} x_i(t) x_{i+1}(t) \\ &= -\sum_{i=1}^{\infty} x_i^2(t) + \left(-\sum_{i=1}^{\infty} x_i^2(t) + \sum_{i=1}^{\infty} x_i(t) x_{i+1}(t) \right). \end{aligned} \tag{55}$$

The term in the brackets on the right-hand side of (55) is always less than or equal to zero because of the Cauchy–Schwarz inequality. Hence,

$$\frac{d}{dt}(B\, e^{At}a, e^{At}a) \leqq -|e^{At}a|^2$$

for all $a \in \ell_2$, and by Theorem 3, e^{At} is uniformly asymptotically stable.

4. A perturbation result. In this section a perturbation result concerning uniformly asymptotically stable semi-groups of operators will be developed. This result is the analogue of similar results in the theory of stability for ordinary differential equations (see, e.g., [5]). The purpose of presenting it here is to demonstrate how the stability theory of ordinary differential equations can be extended to evolutionary processes in a Banach space.

We first establish a lemma.

LEMMA 2. *Let $S(t, t_0)$ be a semi-group of class C_0 and let $R: [0, \infty) \to \mathscr{L}(X, X)$ be strongly differentiable and uniformly bounded by some constant r on $[0, \infty)$. Then, the evolutionary process $U(t, t_0)$ having the representation*

$$U(t, t_0)x = \sum_{n=0}^{\infty} S_n(t, t_0)x, \tag{56a}$$

where

$$S_0(t, t_0)x = S(t, t_0)x \tag{56b}$$

and

$$S_n(t, t_0)x = \int_{t_0}^{t} S(t, x)R(s)S_{n-1}(s)x\, ds, \tag{56c}$$

is of class $C(0, e)$.

Proof. First observe that if (56) makes sense, then it can be written as the integral equation

$$U(t, t_0)x = S(t, t_0)x + \int_{t_0}^{t} S(t, s)R(s)U(s, t_0)x\, ds. \tag{57}$$

That (56) and (57) make sense and $U(t, t_0)$ is an evolutionary process are consequences of Theorem 6.2 in [9, p. 216].

Using the representation (57), the facts that $|S(t, t_0)| \leqq M_1 e^{\omega(t-t_0)}$ and $|R(t)| \leqq r$, we obtain the inequality

$$|U(t, t_0)x|\, e^{\omega t} \leqq M_1 e^{\omega t_0}|x| + \int_{t_0}^{t} M_1 r\, e^{\omega s}|U(s, t_0)x|\, ds. \tag{58}$$

Applying Gronwall's inequality (see, e.g., [2, Prob. 1, p. 37]) to (58), we get

$$|U(t, t_0)x| \leqq M_1 e^{(r+\omega)(t-t_0)}|x| \tag{59}$$

which proves that $U(t, t_0)$ is of class $C(0, e)$.

THEOREM 5. *Let* $S(t, t_0)$ *be a semi-group of class* C_0 *which is uniformly asymptotically stable, and let* $R: [0, \infty) \to \mathscr{L}(X, X)$ *be strongly differentiable and such that* $|R(t)| \to 0$ *as* $t \to \infty$. *Then the evolutionary process described by* (57) *is uniformly asymptotically stable.*

Proof. Since $S(t, t_0)$ is uniformly asymptotically stable, there exist positive constants M and α such that $|S(t, t_0)| \leqq M e^{-\alpha(t-t_0)}$ for all $(t, t_0) \in \Delta$. Hence, we obtain the inequality

$$|U(t, t_0)x| \leqq M e^{-\alpha(t-t_0)}|x| + \int_{t_0}^{t} M e^{-\alpha(t-s)}|R(s)|\,|U(s, t_0)x|\, ds. \tag{60}$$

Multiplying both sides of (60) by $e^{\alpha t}$ and applying Gronwall's inequality to the resulting inequality we obtain, after some rearrangement of terms,

$$\begin{aligned} |U(t, t_0)x| &\leqq M e^{-\alpha(t-t_0)}|x|\left[1 + \int_{t_0}^{t} |R(s)| \exp\left[\left[\int_{s}^{t} |R(u)|\, du\right] ds\right]\right] \\ &= M e^{-\alpha(t-t_0)} \exp\left[\left[\int_{t_0}^{t} |R(u)|\, du\right]\right]|x|. \end{aligned} \tag{61}$$

Since $|R(t)| \to 0$ as $t \to \infty$, it follows that for some $T > 0$, $|R(t)| \leqq \alpha/2$ if $t \geqq T$. Hence, there exists $\overline{M} > 0$ such that for all x in X and $(t, t_0) \in \Delta$,

$$|U(t, t_0)x| \leqq \overline{M} e^{-\alpha/2(t-t_0)},$$

which proves that $U(t, t_0)$ is uniformly asymptotically stable.

5. The Perron condition in Banach spaces. In this section, we shall extend a stability criterion of O. Perron for evolutionary processes in a finite-dimensional space to evolutionary processes of class $C(0, e)$ in a Banach space (see, e.g., [1] or [8]).

Some additional notation must first be established. If $[t_0, t_1]$ is any interval in $[0, \infty)$, then $\chi_{[t_0,t_1]}$ is a real-valued function on $[0, \infty)$ whose value is 1 on $[t_0, t_1]$ and zero everywhere else. It is usually termed the characteristic function of the interval.

If X is a Banach space, then $L^p(X)$, $1 \leqq p < \infty$, will denote the equivalence classes of measurable functions from $[0, \infty)$ into X whose norms are pth power integrable. $L^\infty(X)$ will denote the equivalence classes of measurable functions from $[0, \infty)$ into X which are uniformly bounded on $[0, \infty)$. If $f \in L^p(X)$, $1 \leqq p < \infty$, the norm of f will be denoted by

$$\|f\|_p = \left(\int_0^\infty |f(t)|^p \, dt\right)^{1/p}, \tag{62}$$

and if $f \in L^\infty(X)$, then

$$\|f\|_\infty = \operatorname{ess\,sup} |f(t)|. \tag{63}$$

DEFINITION 8. An evolutionary process $S(t, t_0)$ of class $C(0, e)$ defined on X satisfies the *Perron condition of class* (p, p') if for every $f \in L^p(X)$ the function $\int_0^t S(t, s)f(s)\, ds$ is in $L^{p'}(X)$, where $1 \leqq p, p' \leqq \infty$.

LEMMA 3. *If $S(t, t_0)$ is of class $C(0, e)$ and satisfies the Perron condition of class (p, p'), then the mapping $P\colon L^p(X) \to L^{p'}(X)$ given by*

$$(Pf)(t) = \int_0^t S(t, s)f(s)\, ds \tag{64}$$

is a bounded linear mapping.

Proof. Let $\{f_n\} \to f$ in $L^p(X)$ and $\{Pf_n\} \to g$ in $L^{p'}(X)$. Since we are dealing with L^p-spaces, we can find a subsequence $\{f_\theta\} \subset \{f_n\}$ such that $\{f_\theta(t)\} \to f(t)$ a.e. on $[0, \infty)$ and $\{\int_0^t S(t, s)f_\theta(s)\, ds\} \to g(t)$ a.e. on $[0, \infty)$. Because $S(t, s)$ is strongly continuous on X, this means that $\{\int_0^t S(t, s)f_\theta(s)\, ds\} \to \int_0^t S(t, s)f(s)\, ds$ for all $t \in [0, \infty)$. Hence, $g(t) = \int_0^t S(t, s)f(s)\, ds$ a.e. on $[0, \infty)$. By the closed graph theorem (see, e.g., [10]) this proves that P is continuous.

THEOREM 6. *If $1 \leqq p < \infty$ and $1 \leqq p' < \infty$ and $S(t, t_0)$ is an evolutionary process of class $C(0, e)$ which satisfies the Perron condition of class (p, p'), then $S(t, t_0)$ is uniformly asymptotically stable.*

Proof. Let $x \in X$ and $(T, t_0) \in \Delta$ with $T > t_0$. Define the function

$$t \to \chi_{[t_0,T]}(t)S(t, t_0)x = f(t). \tag{65}$$

Clearly $f \in L^p(X)$, and

$$\int_0^t S(t, s)f(s)\, ds = \begin{cases} 0 & \text{if } t \leqq t_0, \\ (t - t_0)S(t, t_0)x & \text{if } t_0 \leqq t \leqq T, \\ (T - t_0)S(t, t_0)x & \text{if } T \leqq t. \end{cases}$$

By Lemma 3 there exists a finite constant $M(p, p')$ such that

$$\|Pf\|_{p'} \leqq M(p, p')\|f\|_p. \tag{66}$$

Thus, for the function defined by (65),

$$\begin{aligned}\int_0^\infty |Pf(s)|^{p'}\,ds &= \int_{t_0}^T |(s - t_0)S(t, t_0)x|^{p'}\,ds + \int_T^\infty |(T - t_0)S(s, t_0)x|^{p'}\,ds \\ &\leqq [M(p, p')]^{p'}\left(\int_{t_0}^\infty |f(s)|^p\,ds\right)^{p'/p} \\ &\leqq [M(p, p')]^{p'}\left[\frac{1}{p\omega}e^{\omega p[T - t_0]} - 1\right]^{p'/p} M_1^{p'}|x|^{p'}.\end{aligned} \tag{67}$$

Relation (67) can be simplified to

$$(T - t_0)^{p'}\int_T^\infty |S(s, t_0)x|^{p'}\,ds \leqq \overline{M}(p, p')\,e^{\omega p'(T - t_0)}|x|^{p'}. \tag{68}$$

Thus if we choose $T = t_0 + 1$, (68) can be used to show that

$$\int_{t_0+1}^\infty |S(s, t_0)x|^{p'}\,ds \leqq \overline{M}(p, p')\,e^{\omega p'}|x|^{p'}. \tag{69}$$

And thus,

$$\begin{aligned}\int_{t_0}^\infty |S(s, t_0)x|^{p'}\,ds &= \int_{t_0}^{t_0+1} |S(s, t_0)x|^{p'}\,ds + \int_{t_0+1}^\infty |S(s, t_0)x|^{p'}\,ds \\ &\leqq [M_1^{p'} + \overline{M}(p, p')]\,e^{\omega p'}|x|^{p'}.\end{aligned} \tag{70}$$

The inequality (70) shows that $S(t, t_0)$ satisfies the conditions of Remark 3 (equation (18)) and hence is uniformly asymptotically stable.

THEOREM 7. *If* $p' = \infty$ *and* $1 \leqq p < \infty$ *and* $S(t, t_0)$ *is an evolutionary process of class* $C(0, e)$ *which satisfies the Perron condition of class* (p, p'), *then* $S(t, t_0)$ *is uniformly stable.*

Proof. In a manner similar to the proof of Theorem 5, we can obtain the inequality

$$\sup_{t_0 < T \leqq t} |S(t, t_0)x| \leqq \frac{\overline{M}(p, p')}{T - t_0}e^{\omega(T - t_0)}|x| \tag{71}$$

for each $x \in X$, where $\overline{M}(p, p')$ is independent of T and t_0. Setting $T = t_0 + 1$ in (71) we obtain

$$\sup_{t_0 + 1 \leqq t} |S(t, t_0)x| \leqq \overline{M}(p, p')\,e^{\omega}|x|$$

for all $x \in X$ and $t_0 \geqq 0$, which proves $S(t, t_0)$ is uniformly asymptotically stable.

THEOREM 8. *If* $p = \infty$ *and* $p' = \infty$ *and* $S(t, t_0)$ *is of class* $C(0, e)$ *and satisfies the Perron condition of class* (p, p'), *then* $S(t, t_0)$ *is uniformly asymptotically stable.*

Proof. Let $t_0 \geqq 0$, $T > t_0$ and $x \in X$. Define $f(t)$ by equation (65). Then

$$\int_0^t S(t,s)f(s)\,ds = \begin{cases} 0 & \text{if } t \leqq t_0, \\ (t-t_0)S(t,t_0)x & \text{if } t_0 \leqq t \leqq T, \\ (T-t_0)S(t,t_0)x & \text{if } T \leqq t. \end{cases}$$

Thus, if $t \geqq T$,

$$(T-t_0)|S(t,t_0)x| \leqq M(p,p') \sup_{0 \leqq t \leqq \infty} |f(t)| \leqq M(p,p')M_1\, e^{\omega(T-t_0)}|x|.$$

Hence, if $T = t_0 + 1$ and $t \geqq t_0 + 1$,

(72) $$|S(t,t_0)x| \leqq M(p,p')M_1\, e^{\omega}|x|.$$

But (72) implies that there exists a constant $\overline{M}(p, p')$ such that

(73) $$|S(t,t_0)| \leqq \overline{M}(p,p')$$

for all $t \geqq t_0$.

Now set

(74) $$f_1(t) = \begin{cases} S(t,t_0)x & \text{if } t \geqq t_0, \\ 0 & \text{if } t < t_0. \end{cases}$$

Applying the mapping P to f_1 (equation (64)) we obtain

(75) $$\int_0^t S(t,s)f_1(s)\,ds = \begin{cases} 0 & \text{if } t \leqq t_0, \\ (t-t_0)S(t,t_0)x & \text{if } t \geqq t_0. \end{cases}$$

By (74) and (66) this means that

(76) $$\sup_{t_0 \leqq t < \infty} (t-t_0)|S(t,t_0)x| \leqq \overline{M}(p,p')|x|.$$

Hence, if $t > t_0$,

$$|S(t,t_0)x| \leqq \frac{\overline{M}(p,p')|x|}{t-t_0},$$

which proves that $S(t, t_0)$ is uniformly asymptotically stable.

Appendix. We shall prove that the semi-group in Example 1 is asymptotically stable. First observe that for each a in ℓ_2 and $t \geqq 0$,

$$\frac{1}{2}\frac{d}{dt}(|e^{At}a|^2) = -\sum_{i=0}^{\infty} x_i^2(t) + \sum_{i=1}^{\infty} x_i(t)x_{i+1}(t),$$

where the x_i are defined by (45). Hence by the Cauchy–Schwarz inequality, $d(|e^{At}a|^2)/dt \leqq 0$ for all $t \geqq 0$. This means $|e^{At}| \leqq 1$ for all t.

The elements a_f in ℓ_2 with only a finite number of nonzero coordinates are dense in ℓ_2, and from the form of (45) it is easily seen that for such an element a_f, $|e^{At}a_f| \to 0$ as $t \to \infty$. Thus, let a be arbitrary in ℓ_2 and $\varepsilon > 0$. Then there exists an a_f in ℓ_2 such that $|a - a_f| < \varepsilon/2$ and a $t_0 > 0$ such that $|e^{At_0}a_f| < \varepsilon/2$ for all $t \geqq t_0$. Hence

$$|e^{At}a| \leqq |e^{At}(a - a_f)| + |e^{At}a_f| < \varepsilon$$

if $t \geqq t_0$. This shows that e^{At} is asymptotically stable.

Acknowledgment. The author would like to thank the referee and Professor W. Hahn for suggesting improvements in the paper. The lacunae which remain are the fault of the author.

REFERENCES

[1] R. BELLMAN, *On an application of a Banach–Steinhaus theorem to the study of the boundedness of solutions of nonlinear differential and difference equations*, Ann. of Math., 49 (1948), pp. 515–522.

[2] E. A. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[3] R. DATKO, *An extension of a theorem of A. M. Lyapunov to semi-groups of operators*, J. Math. Anal. Appl., 24 (1968), pp. 290–295.

[4] ———, *Extending a theorem of A. M. Lyapunov to Hilbert space*, Ibid., 32 (1970), pp. 610–616.

[5] W. HAHN, *Stability of Motion*, Springer-Verlag, New York, 1967.

[6] A. HALANAY, *Differential Equations—Stability, Oscillations, Time Lags*, Academic Press, New York, 1966.

[7] E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semi-Groups*, Colloquium Publications, Amer. Math. Soc., Providence, 1957.

[8] O. PERRON, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z., 32 (1930), pp. 703–728.

[9] R. S. PHILLIPS, *Perturbation theory for semi-groups of linear operators*, Trans. Amer. Math. Soc., 74 (1953), pp. 199–221.

[10] K. YOSIDA, *Functional Analysis*, Springer-Verlag, New York, 1967.

[11] R. DATKO, *A linear control problem in an abstract Hilbert space*, J. Differential Equations, 9 (1971), pp. 346–359.

[12] R. D. DRIVER, *Existence and stability of solutions of a delay-differential system*, Arch. Rational Mech. Anal., 10 (1962), pp. 401–426.

[13] J. K. HALE, *Linear functional equations with constant coefficients*, Contributions to Differential Equations 1 (1963), pp. 291–317.

[14] T. KATO AND H. TANABE, *On the abstract evolution equation*, Osaka Math. J., 14 (1962), pp. 107–133.

[15] N. N. KRASOVSKII, *Stability in Motion*, Stanford University Press, Stanford, Calif., 1963.

[16] S. G. KREIN, *Linear Differential Equations in a Banach Space*, Izdat. Nauk, Moscow, 1967.

[17] J. L. MASSERA AND J. J. SCHÄFFER, *Linear Differential Equations and Function Spaces*, Academic Press, New York, 1966.

[18] H. TANABE, *On the equations of evolution in a Banach space*, Osaka Math. J., 12 (1960), pp. 363–376.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

INTEGRALS OF PRODUCTS OF BESSEL FUNCTIONS*

A. D. JACKSON† AND L. C. MAXIMON‡

Abstract. Simple expressions for a variety of integrals involving the product of three cylindrical or three spherical Bessel functions are obtained in terms of the angular functions arising in the decomposition of a plane wave in two or three dimensions.

Introduction. Integrals over Bessel functions have been of concern in mathematical physics for well over a hundred years. They arise naturally as solutions of the wave equation in which one has cylindrical or spherical symmetry, the solutions involving the ordinary and spherical Bessel functions, respectively. While there exist extensive investigations of integrals containing the product of two Bessel functions (see, e.g., [1, Chap. XIII]), there is relatively little available in the case of integrals containing the product of three or more Bessel functions. Indeed, in an article in 1934, G. N. Watson [2] begins his consideration of an integral containing three Bessel functions with the statement:

"It seems unlikely that the integral

$$\int_0^\infty t^{\lambda-1} J_\mu(at) J_\gamma(bt) J_\rho(ct)\, dt \tag{1}$$

can be expressed by any simple formula in the general case in which the only restrictions laid on the various parameters are those which are essential to secure convergence. Any special cases of the integral, obtained by assuming relations (neither too numerous nor too trivial) between the parameters, which can be evaluated are consequently of some interest."

While it has been noted by Bailey [3] that the above integral may be expressed in terms of the Appell function F_4 of two variables, it remains desirable to find simple expressions for particular values of the parameters, in the spirit of G. N. Watson.

1. Spherical Bessel functions. In this note we will be concerned primarily with a case of the integral (1) which arises in quantum mechanical problems having spherical symmetry. It is then natural to express wave functions and operators in terms of spherical Bessel functions and spherical harmonics, in which case matrix elements lead to the evaluation of integrals of the form [4, pp. 567–569]

$$\int_0^\infty j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) r^2\, dr, \tag{2}$$

where the $j_l(kr)$ are spherical Bessel functions, which are related to the ordinary Bessel functions by

$$j_l(kr) = \left(\frac{\pi}{2kr}\right)^{1/2} J_{l+1/2}(kr). \tag{3}$$

* Received by the editors June 2, 1971, and in revised form October 4, 1971. This research was supported in part by the United States Atomic Energy Commission.

† Department of Physics, State University of New York, Stony Brook, Long Island, New York 11790.

‡ National Bureau of Standards, Washington, D.C. 20234 and Institute for Theoretical Physics, State University of New York, Stony Brook, Long Island, New York 11790.

Thus in our consideration, the parameters in Watson's integral (1) are restricted to have half-integer values. Our derivation will impose the further restriction that the parameters l_1, l_2, l_3 in (2) satisfy the triangularity condition ($|l_1 - l_2| \leqq l_3 \leqq l_1 + l_2$) and that $l_1 + l_2 + l_3 \equiv L$ is even. Integrals in which the parameters satisfy these restrictions are the ones of greatest physical interest. The first restriction is a consequence of the conservation of angular momentum and the second of parity conservation. We should nonetheless note that in view of the desirability of being able to make analytic continuations of matrix elements in the complex l-plane, one would ultimately like to have simple expressions for this integral for arbitrary, complex values of l_1, l_2 and l_3.

There is an intimate connection between the spherical Bessel functions and the spherical harmonics, and one might expect that simple expressions for such integrals would exploit this connection. Indeed, in both our final expression for (2) and throughout the proof this will be clear. We note an indication of this connection in the expression for a particular case of (1) first noted by MacDonald [5] (cited in [1, (4), p. 412]):

$$\int_0^\infty j_\mu(at) j_\nu(bt) j_\nu(ct) t^{2-\mu}\, dt = \frac{\pi}{4abc}\left(\frac{bc}{a}\sin A\right)^\mu P_\nu^{-\mu}(\cos A) \tag{4}$$

in the case where a, b, c are the sides of a triangle, A is the angle between sides b and c, and μ and ν are complex numbers such that $\operatorname{Re}\mu > -\frac{1}{2}$, $\operatorname{Re}\nu > -\frac{1}{2}$. In (4) we have rewritten the expression given in [1] in terms of the spherical Bessel functions, where the definition (3) is taken to apply for arbitrary (noninteger) l.

We now proceed to the evaluation of the integral

$$I = \int_0^\infty j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) r^2\, dr, \tag{5}$$

where l_1, l_2, l_3 are nonnegative integers such that $l_1 + l_2 + l_3 \equiv L$ is even and the triangularity condition $|l_1 - l_2| \leqq l_3 \leqq l_1 + l_2$ is satisfied. (Our derivation will make extensive use of the functions and techniques of angular momentum algebra. The monograph by Edmonds [6] provides a thorough introduction to the subject.) We start with the decomposition of a plane wave into spherical Bessel functions and spherical harmonics:

$$e^{i\mathbf{k}\cdot\mathbf{r}} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr) Y_{lm}^*(\hat{k}) Y_{lm}(\hat{r}), \tag{6}$$

where $\hat{r}$ and $\hat{k}$ are unit vectors specifying the angles of $\mathbf{r}$ and $\mathbf{k}$. Multiplying both sides of (6) by $Y_{l'm'}(\hat{k})$ and integrating over the angles of $\hat{k}$ we obtain

$$j_l(kr) Y_{lm}(\hat{r}) = \frac{1}{4\pi} i^{-l} \int e^{i\mathbf{k}\cdot\mathbf{r}} Y_{lm}(\hat{k})\, d\Omega_k, \tag{7}$$

where $d\Omega_k = \sin\theta_k\, d\theta_k\, d\phi_k$. From (7) we may write

$$\begin{aligned} &\int j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) r^2\, dr \int Y_{l_1 m_1}(\hat{r}) Y_{l_2 m_2}(\hat{r}) Y_{l_3 m_3}(\hat{r})\, d\Omega_r \\ &\quad = \frac{1}{(4\pi)^3} i^{-(l_1+l_2+l_3)} \int e^{i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{r}} \\ &\quad\quad \cdot Y_{l_1 m_1}(\hat{k}_1) Y_{l_2 m_2}(\hat{k}_2) Y_{l_3 m_3}(\hat{k}_3)\, d\Omega_{k_1}\, d\Omega_{k_2}\, d\Omega_{k_3}\, d^3 r. \end{aligned} \tag{8}$$

The integral over $d\Omega_r$ on the left-hand side of (8) may be evaluated in terms of 3-j symbols [6, (4.6.3), p. 63]:

(9)
$$\int Y_{l_1m_1}(\hat{r})Y_{l_2m_2}(\hat{r})Y_{l_3m_3}(\hat{r})\,d\Omega_r = \left[\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right]^{1/2}\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

We now multiply both sides of (8) by $\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ and sum over all m_1, m_2, m_3.

Using the orthonormality property of the 3-j symbols [6, Eq. (3.7.8), p. 47]

(10)
$$\sum_{m_1,m_2,m_3}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \delta(l_1, l_2, l_3),$$

where $\delta(l_1, l_2, l_3) = 1$ if l_1, l_2, l_3 satisfy the triangularity condition and is zero otherwise, we now obtain

(11)
$$\begin{aligned}\int_0^\infty j_{l_1}(k_1r)j_{l_2}(k_2r)j_{l_3}(k_3r)r^2\,dr &= \left[\frac{4\pi}{(2l_1+1)(2l_2+1)(2l_3+1)}\right]^{1/2} \\ &\cdot \frac{i^{-(l_1+l_2+l_3)}}{(4\pi)^3}\,\frac{1}{\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}} \\ &\cdot \sum_{m_1,m_2,m_3}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\int e^{i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\mathbf{r}}\,Y_{l_1m_1}(\hat{k}_1) \\ &\cdot Y_{l_2m_2}(\hat{k}_2)Y_{l_3m_3}(\hat{k}_3)\,d\Omega_{k_1}\,d\Omega_{k_2}\,d\Omega_{k_3}\,d^3r.\end{aligned}$$

We next perform the $\mathbf{r}$-integration on the right-hand side of (11), which gives a delta function:

(12)
$$\int e^{i\mathbf{k}\cdot\mathbf{r}}\,d^3r = (2\pi)^3\delta^{(3)}(\mathbf{k}).$$

The remaining integral is then

(13)
$$J \equiv \int Y_{l_1m_1}(\hat{k}_1)Y_{l_2m_2}(\hat{k}_2)Y_{l_3m_3}(\hat{k}_3)\delta^{(3)}(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,d\Omega_{k_1}\,d\Omega_{k_2}\,d\Omega_{k_3}.$$

We note that there are six integrals to be performed, two for each of the vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. Three of these integrations should be made so as to exploit the delta function, which serves to fix the angles between $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ so that they form a triangle: $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$. This implies that if the lengths k_1, k_2 and k_3 are such that they cannot form a triangle (either $k_3 < |k_1 - k_2|$ or $k_3 > k_1 + k_2$), the integral is zero provided that l_1, l_2, l_3 satisfy the triangularity condition and that L is an even integer. This conclusion does not hold if these last conditions are not satisfied since (11) contains the ratio of two 3-j symbols, both of which would then be zero. The remaining three integrations represent a rotation of the triangle as a whole. In particular, the orientation of a triangle in space may be defined by the direction of one of its sides and an azimuthal angle about that direction (e.g., the angles θ_3, ϕ_3 of $\hat{k}_3$ and the azimuth ϕ_2 of $\hat{k}_2$). The other three angles (θ_1, ϕ_1 of $\hat{k}_1$ and the polar angle θ_2 of $\hat{k}_2$) will therefore be integrated over first and will serve to determine the triangle.

For the purpose of the first three integrations, and with no loss of generality, it is simplest to choose a coordinate system with z-axis in the direction $\hat{k}_3$ and x-axis in the plane of $\hat{k}_2$ and $\hat{k}_3$ with $\phi_2 = 0$. In this system the angles of $\hat{k}_1$ are θ_{13}, ϕ_{12} and the polar angle of $\hat{k}_2$ is θ_{23}. The delta function can be written in this set of Cartesian coordinates as

$$\delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) = \delta(k_1 \sin\theta_{13} \cos\phi_{12} + k_2 \sin\theta_{23}) \cdot \delta(k_1 \sin\theta_{13} \sin\phi_{12})\delta(k_1 \cos\theta_{13} + k_2 \cos\theta_{23} + k_3). \tag{14}$$

The integrals involving the delta functions may all be evaluated by using

$$\int g(z)\delta(f(z))\,dz = \left.\frac{g(z)}{|f'(z)|}\right|_{f(z)=0}. \tag{15}$$

Integrating first over θ_{13} using $\delta(k_1 \cos\theta_{13} + k_2 \cos\theta_{23} + k_3)$ fixes the value of θ_{13} at

$$\cos\theta_{13} = -\frac{(k_2 \cos\theta_{23} + k_3)}{k_1} \tag{16}$$

and, from (15), introduces the factor $1/(k_1 \sin\theta_{13})$. Integrating next over ϕ_{12} using $\delta(k_1 \sin\theta_{13} \sin\phi_{12})$ fixes the value of ϕ_{12} at

$$\sin\phi_{12} = 0 \tag{17}$$

and introduces the factor $1/(k_1 \sin\theta_{13} |\cos\phi_{12}|)$. We next integrate over θ_{23} using $\delta(k_1 \sin\theta_{13} \cos\phi_{12} + k_2 \sin\theta_{23})$, recognizing that $\sin\theta_{13}$ is a function of θ_{23} and that $\phi_{12} = \pi$ since all other quantities in this last delta function are positive. This determines θ_{23} implicitly by the relation

$$k_1 \sin\theta_{13} = k_2 \sin\theta_{23} \tag{18}$$

and introduces the factor

$$\frac{1}{k_2 \sin\theta_{23} |\cos\theta_{13}/\sin\theta_{13} + \cos\theta_{23}/\sin\theta_{23}|}.$$

Relations (16) and (18) may be combined to express θ_{13} and θ_{23} in terms of the magnitudes of k_1, k_2 and k_3 alone:

$$\cos\theta_{13} = \frac{k_2^2 - k_1^2 - k_3^2}{2k_1k_3}; \qquad \cos\theta_{23} = \frac{k_1^2 - k_2^2 - k_3^2}{2k_2k_3}. \tag{19}$$

The three integrations exhausting the delta functions have served to fix the angles between the vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$. By using (17) and (19) to evaluate the factors, the integral (13) becomes

$$J = \frac{1}{k_1k_2k_3} \int Y_{l_1m_1}(\theta_1, \phi_1) Y_{l_2m_2}(\theta_2, \phi_2) Y_{l_3m_3}(\theta_3, \phi_3)\, d\phi_2\, d\Omega_3 \tag{20}$$

and thus

$$\int_0^\infty j_{l_1}(k_1 r) j_{l_2}(k_2 r) j_{l_3}(k_3 r) r^2\, dr$$

$$= \left[\frac{4\pi}{(2l_1+1)(2l_2+1)(2l_3+1)}\right]^{1/2} \frac{i^{-(l_1+l_2+l_3)}}{\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}} \tag{21}$$

$$\cdot \frac{\Delta}{8k_1k_2k_3} \sum_{m_1,m_2,m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

$$\cdot \int Y_{l_1m_1}(\theta_1, \phi_1) Y_{l_2m_2}(\theta_2, \phi_2) Y_{l_3m_3}(\theta_3, \phi_3)\, d\phi_2\, d\Omega_3.$$

In (20) and (21) the arguments of each of the functions Y_{lm} depend on the angles ϕ_2, θ_3 and ϕ_3, but the relative angles between the vectors $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ are now fixed. In (21) we define

$$\Delta = \begin{cases} 1 \text{ if } \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \text{ form a nondegenerate triangle,} \\ \frac{1}{2} \text{ if } \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \text{ form a degenerate triangle,} \\ 0 \text{ if } \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \text{ do not form a triangle.} \end{cases} \tag{22}$$

The third of these conditions results from the delta function in (13) and has been noted above. That we obtain a factor of $\frac{1}{2}$ when $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ form a degenerate triangle may be shown rigorously if one writes the Bessel functions in terms of sines and cosines and performs the integration directly, leading to an expression in terms of the Appell function F_4. The factor $\frac{1}{2}$ results essentially from the fact that the argument of the delta functions vanishes at the limits of integration.

We observe that the sum of functions Y_{lm} entering into (21) is, apart from a phase factor, the invariant triple product discussed by Fano and Racah [7, Chap. 5, pp. 24, 25 and Chap. 10]. It is the generalization for arbitrary l_1, l_2, l_3 of the vector triple product $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$, for which $l_1 = l_2 = l_3 = 1$. We define

$$S = \sum_{m_1,m_2,m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} Y_{l_1m_1}(\theta_1, \phi_1) Y_{l_2m_2}(\theta_2, \phi_2) Y_{l_3m_3}(\theta_3, \phi_3)$$
$$\equiv [Y^{[l_1]}(\theta_1, \phi_1) \times Y^{[l_2]}(\theta_2, \phi_2) \times Y^{[l_3]}(\theta_3, \phi_3)]^{[0]}. \tag{23}$$

Its value depends only on the relative angles of the triangle; it is independent of the overall orientation of the triangle as determined by ϕ_2, θ_3 and ϕ_3. To see this, we express the functions Y_{lm} in an arbitrary fixed reference frame in terms of Y_{lm} in the coordinate system used for the first three integrations and the matrix elements of the rotation (ω) which takes the fixed frame into the latter frame:

$$Y_{l_1m_1}(\theta_1, \phi_1) = \sum_{m_1'} Y_{l_1m_1'}(\theta_{13}, \phi_{12}) \mathscr{D}^{(l_1)}_{m_1'm_1}(\omega),$$
$$Y_{l_2m_2}(\theta_2, \phi_2) = \sum_{m_2'} Y_{l_2m_2'}(\theta_{23}, 0) \mathscr{D}^{(l_2)}_{m_2'm_2}(\omega), \tag{24}$$
$$Y_{l_3m_3}(\theta_3, \phi_3) = \sum_{m_3'} Y_{l_3m_3'}(0, 0) \mathscr{D}^{(l_3)}_{m_3'm_3}(\omega).$$

The sum (23) can thus be written

$$S = \sum_{m_1',m_2',m_3'} Y_{l_1m_1'}(\theta_{13}, \phi_{12}) Y_{l_2m_2'}(\theta_{23}, 0) Y_{l_3m_3'}(0, 0) \cdot \sum_{m_1,m_2,m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \mathscr{D}^{(l_1)}_{m_1'm_1}(\omega) \mathscr{D}^{(l_2)}_{m_2'm_2}(\omega) \mathscr{D}^{(l_3)}_{m_3'm_3}(\omega). \tag{25}$$

As shown by Edmonds [6], the sum over m_1, m_2, m_3 can be performed:

$$\sum_{m_1,m_2,m_3} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \mathscr{D}^{(l_1)}_{m_1'm_1}(\omega) \mathscr{D}^{(l_2)}_{m_2'm_2}(\omega) \mathscr{D}^{(l_3)}_{m_3'm_3}(\omega) = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1' & m_2' & m_3' \end{pmatrix} \tag{26}$$

and is independent of (ω). Using the fact that

$$Y_{l_3m_3}(0, 0) = \left[\frac{2l_3 + 1}{4\pi}\right]^{1/2} \delta_{m_30} \tag{27}$$

we find the sum S becomes

$$S = \left[\frac{2l_3 + 1}{4\pi}\right]^{1/2} \sum_m Y_{l_1m}(\theta_{13}, \phi_{12}) Y_{l_2-m}(\theta_{23}, 0) \begin{pmatrix} l_1 & l_2 & l_3 \\ m & -m & 0 \end{pmatrix}. \tag{28}$$

Thus we can see that the integrand in (21) is a function only of the relative angles θ_{13}, ϕ_{12} and θ_{23}, whose values have been fixed by the delta function. It is clearly independent of the angles ϕ_2, θ_3 and ϕ_3. The invariant triple product may thus be removed from under the integral and evaluated by (28), or in any other convenient coordinate system. The remaining integral,

$$\int d\phi_2 \, d\Omega_3,$$

is now trivial and equal to $8\pi^2$.

Thus we obtain the final result

$$\int_0^\infty j_{l_1}(k_1r) j_{l_2}(k_2r) j_{l_3}(k_3r) r^2 \, dr = \frac{\pi^2 \Delta}{k_1k_2k_3} \left[\frac{4\pi}{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}\right]^{1/2} \frac{i^{-(l_1+l_2+l_3)}}{\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}} \cdot [Y^{[l_1]}(\theta_1, \phi_1) \times Y^{[l_2]}(\theta_2, \phi_2) \times Y^{[l_3]}(\theta_3, \phi_3)]^{[0]}. \tag{29}$$

This result may be expressed as a sum of associated Legendre functions by rewriting the invariant triple product with the aid of (28) and the definition of the spherical harmonics

$$Y_{lm}(\theta, \phi) = (-1)^m \left[\frac{(2l + 1)(l - m)!}{4\pi(l + m)!}\right]^{1/2} P_l^m(\cos\theta) \, e^{im\phi}. \tag{30}$$

We then obtain

$$\int j_{l_1}(k_1r)j_{l_2}(k_2r)j_{l_3}(k_3r)r^2\,dr$$

(31)
$$= \frac{\pi\Delta}{4k_1k_2k_3}\frac{i^{-(l_1+l_2+l_3)}}{\begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0\end{pmatrix}}\sum_m(-1)^m\left[\frac{(l_1-m)!(l_2+m)!}{(l_1+m)!(l_2-m)!}\right]^{1/2}\begin{pmatrix} l_1 & l_2 & l_3 \\ m & -m & 0\end{pmatrix}$$
$$\cdot P_{l_1}^m(\cos\theta_{13})P_{l_2}^{-m}(\cos\theta_{23}).$$

Since the terms in this sum are nonzero only when $|m| \leqq l_1$ and $|m| \leqq l_2$, the simplest expression in terms of associated Legendre functions will result when l_3 is chosen to be the largest of l_1, l_2, l_3. A recursive scheme for obtaining (29) in terms of Legendre polynomials is described in the Appendix.

2. Cylindrical Bessel functions. The essential elements of the derivation given in the case of three dimensions may be followed with two dimensions, resulting in the evaluation of the integral of the product of three cylindrical Bessel functions:

(32)
$$I = \int_0^\infty J_{n_1}(k_1\rho)J_{n_2}(k_2\rho)J_{n_3}(k_3\rho)\rho\,d\rho.$$

The principal steps and comparison with the previous discussion follow.

Starting with the generating function for the cylindrical Bessel functions [1, (1), p. 14]

(33)
$$e^{z(t-1/t)/2} = \sum_{n=-\infty}^{\infty} t^n J_n(z)$$

we obtain, with $t = i\,e^{i\theta}$,

(34)
$$e^{iz\cos\theta} = \sum_{n=-\infty}^{\infty} i^n\,e^{in\theta}J_n(z).$$

Now if we consider two 2-dimensional vectors, $\mathbf{k} = (k, \theta_k)$ and $\boldsymbol{\rho} = (\rho, \theta_\rho)$, then (34) gives the decomposition of a plane wave into cylindrical Bessel functions (cf. (6)):

(35)
$$e^{i\mathbf{k}\cdot\boldsymbol{\rho}} = \sum_{n=-\infty}^{\infty} i^n J_n(k\rho)\,e^{-in\theta_k}\,e^{in\theta_\rho}.$$

Multiplying both sides of (35) by $e^{in'\theta_k}$ and integrating over θ_k, one obtains (cf. (7))

(36)
$$J_n(k\rho)\,e^{in\theta_\rho} = \frac{1}{2\pi}i^{-n}\int_0^{2\pi} e^{i\mathbf{k}\cdot\boldsymbol{\rho}}\,e^{in\theta_k}\,d\theta_k.$$

From (36) we may write (cf. (8))

(37)
$$\int_0^\infty J_{n_1}(k_1\rho)J_{n_2}(k_2\rho)J_{n_3}(k_3\rho)\rho\,d\rho\int_0^{2\pi} e^{in_1\theta_\rho}\,e^{in_2\theta_\rho}\,e^{in_3\theta_\rho}\,d\theta_\rho$$
$$= \frac{1}{(2\pi)^3}i^{-(n_1+n_2+n_3)}\int e^{i(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\cdot\boldsymbol{\rho}}\,e^{in_1\theta_{k_1}}\,e^{in_2\theta_{k_2}}\,e^{in_3\theta_{k_3}}$$
$$\cdot\,d\theta_{k_1}\,d\theta_{k_2}\,d\theta_{k_3}\,d^2\rho.$$

The integral over θ_ρ on the left-hand side of (37) gives

$$\int_0^{2\pi} e^{i(n_1+n_2+n_3)\theta_\rho}\, d\theta_\rho = 2\pi\delta_{n_1+n_2+n_3,0}, \tag{38}$$

a far more restrictive condition on the order of the Bessel functions than that obtained in (9) in the case of three dimensions. We next perform the ρ-integration on the right-hand side of (37), which gives a two-dimensional delta function:

$$\int e^{i\mathbf{k}\cdot\boldsymbol{\rho}}\, d^2\rho = (2\pi)^2\delta^{(2)}(\mathbf{k}). \tag{39}$$

The remaining integral is then

$$J = \int e^{in_1\theta_{k_1}}\, e^{in_2\theta_{k_2}}\, e^{in_3\theta_{k_3}}\, \delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_3 + \mathbf{k}_3)\, d\theta_{k_1}\, d\theta_{k_2}\, d\theta_{k_3}. \tag{40}$$

Here there are three integrations to be performed, one for each of the vectors $\mathbf{k}_1$, $\mathbf{k}_2, \mathbf{k}_3$. Two of these integrations should be made so as to exploit the delta function, which serves to fix the angles between $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$ so that they form a triangle: $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$. This implies that if the lengths k_1, k_2 and k_3 are such that they cannot form a triangle, the integral (32) is zero provided that n_1, n_2, n_3 satisfy the condition $n_1 + n_2 + n_3 = 0$. This conclusion does not hold if this last condition is not satisfied. The remaining integral represents a rotation of the triangle as a whole. For the purpose of the first two integrations we choose a coordinate system with x-axis in the direction of $\mathbf{k}_3$. Denoting the angles of $\mathbf{k}_1$ and $\mathbf{k}_2$ relative to $\mathbf{k}_3$ by θ_{13} and θ_{23}, respectively, the delta function can be written in this set of Cartesian coordinates as

$$\delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) = \delta(k_1\cos\theta_{13} + k_2\cos\theta_{23} + k_3)\delta(k_1\sin\theta_{13} + k_2\sin\theta_{23}). \tag{41}$$

Again by following the procedure used in the three-dimensional case, integrating first over θ_{13} using $\delta(k_1\cos\theta_{13} + k_2\cos\theta_{23} + k_3)$ fixes the values of θ_{13} at

$$\cos\theta_{13} = -\frac{(k_2\cos\theta_{23} + k_3)}{k_1} \tag{42}$$

and introduces the factor $1/(k_1|\sin\theta_{13}|)$. We next integrate over θ_{23} using $\delta(k_1\sin\theta_{13} + k_2\sin\theta_{23})$, which fixes the value of θ_{23} at

$$\sin\theta_{23} = -\frac{k_1\sin\theta_{13}}{k_2} \tag{43}$$

and introduces the factor $1/|k_1\cos\theta_{13} + k_2\cos\theta_{23}| = 1/k_3$. (Note, from (42) and (43), that $d(\sin\theta_{13})/d\theta_{23} = \cos\theta_{13}$.) The overall factor introduced by the first two integrations is thus

$$(k_1k_3\sin\theta_{13})^{-1} = (2A)^{-1}, \tag{44}$$

where A is the area of the triangle defined by $\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0$.

Now in order to perform the remaining integral over θ_{k_3}, we must express θ_{k_1} and θ_{k_2} in (40) in terms of θ_{k_3} (and k_1, k_2 and k_3) using (42) and (43). However, these latter conditions define two distinct triangles, denoted by I and II in Fig. 1, and both must be included.[1]

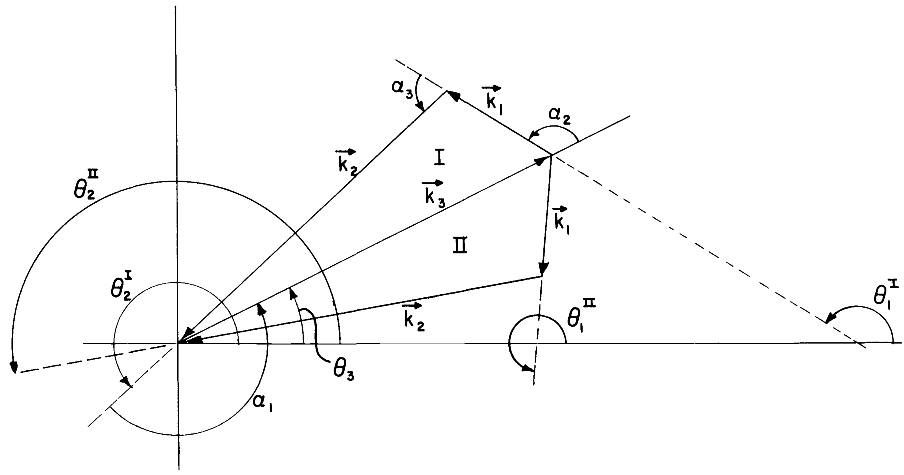

FIG. 1. *The two triangles with sides k_1, k_2, k_3 formed by the delta function $\delta^{(2)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)$ in the two-dimensional case. Note that these two triangles are not related by a rotation in two-dimensional space.*

In I,

$$\theta_1 = \theta_1^{\mathrm{I}} = \theta_3 + \alpha_2, \qquad \theta_2 = \theta_2^{\mathrm{I}} = \theta_3 + 2\pi - \alpha_1. \tag{45}$$

In II,

$$\theta_1 = \theta_1^{\mathrm{II}} = \theta_3 + 2\pi - \alpha_2, \qquad \theta_2 = \theta_2^{\mathrm{II}} = \theta_3 + \alpha_1, \tag{46}$$

the exterior angles $\alpha_1, \alpha_2, \alpha_3$ being given uniquely by the lengths of the sides k_1, k_2, k_3. The integral (40) may thus be written as

$$\begin{aligned} J &= \tfrac{1}{2}A^{-1}\, e^{in_1\alpha_2 + in_2(2\pi - \alpha_1)} \int_0^{2\pi} e^{i(n_1+n_2+n_3)\theta_3}\, d\theta_3 \\ &\quad + \tfrac{1}{2}A^{-1}\, e^{in_1(2\pi-\alpha_2) + in_2\alpha_1} \int_0^{2\pi} e^{i(n_1+n_2+n_3)\theta_3}\, d\theta_3 \\ &= \frac{2\pi}{A}\cos(n_1\alpha_2 - n_2\alpha_1) \end{aligned} \tag{47}$$

since $n_1 + n_2 + n_3 = 0$. An evaluation of the integral (32) using group-theoretic methods, leading to the result given by the right-hand side of (47), has been given by Sharp [9]. Note, in particular, p. 305, Eq. (6.8). He relates this integral to a more

---

[1] This point seems to have been overlooked in an alternative derivation given in [8], leading to the complex expression given [8, (12), p. 220]. The correct result, given below in (48), is the real part of that expression.

general integral over the product of three matrix elements of irreducible unitary group representations [9, (3.9), p. 89]. The expression (47) is in fact invariant under the permutation of any two subscripts ($n_1 \rightleftarrows n_3$ and $\alpha_1 \rightleftarrows \alpha_3$, for example), as it should be considering the original integral (32). This may be made manifest if we note, in view of $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$ and $n_1 + n_2 + n_3 = 0$, that

$$\begin{aligned} n_1\alpha_2 - n_2\alpha_1 &= n_2\alpha_3 - n_3\alpha_2 - 2\pi n_2 \\ &= n_3\alpha_1 - n_1\alpha_3 + 2\pi n_1 . \end{aligned}$$

One may, therefore, write the integral (32) in the manifestly invariant form (with Δ defined in (22)),

$$\begin{aligned} &\int_0^\infty J_{n_1}(k_1\rho)J_{n_2}(k_2\rho)J_{n_3}(k_3\rho)\rho\, d\rho \\ &= \frac{\Delta}{6\pi A}[\cos(n_1\alpha_2 - n_2\alpha_1) + \cos(n_2\alpha_3 - n_3\alpha_2) + \cos(n_3\alpha_1 - n_1\alpha_3)], \end{aligned} \tag{48}$$

where $n_1 + n_2 + n_3 = 0$.

3. Extension to other integrals. We may use the result of (29) in order to evaluate integrals of the form

$$\int_0^\infty j_{l_1}(k_1r)j_{l_2}(k_2r)j_{l_3}(k_3r)r^{2-M}\, dr,$$

where M is an integer such that

$$\begin{aligned} 0 \leqq M &\leqq l_1 + l_2 + l_3 - 2l_M, \\ l_M &= \max(l_1, l_2, l_3). \end{aligned}$$

These integrals may be expressed as linear combinations of integrals of the form of (29) with the aid of the recurrence relation for spherical Bessel functions

$$j_{l-1}(kr) + j_{l+1}(kr) = \frac{(2l+1)}{kr} j_l(kr).$$

No similar extension may be made of the result (48) for cylindrical Bessel functions due to the highly restrictive nature of the condition $n_1 + n_2 + n_3 = 0$.

The results of this paper may also be generalized by analogous derivations in higher dimensional spaces. Consideration of the form of the expansion of a plane wave in higher dimensional spaces [1, pp. 128–130, 363, 368]; [10]–[12]; [13, Appendix IV, pp. 227–235]; [14, Chap. XI, pp. 232–263] suggests that such an analysis will again lead to the product of three Bessel functions, of integer order (for even dimensions) or half-integer order (for odd dimensions), with conditions on the order of the Bessel functions less restrictive than those imposed in (29) and (48).

Appendix. In our principal result (29), we have shown that the integral over the product of three spherical Bessel functions may be expressed in terms of the invariant triple product, which is a sum of products of three associated Legendre functions. We now show that one can in fact express the integral in (29) in terms of products of three ordinary Legendre polynomials. We begin these considerations by writing a single product of three Legendre polynomials, the arguments of which are the cosines of the external angles of the triangle formed by $\mathbf{k}_1$, $\mathbf{k}_2$ and $\mathbf{k}_3$, as a sum over scalar triple products. We will then construct a recursive scheme by which this sum may be inverted, thus expressing the single invariant triple product which appears in (29) in terms of a sum of products of Legendre polynomials.

We start with the addition theorem for Legendre polynomials [6, (4.6.6), p. 63] written for each of the exterior angles of the triangle:

$$\begin{aligned} P_{l_1}(\cos\theta_{23}) &= \frac{4\pi}{2l_1+1}\sum_{m_1} Y^*_{l_1m_1}(\hat{k}_2)Y_{l_1m_1}(\hat{k}_3), \\ P_{l_2}(\cos\theta_{13}) &= \frac{4\pi}{2l_2+1}\sum_{m_2} Y^*_{l_2m_2}(\hat{k}_3)Y_{l_2m_2}(\hat{k}_1), \\ P_{l_3}(\cos\theta_{12}) &= \frac{4\pi}{2l_3+1}\sum_{m_3} Y^*_{l_3m_3}(\hat{k}_1)Y_{l_3m_3}(\hat{k}_2). \end{aligned} \tag{A.1}$$

(The indices here, l_1, l_2 and l_3, are arbitrary; they are not necessarily the same as those appearing in (29).)

Multiplying these three equations gives, on the right-hand side, products of spherical harmonics which may be grouped in pairs having the same angles for arguments. For these pairs we may use the expansion [6, (4.6.5), p. 63]

$$\begin{aligned} Y_{l_1m_1}(\hat{k})Y_{l_2m_2}(\hat{k}) = \sum_{l,m}&\left[\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}\right]^{1/2} \\ &\cdot\begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m\end{pmatrix} Y^*_{lm}(\hat{k})\begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0\end{pmatrix}. \end{aligned} \tag{A.2}$$

In both (A.1) and (A.2) we may interchange the function Y_{lm} and its complex conjugate Y^*_{lm}, using [6, (2.5.6), p. 21]

$$Y^*_{lm}(\hat{k}) = (-1)^m Y_{l\,-m}(\hat{k}). \tag{A.3}$$

We then have

$$\begin{aligned} P_{l_1}(\cos\theta_{23})P_{l_2}(\cos\theta_{13})P_{l_3}(\cos\theta_{12}) = (4\pi)^{3/2} &\sum_{\substack{m_1,m_2,m_3\\ l'_1,l'_2,l'_3\\ m'_1,m'_2,m'_3}} (-1)^{m_1+m_2+m_3+m'_1+m'_2+m'_3} \\ &\cdot[(2l'_1+1)(2l'_2+1)(2l'_3+1)]^{1/2}\begin{pmatrix} l_2 & l_3 & l'_1 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_3 & l_1 & l'_2 \\ 0 & 0 & 0\end{pmatrix} \\ &\cdot\begin{pmatrix} l_1 & l_2 & l'_3 \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l_2 & l_3 & l'_1 \\ m_2 & -m_3 & m'_1\end{pmatrix}\begin{pmatrix} l_3 & l_1 & l'_2 \\ m_3 & -m_1 & m'_2\end{pmatrix}\begin{pmatrix} l_1 & l_2 & l'_3 \\ m_1 & -m_2 & m'_3\end{pmatrix} \\ &\cdot Y_{l'_1\,-m'_1}(\hat{k}_1)Y_{l'_2\,-m'_2}(\hat{k}_2)Y_{l'_3\,-m'_3}(\hat{k}_3). \end{aligned} \tag{A.4}$$

We now make use of the symmetry properties of the 3-j coefficients [6, (3.7.4), p. 46] and write

(A.5)
$$\begin{pmatrix} l_2 & l_3 & l'_1 \\ m_2 & -m_3 & m'_1 \end{pmatrix} = \begin{pmatrix} l'_1 & l_2 & l_3 \\ m'_1 & m_2 & -m_3 \end{pmatrix},$$
$$\begin{pmatrix} l_3 & l_1 & l'_2 \\ m_3 & -m_1 & m'_2 \end{pmatrix} = \begin{pmatrix} l_1 & l'_2 & l_3 \\ -m_1 & m'_2 & m_3 \end{pmatrix}.$$

In (A.4), the sum over m_1, m_2, m_3 may now be performed, which gives a result involving the 6-j coefficient [6, (6.2.8), p. 95]

(A.6)
$$\sum_{m_1,m_2,m_3} (-1)^{m_1+m_2+m_3+l_1+l_2+l_3} \begin{pmatrix} l'_1 & l_2 & l_3 \\ m'_1 & m_2 & -m_3 \end{pmatrix} \begin{pmatrix} l_1 & l'_2 & l_3 \\ -m_1 & m'_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l'_3 \\ m_1 & -m_2 & m'_3 \end{pmatrix}$$
$$= \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix} \begin{Bmatrix} l'_1 & l'_2 & l'_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}.$$

Substituting (A.6) in (A.4), we replace m'_1, m'_2, m'_3 by $-m'_1$, $-m'_2$, $-m'_3$ and then use [6, (3.7.6), p. 47]

(A.7)
$$\begin{pmatrix} l'_1 & l'_2 & l'_3 \\ -m'_1 & -m'_2 & -m'_3 \end{pmatrix} = (-1)^{-l'_1-l'_2-l'_3} \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}.$$

We note further, in (A.4), that in view of the first three 3-j coefficients appearing there, we have that $l'_1 + l'_2 + l'_3$ is an even integer (see [6, (3.7.14), p. 49]); and in view of the last three 3-j coefficients there, we have $m'_1 + m'_2 + m'_3 = 0$. Thus (A.4) can now be written in the form

(A.8)
$$P_{l_1}(\cos\theta_{23})P_{l_2}(\cos\theta_{13})P_{l_3}(\cos\theta_{12})$$
$$= (4\pi)^{3/2}(-1)^{l_1+l_2+l_3} \sum_{\substack{l'_1,l'_2,l'_3 \\ m'_1,m'_2,m'_3}} [(2l'_1+1)(2l'_2+1)(2l'_3+1)]^{1/2}$$
$$\cdot \begin{pmatrix} l'_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l'_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l'_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l'_1 & l'_2 & l'_3 \\ m'_1 & m'_2 & m'_3 \end{pmatrix}$$
$$\cdot \begin{Bmatrix} l'_1 & l'_2 & l'_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} Y_{l'_1m'_1}(\hat{k}_1)Y_{l'_2m'_2}(\hat{k}_2)Y_{l'_3m'_3}(\hat{k}_3).$$

The sum over m'_1, m'_2, m'_3 gives the invariant triple product defined in (23).

In (A.8) we have expressed a single product of three Legendre polynomials as a sum over invariant triple products. For the purpose of inverting this sum, it is convenient to write it in the more compact form

(A.9)
$$(P_{\nu_1} \quad P_{\nu_2} \quad P_{\nu_3}) = \sum_{l_1,l_2,l_3} C^{l_1l_2l_3}_{\nu_1\nu_2\nu_3}[Y_{l_1} \quad Y_{l_2} \quad Y_{l_3}],$$

where

(A.10)
$$(P_{\nu_1} \quad P_{\nu_2} \quad P_{\nu_3}) \equiv P_{\nu_1}(\cos\theta_{23})P_{\nu_2}(\cos\theta_{13})P_{\nu_3}(\cos\theta_{12}),$$

(A.11)
$$[Y_{l_1} \quad Y_{l_2} \quad Y_{l_3}] \equiv [Y^{[l_1]}(\hat{k}_1) \times Y^{[l_2]}(\hat{k}_2) \times Y^{[l_3]}(\hat{k}_3)]^{[0]}$$

and

$$C^{l_1 l_2 l_3}_{\nu_1 \nu_2 \nu_3} = (4\pi)^{3/2}(-1)^{\nu_1+\nu_2+\nu_3}[(2l_1+1)(2l_2+1)(2l_3+1)]^{1/2}$$

$$\cdot \begin{pmatrix} l_1 & \nu_2 & \nu_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \nu_1 & l_2 & \nu_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \nu_1 & \nu_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{Bmatrix} l_1 & l_2 & l_3 \\ \nu_1 & \nu_2 & \nu_3 \end{Bmatrix}. \tag{A.12}$$

We now wish to invert (A.9), expressing a single term $[Y_{L_1} \quad Y_{L_2} \quad Y_{L_3}]$ as a sum of terms of the form appearing in (A.10) for L_1, L_2 and L_3 which satisfy the triangularity condition ($|L_1 - L_2| \leqq L_3 \leqq L_1 + L_2$), and such that $L_1 + L_2 + L_3$ is an even integer.

We define

$$\begin{aligned} \lambda_1 &= \tfrac{1}{2}(L_2 + L_3 - L_1), \\ \lambda_2 &= \tfrac{1}{2}(L_3 + L_1 - L_2), \\ \lambda_3 &= \tfrac{1}{2}(L_1 + L_2 - L_3), \\ L &= L_1 + L_2 + L_3 \end{aligned} \tag{A.13}$$

and $l = l_1 + l_2 + l_3$ for any set of (l_1, l_2, l_3) in the sum (A.9). We may then write, in particular,

$$(P_{\lambda_1} \quad P_{\lambda_2} \quad P_{\lambda_3}) = \sum_{l_1, l_2, l_3} C^{l_1 l_2 l_3}_{\lambda_1 \lambda_2 \lambda_3} [Y_{l_1} \quad Y_{l_2} \quad Y_{l_3}]. \tag{A.14}$$

In (A.14), the largest value that each of the summation indices l_1, l_2 and l_3 can attain is determined by the 6-j coefficient in (A.12), viz., by using (A.13),

$$\begin{aligned} l_1 &= \lambda_2 + \lambda_3 = L_1, \\ l_2 &= \lambda_3 + \lambda_1 = L_2, \\ l_3 &= \lambda_1 + \lambda_2 = L_3; \end{aligned} \tag{A.15}$$

and, since L_1, L_2, L_3 satisfy the triangularity condition and $L_1 + L_2 + L_3$ is an even integer, the coefficient $C^{l_1 l_2 l_3}_{\lambda_1 \lambda_2 \lambda_3}$ is nonzero for these values of l_1, l_2 and l_3. For any other terms in the sum (A.14), at least one of the summation indices l_1, l_2, l_3 is smaller than the values given in (A.15), and hence $l = l_1 + l_2 + l_3 < L = L_1 + L_2 + L_3$ for all other terms in (A.14). Thus we may write (A.14) in the form

$$C^{L_1 L_2 L_3}_{\lambda_1 \lambda_2 \lambda_3}[Y_{L_1} \quad Y_{L_2} \quad Y_{L_3}] = (P_{\lambda_1} \quad P_{\lambda_2} \quad P_{\lambda_3}) - \sum_{\substack{l_1, l_2, l_3 \\ l < L}} C^{l_1 l_2 l_3}_{\lambda_1 \lambda_2 \lambda_3}[Y_{l_1} \quad Y_{l_2} \quad Y_{l_3}]. \tag{A.16}$$

In (A.16) we have an integral equation for $[Y_{L_1} \quad Y_{L_2} \quad Y_{L_3}]$ which we may solve recursively and obtain $[Y_{L_1} \quad Y_{L_2} \quad Y_{L_3}]$ as a sum of terms of the form $(P_{\nu_1} \quad P_{\nu_2} \quad P_{\nu_3})$: Each of the elements $[Y_{l_1} \quad Y_{l_2} \quad Y_{l_3}]$ on the right-hand side of (A.16) is of the same character as the single term $[Y_{L_1} \quad Y_{L_2} \quad Y_{L_3}]$ in that any set of indices l_1, l_2, l_3 satisfies the triangularity condition (which follows from the 6-j coefficient in (A.12)) and moreover $l_1 + l_2 + l_3$ is an even integer (which follows from the three 3-j coefficients in (A.12)). Now the largest value of l in the sum in (A.15) is $l = L - 2$, and there are at most three terms for which $l = L - 2$: ($l_1 = L_1$,

$l_2 = L_2, l_3 = L_3 - 2), (l_1 = L_1, l_2 = L_2 - 2, l_3 = L_3)$, and $(l_1 = L_1 - 2, l_2 = L_2, l_3 = L_3)$. If one or more of the indices L_1, L_2, L_3 is zero, then there are clearly fewer than three terms for which $l = L - 2$. In either case, for each of the terms for which $l = L - 2$ we can again substitute (A.16), where now in place of (A.13) we write

$$\lambda_1 = \tfrac{1}{2}(l_2 + l_3 - l_1), \quad \lambda_2 = \tfrac{1}{2}(l_3 + l_1 - l_2), \quad \lambda_3 = \tfrac{1}{2}(l_1 + l_2 - l_3) \tag{A.17}$$

for each of the set of indices l_1, l_2, l_3. We are now left with a sum of the same form as that in (A.16), except that now $l \leqq L - 4$. This process may be continued until we come to a term on the right-hand side for which one of the indices l_1, l_2, l_3 is zero. Then, since we have already seen that all sets of indices l_1, l_2, l_3 in (A.16) obey the triangularity condition, the other two indices l must be equal, and from (A.17) two of the λ's are zero. In that case the sum in (A.16) has only one term, and we have simply (taking, for example, $l_1 = l_2, l_3 = 0$),

$$C_{00l_1}^{l_1 l_1 0}[Y_{l_1} \quad Y_{l_1} \quad Y_0] = (P_0 \quad P_0 \quad P_{l_1}). \tag{A.18}$$

The recursive process thus ends at this point; and when it has ended for each of the terms on the right-hand side of (A.16) we have expressed $[Y_{L_1} \quad Y_{L_2} \quad Y_{L_3}]$ completely in terms of the $(P_{v_1} \quad P_{v_2} \quad P_{v_3})$.

We note that in the course of the recursion process, whenever we express $[Y_{l_1} \quad Y_{l_2} \quad Y_{l_3}]$ in terms of $(P_{v_1} \quad P_{v_2} \quad P_{v_3})$ and a sum, as written in (A.16), the coefficient $C_{\lambda_1\lambda_2\lambda_3}^{l_1 l_2 l_3}$ appearing on the left-hand side is always "fully stretched," i.e., the indices satisfy the equations $l_1 = \lambda_2 + \lambda_3$, $l_2 = \lambda_3 + \lambda_1$, $l_3 = \lambda_1 + \lambda_2$. These coefficients have a particularly simple form: From [6, (6.3.1), p. 97] and [6, (3.7.17), p. 50],

$$\begin{Bmatrix} l_1 & l_2 & l_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{Bmatrix} = (-1)^{2(\lambda_1+\lambda_2+\lambda_3)}\left[\frac{(2\lambda_1)!(2\lambda_2)!(2\lambda_3)!(2\lambda_1 + 2\lambda_2 + 2\lambda_3 + 1)!}{(2\lambda_1 + 2\lambda_2 + 1)!(2\lambda_2 + 2\lambda_3 + 1)!(2\lambda_3 + 2\lambda_1 + 1)!}\right]^{1/2} \tag{A.19}$$

and

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_1 + \lambda_2 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{\lambda_1+\lambda_2}\left[\frac{(2\lambda_1)!(2\lambda_2)!}{(2\lambda_1 + 2\lambda_2 + 1)!}\right]\frac{(\lambda_1 + \lambda_2)!}{\lambda_1!\lambda_2!} \tag{A.20}$$

and hence (A.12) becomes

$$C_{\lambda_1\lambda_2\lambda_3}^{l_1 l_2 l_3} = \frac{(4\pi)^{3/2}(-1)^{l/2}l_1!l_2!l_3![(l+1)!]^{1/2}}{(2l_1 + 1)!(2l_2 + 1)!(2l_3 + 1)!} \cdot \frac{[(2\lambda_1)!(2\lambda_2)!(2\lambda_3)!]^{3/2}[(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)]^{1/2}}{[\lambda_1!\lambda_2!\lambda_3!]^2}, \tag{A.21}$$

where $l = l_1 + l_2 + l_3$, and $\lambda_1, \lambda_2, \lambda_3$ are given in terms of l_1, l_2, l_3 by (A.17).

Finally, it should be noted that the decomposition of a given $[Y_{l_1} \quad Y_{l_2} \quad Y_{l_3}]$ into a sum of terms of the form $(P_{\nu_1} \quad P_{\nu_2} \quad P_{\nu_3})$ is not unique, the angles $\theta_{12}, \theta_{23}, \theta_{13}$ being related by $\theta_{12} + \theta_{23} + \theta_{13} = 2\pi$. Thus, for example,

$$3(P_1 \quad P_1 \quad P_1) = (P_2 \quad P_0 \quad P_0) + (P_0 \quad P_2 \quad P_0) + (P_0 \quad P_0 \quad P_2) \tag{A.22}$$

so that, with $l_1 = l_2 = l_3 = 2$, we have

$$\begin{aligned} &\int_0^\infty j_2(k_1 r) j_2(k_2 r) j_2(k_3 r) r^2 \, dr \\ &= \frac{\pi\Delta}{8k_1k_2k_3}\{-3\alpha P_1(\cos\theta_{23})P_1(\cos\theta_{13})P_1(\cos\theta_{12}) \\ &\quad - (1-\alpha)[P_2(\cos\theta_{23}) + P_2(\cos\theta_{13}) + P_2(\cos\theta_{12})] + 1\}, \end{aligned} \tag{A.23}$$

where α is an arbitrary constant.

Acknowledgment. We would like to thank L. C. Biedenharn for bringing the references [11]–[14] to our attention.

REFERENCES

[1] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge, 1944.
[2] ———, *An infinite integral involving Bessel functions*, J. London Math. Soc., 9 (1934), pp. 16–22.
[3] W. N. BAILEY, *Some infinite integrals involving Bessel functions*, Proc. London Math. Soc., 2nd Ser., 40 (1936), pp. 37–48.
[4] M. DANOS, *Short range correlations*, Nuclear Physics, C. DeWitt and V. Gillet, eds., Gordon and Breach, New York, 1969, pp. 543–584.
[5] H. M. MACDONALD, *Note on the evaluation of a certain integral containing Bessel's functions*, Proc. London Math. Soc., 2nd Ser., 7 (1909), pp. 142–149.
[6] A. R. EDMONDS, *Angular Momentum in Quantum Mechanics*, 2nd ed., Princeton University Press, Princeton, 1960.
[7] U. FANO AND G. RACAH, *Irreducible Tensorial Sets*, Academic Press, New York, 1959.
[8] N. YA. VILENKIN, *Special functions and the theory of group representations*, Translations of Mathematical Monographs 22, American Mathematical Society, Providence, 1968.
[9] W. T. SHARP, *Racah algebra and the contraction of groups*, CRT-935, Atomic Energy of Canada Limited, Chalk River, Ontario, 1960.
[10] E. W. HOBSON, *On Bessel's functions, and relations connecting them with hyper-spherical and spherical harmonics*, Proc. London Math. Soc., 25 (1894), pp. 49–75.
[11] V. FOCK, *Zur Theorie des Wasserstoffatoms*, Z. Physik, 98 (1936), pp. 145–154.
[12] A. SOMMERFIELD, *Die ebene und sphärische Welle in polydimensionalen Raum*, Math. Ann., 119 (1943), pp. 1–20.
[13] ———, *Partial Differential Equations in Physics*, Academic Press, New York, 1949.
[14] H. BATEMAN, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

SERIES EXPANSIONS OF SOLUTIONS OF $U_{xx} + U_{yy} + \varepsilon^2 U_{tt} = U_t$*

CHI YEUNG LO†

Abstract. In this paper, the properties of a set of polynomial solutions of the equation

$$U_{xx} + U_{yy} + \varepsilon^2 U_{tt} = U_t$$

are discussed, where ε is a parameter. As $\varepsilon \to 0$, this set of polynomials reduces to the corresponding heat polynomials. Necessary and sufficient conditions are developed to enable any solution to be expanded in series of this set of polynomials.

1. Introduction. In a recent paper [3], this author discussed the properties of a set of polynomial solutions of the equation

$$\frac{\partial^2 U}{\partial X^2} + \varepsilon^2 \frac{\partial^2 U}{\partial t^2} = \frac{\partial U}{\partial t} \tag{1.1}$$

defined by

$$U_n(X, t, \varepsilon) = X^n + n! \sum_{k=1}^{[n/2]} \frac{X^{n-2k}}{(n-2k)!k!} \sum_{m=0}^{k-1} \frac{(k+m-1)!}{m!(k-m-1)!} t^{k-m} \varepsilon^{2m}, \tag{1.2}$$

where ε is a parameter. As $\varepsilon \to 0$, this set of polynomials reduces to the heat polynomials whose properties were studied in a paper of Rosenbloom and Widder [5]. The results obtained in [3] agree, in general, with those in [5] as $\varepsilon \to 0$.

In this paper we shall consider the polynomial solutions $\{U_{mn}(x, y, t; \varepsilon)\}$ of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \varepsilon^2 \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial t}. \tag{1.3}$$

Our approach will be the same as in [3] except for one major difference. In the two-dimensional case, the concept of conformal mapping is used, whereas in this case, we have to use different methods. In § 3, the generating function for the set of polynomial functions $\{u_{mn}(x, y, t; \varepsilon)\}$ is obtained by the application of the uniqueness theorem (i.e., Theorem 3.1). Similarly, in § 4, the uniqueness theorem also provides a sufficient condition for the polynomial expansions of any solution of (1.3). Our approach is general enough to be applied to the $(n + 1)$-dimensional equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \varepsilon^2 \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial t}, \tag{1.4}$$

as discussed briefly in § 5.

* Received by the editors March 11, 1971, and in revised form November 18, 1971.

† Department of Mathematics, Michigan State University, East Lansing, Michigan 48823.

An interesting comparison is made throughout this paper with the series developments of the solution of the heat equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial u}{\partial t} \tag{1.5}$$

in Widder's paper [7] for $n = 2$.

2. Integral representation. Let $t = \varepsilon z$. Then (1.3) is transformed to

$$\tilde{u}_{xx} + \tilde{u}_{yy} + \tilde{u}_{zz} = \frac{1}{\varepsilon}\tilde{u}_z, \tag{2.1}$$

where $\tilde{u}(x, y, z) = \tilde{u}(x, y, t/\varepsilon) = u(x, y, t; \varepsilon)$.

Introduce a new function $w(x, y, z) = \tilde{u}(x, y, z)\, e^{-z/2\varepsilon}$. By elementary calculations, (2.1) is transformed to

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} - \lambda^2 w = 0, \tag{2.2}$$

where $\lambda = 1/2\varepsilon$.

It can be shown that Green's function of (2.2) for the upper half-space $z > 0$ is given by

$$G(x, y, z; \xi, \eta, \zeta) = \frac{1}{4\pi}\left(\frac{e^{-\lambda r}}{r} - \frac{e^{-\lambda r_1}}{r_1}\right), \tag{2.3}$$

where

$$r = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2}$$

and

$$r_1 = \sqrt{(x - \xi)^2 + (y - \eta)^2 + (z + \zeta)^2}$$

(see [6, p. 472]).

Let $w(x, y, z)$ be a regular solution of (2.2). Then the Green's representation formula [6, pp. 470–472] yields

$$w(x, y, z) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-\lambda R}(\lambda R + 1)}{R^3} w(\xi, \eta, 0)\, d\xi\, d\eta, \tag{2.4}$$

where

$$R = \sqrt{(x - \xi)^2 + (y - \eta)^2 + z^2}.$$

Define

$$\begin{aligned} H(x, y, t; \varepsilon) = {} & \frac{t}{2\pi\varepsilon}\left\{\frac{1}{2\varepsilon}\left(\frac{\varepsilon^2 x^2 + \varepsilon^2 y^2 + t^2}{\varepsilon^2}\right)^{-1} + \left(\frac{\varepsilon^2 x^2 + \varepsilon^2 y^2 + t^2}{\varepsilon^2}\right)^{-3/2}\right\} \\ & \cdot \exp\left[\frac{1}{2\varepsilon^2}\left(t - \sqrt{\varepsilon^2 x^2 + \varepsilon^2 y^2 + t^2}\right)\right]. \end{aligned} \tag{2.5}$$

Then the solution of (1.3) can be represented by

$$(2.6) \qquad u(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x - \xi, y - \eta, t; \varepsilon) u(\xi, \eta, 0)\, d\xi\, d\eta.$$

It is easily seen that

$$\lim_{\varepsilon \to 0} H(x, y, t; \varepsilon) = \frac{1}{4\pi t} \exp\left[-\frac{1}{4t}(x^2 + y^2)\right] = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}} \cdot \frac{e^{-y^2/4t}}{\sqrt{4\pi t}},$$

which agrees with the kernel in the Poisson representation of the two-dimensional heat polynomial [7, p. 300].

In order to apply the integral representation (2.6), we have to impose an additional condition on $u(\xi, \eta, 0)$ such that the improper integral converges absolutely. Set $u(\xi, \eta, 0) = f(\xi, \eta) = \exp(\alpha\xi + \beta\eta)$ in (2.6), where α and β are parameters to be determined.

Then

$$I(x, y, t; \varepsilon, \alpha, \beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x - \xi, y - \eta, t; \varepsilon) \exp(\alpha\xi + \beta\eta)\, d\xi\, d\eta$$

$$(2.7) \qquad = \frac{2t}{\pi\varepsilon} \exp\left(\frac{t}{2\varepsilon^2} + \alpha x + \beta y\right) \cdot \int_0^{\infty} \int_0^{\infty} \exp\left[\alpha\xi + \beta\eta - \frac{1}{2\varepsilon^2}\sqrt{\varepsilon^2\xi^2 + \varepsilon^2\eta^2 + t^2}\right] \cdot \left\{\frac{1}{2\varepsilon}\left(\xi^2 + \eta^2 + \frac{t^2}{\varepsilon^2}\right)^{-1} + \left(\xi^2 + \eta^2 + \frac{t^2}{\varepsilon^2}\right)^{-3/2}\right\} d\xi\, d\eta$$

The improper integral will converge absolutely if

$$\alpha\xi + \beta\eta < \frac{1}{2\varepsilon^2}\sqrt{\varepsilon^2\xi^2 + \varepsilon^2\eta^2}.$$

Hence, one sufficient condition is

$$\alpha^2 + \beta^2 < \frac{1}{4\varepsilon^2}.$$

We state this as the following lemma.

LEMMA 2.1. *The integral*

$$I(x, y, t; \varepsilon, \alpha, \beta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x - \xi, y - \eta, t)\, e^{\alpha\xi + \beta\eta}\, d\xi\, d\eta$$

converges absolutely if $\alpha^2 + \beta^2 < 1/4\varepsilon^2$.

For the existence of the solution of (1.3) in the upper half-space $t > 0$, we have the following theorem.

THEOREM 2.2. *Let $f(x, y)$ be continuous and $|f(x, y)| \leqq M \exp(\alpha|x| + \beta|y|)$ for all x and y, where M, α, β are positive constants such that $\alpha^2 + \beta^2 < 1/4\varepsilon^2$. Then*

$$u(x, y, t) = L[f] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x - \xi, y - \eta, t) f(\xi, \eta)\, d\xi\, d\eta \tag{2.8}$$

is a solution of (1.3) *with*

$$u(x, y, 0) = f(x, y), \tag{2.9}$$

where $H(x, y, t)$ is defined by (2.5).

Proof. The existence of the integral (2.8) is assumed by Lemma 2.1. It is easily verified that the function $u(x, y, t)$ satisfies (1.3) since differentiation with respect to x, y and t is justified under the integral sign. By elementary calculations, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x - \xi, y - \eta, t)\, d\xi\, d\eta = 1.$$

To show (2.9), we need to establish that

$$|u(x, y, t) - f(x, y)| = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x - \xi, y - \eta, t)(f(\xi, \eta) - f(x, y))\, d\xi\, d\eta \right|$$

can be made as small as possible within a suitable neighborhood of the point $(x, y, 0)$. Since this approach is quite standard, the details are omitted.

3. Generalized Helmholtz polynomials. Let $f(\xi, \eta) = \xi^m \eta^n$, $m + n \geqq 1$ in (2.8). Then

$$\begin{aligned} u_{mn}(x, y, t; \varepsilon) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x - \xi, y - \eta, t) \xi^m \eta^n\, d\xi\, d\eta \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(\xi, \eta, t)(x + \xi)^m (y + \eta)^n\, d\xi\, d\eta \\ &= \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} x^{m-i} y^{n-j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(\xi, \eta, t) \xi^i \eta^j\, d\xi\, d\eta \\ &= \sum_{i=0}^{[m/2]} \sum_{j=0}^{[n/2]} \binom{m}{2i} \binom{n}{2j} \frac{2t}{\pi\varepsilon} \exp(t/2\varepsilon^2) x^{m-2i} y^{n-2j} I, \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} I = \int_0^{\infty} \int_0^{\infty} & \frac{\exp[-(1/2\varepsilon)\sqrt{\xi^2 + \eta^2 + t^2/\varepsilon^2}]}{[\xi^2 + \eta^2 + t^2/\varepsilon^2]^{3/2}} \\ & \cdot \left\{ \frac{1}{2\varepsilon} \sqrt{\xi^2 + \eta^2 + t^2/\varepsilon^2} + 1 \right\} \xi^{2i} \eta^{2j}\, d\xi\, d\eta. \end{aligned} \tag{3.2}$$

For $i + j \geqq 1$, we set $\xi = r\cos\theta$, $\eta = r\sin\theta$ in (3.2). Then

$$\begin{aligned} I = \int_0^{\pi/2} \cos^{2i}\theta \sin^{2j}\theta\, d\theta \int_0^{\infty} & \frac{\exp[(1/2\varepsilon)\sqrt{r^2 + t^2/\varepsilon^2}]}{(r^2 + t^2/\varepsilon^2)^{3/2}} \\ & \cdot \left[\frac{1}{2\varepsilon} \sqrt{r^2 + t^2/\varepsilon^2} + 1 \right] r^{2i+2j+1}\, dr. \end{aligned}$$

It is easily shown that

$$I_1 = \int_0^{\pi/2} \cos^{2i} \sin^{2j} \theta \, d\theta = \frac{(2i)!(2j)!}{2^{2i+2j+1}(i+j)!i!j!}\pi.$$

To evaluate

$$I_2 = \int_0^{\infty} \frac{\exp[-(1/2\varepsilon)\sqrt{r^2 + t^2/\varepsilon^2}]}{(r^2 + t^2/\varepsilon^2)^{3/2}} \left[\frac{1}{2\varepsilon}\sqrt{r^2 + t^2/\varepsilon^2} + 1\right] r^{2i+2j+1} \, dr,$$

we set $r = (t/\varepsilon)\sinh p$.

Then by transformation and integration by parts, we obtain

$$I_2 = 2(i+j)\left(\frac{t}{\varepsilon}\right)^{2i+2j-1} \int_0^{\infty} \exp\left(-\frac{t}{2\varepsilon^2}\cosh p\right)(\sinh p)^{2i+2j-1} \, dp.$$

Using a formula on the modified Bessel function of the third kind of νth order $K_\nu(z)$, $\Gamma(1/2 + \nu)K_\nu(z) = \pi^{1/2}(z/2)^\nu \int_0^{\infty} \exp(-z\cosh t)(\sinh t)^{2\nu} \, dt$ (see [2, p. 82]) we obtain

$$I_2 = \frac{2}{\pi^{1/2}}(i+j)!(2t)^{i+j-1/2} K_{i+j-1/2}\left(\frac{t}{2\varepsilon^2}\right).$$

Since $K_{n+1/2}(z) = (\pi/2z)^{1/2} e^{-z} \sum_{m=0}^{n} (n + 1/2, m)(2z)^{-m}$ (see [2, p. 10]), then

$$I_2 = \frac{2}{\pi^{1/2}}(i+j)!(2t)^{i+j-1/2}\left(\frac{\pi\varepsilon^2}{t}\right)^{1/2} e^{-t/2\varepsilon^2} \sum_{k=0}^{i+j-1} \frac{\Gamma(i+j+k)}{k!\,\Gamma(i+j-k)}\left(\frac{t}{\varepsilon^2}\right)^{-k}.$$

Hence,

$$I = I_1 I_2 = e^{-t/2\varepsilon^2}\pi\frac{(2i)!(2j)!}{i!j!}\,\frac{(2\varepsilon)}{2^{i+j+1}} \cdot \sum_{k=0}^{i+j-1} \frac{(i+j+k-1)!}{k!(i+j-k-1)!} t^{-k}\varepsilon^{2k}$$

and

$$\begin{aligned} &u_{mn}(x, y, t; \varepsilon) \\ (3.3)\quad &= m!n! \sum_{i=0}^{[m/2]} \sum_{j=0}^{[n/2]} \frac{x^{m-2i} y^{n-2j}}{(m-2i)!(n-2j)!i!j!} \sum_{k=0}^{i+j-1} \frac{(i+j+k-1)!}{k!(i+j-k-1)!} t^{i+j-k}\varepsilon^{2k}. \end{aligned}$$

(Here we adopt the convention that whenever $i + j - 1 < 0$,

$$\sum_{k=0}^{i+j-1} \frac{(i+j+k-1)!}{k!(i+j-k-1)!} t^{i+j-k}\varepsilon^{2k} = 1.)$$

Now

$$\lim_{\varepsilon \to 0} u_{mn}(x, y, t; \varepsilon) = m!n! \sum_{i=0}^{[m/2]} \sum_{j=0}^{[n/2]} \frac{x^{m-2i} y^{n-2j}}{(m-2i)!(n-2j)!i!j!},$$

which are two-dimensional heat polynomials [7, p. 391] and, when $n = 0$,

$$u_{m0}(x, y, t; \varepsilon) = m! \sum_{i=0}^{[m/2]} \frac{x^{m-2i}}{(m-2i)!i!} \sum_{k=0}^{i-1} \frac{(i+k-1)!}{k!(i-k-1)!} t^{i-k}\varepsilon^{2k},$$

which are two-dimensional generalized Helmholtz polynomials [3].

We shall call (3.3) the generalized Helmholtz polynomials of dimension 3.

The generating function for the generalized Helmholtz polynomials defined by (1.2) was obtained in [3], i.e.,

$$F(x, t; \varepsilon, z) = \exp\left[x + \frac{t}{2\varepsilon^2}(1 - \sqrt{1 - 4\varepsilon^2 z^2})\right] = \sum_{n=0}^{\infty} \frac{u_n(x, t; \varepsilon) z^n}{n!} \tag{3.4}$$

for all values of z such that $|z| < 1/2\varepsilon$. For the determination of the generating function for the set $\{u_{mn}(x, y, t; \varepsilon)\}$, we observe first that the function

$$F(x, y, t; \varepsilon, \alpha, \beta) = \exp\left[x\alpha + y\beta + \frac{t}{2\varepsilon^2}(1 - \sqrt{1 - 4\varepsilon^2(\alpha^2 + \beta^2)})\right] \tag{3.5}$$

satisfies (1.3) for all complex values of α and β, and can be expanded in a power series in α and β provided that $\alpha^2 + \beta^2 < 1/4\varepsilon^2$. We have to show that the coefficients of $\alpha^m\beta^n$ are $u_{mn}(x, y, t; \varepsilon)/m!n!$ for any positive integers m and n.

To this end, we want to establish the following uniqueness theorem.

THEOREM 3.1. *If $w(x, y, z)$ is continuously differentiable in $z \geqq 0$, and satisfies*

$$|w(x, y, 0)|(\rho^2 + 1)^{-1} \exp(-\lambda\rho) \in L_1(R^2), \tag{3.6}$$

where $\rho = \sqrt{x^2 + y^2}$, $2\varepsilon\lambda = 1$, and if

$$\iiint_{a \leqq r \leqq a+1} |w(x, y, z)| r^{-1} \exp(-\lambda r)\, dx\, dy\, dz \to 0 \tag{3.7}$$

as $a \to \infty$, where $r = (x^2 + y^2 + z^2)^{1/2}$, then $w(x, y, z)$ is given by the Poisson formula (2.4).

Proof. Apply Green's formula in the form

$$\iint_{\partial\Omega} \left\{ v\frac{\partial G}{\partial n} - G\frac{\partial v}{\partial n} \right\} dA = v(P_0) - \iiint_{\Omega} G(\Delta v - \lambda^2 v)\, dV, \tag{3.8}$$

where Ω is the hemisphere $R^2 = \xi^2 + \eta^2 + \zeta^2 \leqq (a + 2)^2$, $\zeta \geqq 0$, $P_0 = (x, y, z)$, and G is Green's function of (2.3), and $a > r = (x^2 + y^2 + z^2)^{1/2}$. Now take

$$v = (1 - \Psi)w,$$

where Ψ is a function of class $C^2(\Omega)$ such that

$$\Psi = 0 \quad \text{for} \quad R \leqq a,$$

$$\Psi = 1 \quad \text{for} \quad a + 1 \leqq R \leqq a + 2.$$

For example, we may take

$$\Psi = \int_0^{R-a} h(s)\, ds,$$

where $h(s) = 30s^2(1 - s)^2$.

Then the left-hand side of (3.8) approaches the Poisson integral (2.4). The integrand on the right vanishes outside the hemispherical shell $\Omega_1 : a \leqq R \leqq a + 1$,

$\zeta > 0$. The only slightly troublesome term is

$$\iiint_{\Omega_1} G\nabla\Psi\cdot\nabla w\,dV = \iiint_{\Omega_1} \{\nabla\cdot(wG\nabla\Psi) - w\nabla\cdot(G\nabla\Psi)\}\,dV$$

$$= \iint_{\partial\Omega_1} wG\frac{\partial\Psi}{\partial n}dA - \iiint_{\Omega_1} w(G\nabla\Psi + \nabla G\cdot\nabla\Psi)\,dV$$

$$= -\iiint_{\Omega_1} w(G\nabla\Psi + \nabla G\cdot\nabla\Psi)\,dV,$$

since $G\partial\Psi/\partial n$ vanishes on $\partial\Omega_1$. Since both G and ∇G are $O(R^{-1}\exp(-\lambda R))$, the integral approaches 0 as $a \to \infty$.

The above uniqueness theorem applied to $u = 1$ ($w = \exp(-\lambda z)$) yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x-\xi, y-\eta, t)\,d\xi\,d\eta = 1,$$

and applied to the solution $H(x - x_1, y - y_1, t + \tau)$ yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x-\xi, y-\eta, \tau)H(\xi - x_1, \eta - y_1, t)\,d\xi\,d\eta, \tag{3.9}$$

corresponding to the semigroup property of the integral transformation (2.6).

If we define

$$N(t,\tau)(u) = \iint_{R^2} |u(x,y,t)|H(x'-x, y'-y, \tau)\,dx\,dy,$$

then we have

$$N(t,\tau)(u) \leqq N(0, t+\tau)(u) \tag{3.10}$$

so that the integral transformation (2.6) preserves the property (3.6).

Similarly, if $w = \exp[x\alpha + y\beta - z(\lambda^2 - \alpha^2 - \beta^2)^{1/2}]$, $u = \exp[x\alpha + y\beta + 2\lambda t(\lambda - (\lambda^2 - \alpha^2 - \beta^2)^{1/2})]$, then we find that the integral in Lemma 2.1 is given by (3.5) for $\alpha^2 + \beta^2 < \lambda^2$. The series for $u(\xi, \eta, 0)$ in powers of α and β is dominated by $\exp(|\alpha|\,|\xi| + |\beta|\,|\eta|)$ which satisfies (3.6), so that term-by-term integration is justified. It follows that (3.5) is indeed the generating function for the polynomials $U_{mn}(x, t, t; \varepsilon)$ in (3.1).

Thus

$$\exp\left[x\alpha + y\beta + \frac{t}{2\varepsilon^2}(1 - (1 - 4\varepsilon^2(\alpha^2 + \beta^2))^{1/2})\right] = \sum_{m,n=0}^{\infty} \frac{\alpha^m\beta^n}{m!n!}u_{mn}(x, y, t; \varepsilon). \tag{3.11}$$

Remark. It is worthwhile to note that (3.11) can be obtained directly from (3.4) without use of Theorem 3.1. If we set $\alpha = z\cos\theta$, $\beta = z\sin\theta$ in (3.5), we have

$$F(x, y, t; \varepsilon, \alpha, \beta) = \exp\left[xz\cos\theta + yz\sin\theta + \frac{t}{2\varepsilon^2}(1 - (1 - 4\varepsilon^2 z^2)^{1/2})\right] \tag{3.12}$$
$$= \sum_{m=0}^{\infty} \frac{x^m z^m \cos^m\theta}{m!} \sum_{n=0}^{\infty} u_n(y\sin\theta, t; \varepsilon)\frac{z^n}{n!}.$$

Applying (3.4) to the second series, and rearranging (3.12) in powers of α and β, we can show that the coefficients of $\alpha^m\beta^n$ are $u_m(x, y, t; \varepsilon)/m!n!$.

4. Polynomial expansions. In this section we shall establish necessary and sufficient conditions for series expansions of any solution of (1.3) in terms of the generalized Helmholtz polynomials, $\{u_{mn}(x, y, t; \varepsilon)\}$. To do this, we need to know the asymptotic behavior of $u_{mn}(x, y, t; \varepsilon)$ when $m,n \to \infty$.

LEMMA 4.1. For $0 \leqq z < \infty, 0 < \delta < \infty, n = 1, 2, \cdots$,

$$z^n \leqq e^{\delta z}\left(\frac{n}{e\delta}\right)^n. \tag{4.1}$$

Proof. This is proved by computing the maximum of $f(z) = e^{-\delta z}z^n$.

Theorem 4.2. *If* $t > 0$, $-\infty < x, y < \infty$, $m,n = 1, 2, \cdots$, *then*

$$|u_{mn}(x, y, t; \varepsilon)| < K(\varepsilon)t\exp\left\{\delta_1|x| + \delta_2|y| + \frac{t}{2\varepsilon^2}\right\}\left(\frac{m}{e\delta_1}\right)^m\left(\frac{n}{e\delta_2}\right)^n, \tag{4.2}$$

where $K(\varepsilon)$ *is a suitably chosen constant and* $\delta_1^2 + \delta_2^2 < 1/4\varepsilon^2$.

Proof.

$$\begin{aligned}
|u_{mn}(x, y, t; \varepsilon)| &= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(x - \xi, y - \eta, t)\xi^m\eta^n\, d\xi\, d\eta\right| \\
&= \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\xi, \eta, t)(x + \xi)^m(y + \eta)^n\, d\xi\, d\eta\right| \\
&\leqq 4\int_{0}^{\infty}\int_{0}^{\infty} H(\xi, \eta, t)(|x| + \xi)^m(|y| + \eta)^n\, d\xi\, d\eta \\
&\leqq 4\exp[\delta_1|x| + \delta_2|y|]\left(\frac{m}{e\delta_1}\right)^m\left(\frac{n}{e\delta_2}\right)^n \\
&\quad \cdot\int_{0}^{\infty}\int_{0}^{\infty} H(\xi, \eta, t)\exp[\delta_1\xi + \delta_2\eta]\, d\xi\, d\eta \\
&\leqq \frac{2t}{\pi\varepsilon}\exp\left[\delta_1|x| + \delta_2|y| + \frac{t}{2\varepsilon^2}\right]\left(\frac{m}{e\delta_1}\right)^m\left(\frac{n}{e\delta_2}\right)^n \\
&\quad \cdot\int_{0}^{\infty}\int_{0}^{\infty}\exp\left[-\frac{1}{2\varepsilon}\sqrt{\xi^2 + \eta^2}\right]\left\{\frac{1}{2\varepsilon}(\xi^2 + \eta^2)^{-1} + (\xi^2 + \eta^2)^{-3/2}\right\} \\
&\quad \cdot\exp[\delta_1\xi + \delta_2\eta]\, d\xi\, d\eta \\
&= K(\varepsilon)t\exp\left[\delta_1|x| + \delta_2|y| + \frac{t}{2\varepsilon^2}\right]\left(\frac{m}{e\delta_1}\right)^m\left(\frac{n}{e\delta_2}\right),
\end{aligned}$$

where

$$K(\varepsilon) = \frac{2}{\pi\varepsilon}\int_0^\infty\int_0^\infty \exp\left[\delta_1\xi + \delta_2\eta - \frac{1}{2\varepsilon}\sqrt{\xi^2 - \eta^2}\right]\left\{\frac{1}{2\varepsilon}(\xi^2 + \eta^2)^{-1} + (\xi^2 + \eta^2)^{-3/2}\right\} d\xi\, d\eta < \infty.$$

LEMMA 4.3. *If the series*

$$\sum_{m,n=0}^{\infty} a_{mn} u_{mn}(x, y, t; \varepsilon) \tag{4.3}$$

converges at (x_0, y_0, t_0), $x_0 > 0$, $y_0 > 0$, $t_0 > 0$, *then*

$$a_{mn} = O\left[(m + n)\left(\frac{e}{2m\varepsilon}\right)^m\left(\frac{e}{2n\varepsilon}\right)^n\right]. \tag{4.4}$$

Proof. We consider only the case $m = 2p$, $n = 2q$; the other cases can be treated similarly.

The coefficients in the expansion of $u_{mn}(x, y, t; \varepsilon)$ are always positive. Hence for $x_0 > 0, y_0 > 0, t_0 > 0, u_{mn}(x_0, y_0, t_0; \varepsilon)$ is larger than any term of the expansion, in particular, the term involving $\varepsilon^{2p+2q-2}$. Hence

$$u_{mn}(x_0, y_0, t_0; \varepsilon) \geqq \frac{(2p)!(2q)!}{p!q!}\frac{(2p + 2q - 2)!}{(p + q - 1)!} t_0 \varepsilon^{2p+2q-2}. \tag{4.5}$$

Using Stirling's formula $n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n}$ as $n \to \infty$, and the fact that $a_{mn}u_{mn}(x_0, y_0, t_0; \varepsilon) \to 0$ as $m,n \to \infty$, we obtain

$$a_{(2p)(2q)} = O\left[(2p + 2q)\left(\frac{e}{4p\varepsilon}\right)^{2p}\left(\frac{e}{4q\varepsilon}\right)^{2q}\right]$$

or

$$a_{mn} = O\left[(m + n)\left(\frac{e}{2m\varepsilon}\right)^m\left(\frac{e}{2n\varepsilon}\right)^n\right].$$

THEOREM 4.4. *If*

$$\limsup_{m,n\to\infty}\left(\frac{|a_{mn}|m^m n^n}{e}\right)^{1/(m+n)} = \sigma < \frac{1}{2\sqrt{2\varepsilon}}, \tag{4.6}$$

then the series

$$\sum_{m,n=0}^{\infty} a_{mn} u_{mn}(x, y, t; \varepsilon) \tag{4.7}$$

converges absolutely in the upper half-space $t \geqq 0$.

Proof. For $0 < \theta < 1$, the condition (4.6) is equivalent to $|a_{mn}|m^m n^n \leqq (e\sigma/\theta)^{m+n}$, $m,n \geqq N$, for a suitable $N(\theta)$, or

$$|a_{mn}| \leqq \left(\frac{e\sigma}{m\theta}\right)^m\left(\frac{e\sigma}{n\theta}\right)^n. \tag{4.8}$$

By (4.2) and (4.8), the series (4.7) is dominated by

$$(4.9)\qquad \sum_{m,n=1}^{\infty} K(\varepsilon)t \exp\left\{\delta_1|x| + \delta_2|y| + \frac{t}{2\varepsilon^2}\right\}\left(\frac{\sigma}{\theta\delta_1}\right)^m\left(\frac{\sigma}{\theta\delta_2}\right)^n$$

$$\text{for}\quad \delta_1^2 + \delta_2^2 < 1/4\varepsilon^2.$$

Choose δ_1, δ_2 such that $\sigma < \delta_1, \delta_2 < 1/2\sqrt{2}\varepsilon$. Since $\sigma/\delta_1\theta < 1$, $\sigma/\delta_2\theta < 1$ for θ near 1, the series (4.9) converges. This implies (4.7) converges absolutely for $t > 0$.

For $t = 0$, $u_{mn}(x, y; \varepsilon) = x^m y^n$. Then by Lemma 4.1,

$$|x|^m|y|^n \leqq e^{\rho_1|x| + \rho_2|y|}\left[\frac{m}{e\rho_1}\right]^m\left[\frac{n}{e\rho_2}\right]^n.$$

By the same arguments, the absolute convergence of (4.7) can also be proved for $t = 0$.

Remark. In the case of heat polynomials, that is, $\varepsilon = 0$, Widder and Rosenbloom have investigated their asymptotic behavior for the cases $t \geqq 0$ and $t < 0$, and shown that the series (4.7) $\sum_{m,n=0}^{\infty} a_{mn}v_{mn}(x, y, t)$, where $v_{mn}(x, y, t) = u_{mn}(x, y, t; 0)$, converges in a region of the type $|t| < \rho$, where ρ is a constant which depends on different growth condition on the coefficients a_{mn} (see [5] and [7]). In our approach, we consider the asymptotic behavior of the generalized Helmholtz polynomials only for $t \geqq 0$ (see Theorem 4.2 and Lemma 4.3). The condition for the convergence of the series $\sum a_{mn}u_{mn}(x, y, t; \varepsilon)$ in the lower half-space is still under investigation.

DEFINITION. A function $f(x, y)$ belongs to the class $(1, \sigma)$ if it is equal to a double power series

$$(4.10)\qquad f(x, y) = \sum a_{mn}x^m y^n$$

whose coefficients satisfy the inequality

$$(4.11)\qquad \limsup\left(\frac{|a_{mn}|m^m n^n}{e}\right)^{1/(m+n)} \leqq \sigma.$$

COROLLARY 4.5. *Let $f(x, y)$ belong to the class $(1, \sigma)$, where $\sigma < 1/2\sqrt{2}\varepsilon$. Then*

$$u(x, y, t; \varepsilon) = L[f]$$

is a solution of (1.3) *in the upper half-space $t > 0$ and $u(x, y, 0; \varepsilon) = f(x, y)$.*

Proof. Let $0 < \theta < 1$ and

$$|f(x, y)| \leqq \sum |a_{mn}|\,|x|^m|y|^n$$
$$\leqq \sum\left(\frac{\sigma e}{m\theta}\right)^m\left(\frac{\sigma e}{n\theta}\right)^n e^{\rho_1|x| + \rho_2|y|}\left(\frac{m}{e\rho_1}\right)^m\left(\frac{n}{e\rho_2}\right)^n$$

for $m,n \geqq N(\theta)$. Choose $p_1, p_2 < \sigma < 1/2\sqrt{2}\varepsilon$ and θ near 1. Then

$$|f(x, y)| \leqq e^{\rho_1|x| + \rho_2|y|}\sum\left(\frac{\sigma}{\rho_1\theta}\right)^m\left(\frac{\sigma}{\rho_2\theta}\right)^n$$
$$= Me^{\rho_1|x| + \rho_2|y|},$$

where M is a constant.

Hence $f(x, y)$ satisfies the hypothesis of Theorem 2.2 and our result follows from the same theorem.

COROLLARY 4.6. *The series*

$$\sum_{m,n=0}^{\infty} a_{mn} u_{mn}(x, y, t; \varepsilon)$$

is a solution of (1.3) *if its coefficients satisfy the condition* (4.6).

Proof. By Theorem 4.4 and Corollary 4.5.

THEOREM 4.7. *Let* $u(x, y, t; \varepsilon)$ *be a solution of* (1.3) *in* $t > 0$ *and be continuous in* $t \geqq 0$. *Suppose*

$$u(x, y, t; \varepsilon) = \sum_{m,n} a_{mn} u_{mn}(x, y, t; \varepsilon).$$

Then $u(x, y, 0; \varepsilon)$ *belongs to the class* $(1, 1/2\sqrt{2\varepsilon})$.

Proof. Let $x = x_0 > 0$, $y = y_0 > 0$, $t = t_0 > 0$. The series converges at (x_0, y_0, t_0). By Lemma 4.3,

$$a_{mn} = O\left[(m + n)\left(\frac{e}{2m\varepsilon}\right)^m \left(\frac{e}{2n\varepsilon}\right)^n\right].$$

Now

$$u(x, y, 0; \varepsilon) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n,$$

where

$$\limsup \left(\frac{|a_{mn}| m^m n^n}{e}\right)^{1/(m+n)} \leqq \frac{1}{2\sqrt{2\varepsilon}}.$$

Hence $u(x, y, 0; \varepsilon)$ belongs to the class $(1, 1/2\sqrt{2\varepsilon})$.

THEOREM 4.8. *Let* $u(x, y, t)$ *be a solution of* (1.3) *in* $t > 0$ *and be continuous in* $t \geqq 0$. *Suppose* $u(x, y, 0)$ *belongs to the class* $(1, \sigma)$, *where* $\sigma < 1/2\sqrt{2\varepsilon}$ *and* $w(x, y, z) = \exp[-z/2\varepsilon]u(x, y, t; \varepsilon)$, $t = \varepsilon z$, *satisfies condition* (3.7). *Then*

$$u(x, y, t) = \sum_{m,n=0}^{\infty} a_{mn} u_{mn}(x, y, t; \varepsilon) \tag{4.12}$$

converges absolutely in $t > 0$, *and*

$$m!n!a_{mn} = \frac{\partial^{m+n} u(x, y, 0)}{\partial x^m \partial y^n}. \tag{4.13}$$

Proof. Let $u(x, y, 0) = w(x, y, 0) = f(x, y)$.

The computation in Corollary 4.5 shows that $f(x, y)$ satisfies the hypothesis of Theorem 2.2, and, in particular, the condition (3.6). Hence $w(x, y, z)$ satisfies the hypothesis of Theorem 3.1 and can be represented by the Poisson formula (2.4). Equivalently, $u(x, y, t)$ can be represented by the integral (2.6).

Let $u(x, y, 0) = \sum_{m,n=0}^{\infty} a_{mn} x^m y^n$ in (2.6), where a_{mn} is given by (4.13). The series (4.12) is then obtained by termwise integration of the series $u(x, y, 0)$ which is permissible.

5. Generalization. The polynomial expansions of any solution of the equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \varepsilon^2 \frac{\partial^2 u}{\partial t^2} = \frac{\partial u}{\partial t} \tag{5.1}$$

can be obtained in the same way as for $n = 2$. Equation (5.1) can be transformed into

$$\sum_{i=1}^{n+1} \frac{\partial^2 w}{\partial x_i^2} - \lambda^2 w = 0, \tag{5.2}$$

where $\lambda = 1/2\varepsilon$, $x_{n+1} = t/\varepsilon$ and

$$w(x_1, \cdots, x_n, x_{n+1}) = u(x_1, \cdots, x_u, t) \exp[-t/2\varepsilon^2].$$

The fundamental solution $w(r)$, where $r = \sqrt{\sum_{i=1}^{n+1} x_i^2}$, can be obtained from

$$\frac{\partial^2 w}{\partial r^2} + \frac{n}{r}\frac{\partial w}{\partial r} - \lambda^2 w = 0. \tag{5.3}$$

As is easily verified, the function

$$p(r) = \frac{1}{r}\frac{\partial w}{\partial r}$$

satisfies the same equation as $w(r)$ with n replaced by $n + 2$, that is,

$$\frac{\partial^2 p}{\partial r^2} + \frac{n+2}{r}\frac{\partial p}{\partial r} - \lambda^2 p = 0. \tag{5.4}$$

The fundamental solution for (5.3) for any n can be obtained by recursion from $n = 1$ and $n = 2$. Then, by method of images, Green's function $G(x_1, \cdots, x_{n+1}, \xi_1, \cdots, \xi_{n+1})$ in the upper half-space can be obtained. The Poisson representation formula is given as

$$w(x_1, \cdots, x_{n+1}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\partial G}{\partial \xi_{n+1}}(x_1, \cdots, x_{n+1}, \xi_1, \cdots, \xi_n, 0) \cdot w(\xi_1, \cdots, \xi_n, 0)\, d\xi_1 \cdots d\xi_n. \tag{5.5}$$

The generalized Helmholtz polynomials $u_{m_1,\cdots,m_n}(x_1, \cdots, x_n, t) = e^{t/2\varepsilon^2} \cdot w(x_1, \cdots, x_n, t/\varepsilon)$, where $w(x_1, \cdots, x_n, x_{n+1})$ is generated by (5.5) with $w(\xi_1, \cdots, \xi_n, 0) = \xi_1^{m_1}, \cdots, \xi_n^{m_n}$, and their generating function $F(x_1, \cdots, x_n, t; \cdot \varepsilon, \alpha_1, \cdots, \alpha_u)$ can be shown as $\exp[\alpha_1 x_1 + \cdots + \alpha_n x_n + t/2\varepsilon^2]$ for $\alpha_1^2 + \cdots + \alpha_n^2 < 1/4\varepsilon^2$. However, it should be pointed out that since the kernel $\partial G/\partial \xi_{n+1}$ in (5.5) involves both exponential functions and Bessel functions of different orders, it is impossible to give an explicit formula for any *arbitrary n*. If we are interested in obtaining the set of polynomials for a fixed n, say $n = 3$ or 4, the Poisson formula (5.5) can be given in closed form and the generalized Helmholtz polynomials are given through detailed calculations as

$$m_1! \cdots m_n! \sum_{i_1=0}^{[m_1/2]} \cdots \sum_{i_n=0}^{[m_n/2]} \frac{x_1^{m-2i_1} \cdots x_n^{m_n-2i_n}}{(m_1 - 2i_1)! \cdots (m_n - 2i_n)! i_1! \cdots i_n!}$$
$$\cdot \sum_{k=0}^{i_1+\cdots+i_n-1} \frac{(i_1 + \cdots + i_n + k - 1)!}{k!(i_1 + \cdots + i_n - k - 1)!} t^{i_1+i_2+\cdots+i_n-k} \varepsilon^{2k}$$

for a *fixed* n. All the theorems in this paper can be extended to the general case with obvious modifications.

REFERENCES

[1] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics. Vol. II: Partial Differential Equations* (by R. Courant), Interscience, New York, 1962.

[2] A. ERDELYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions II*, McGraw-Hill, New York, 1955.

[3] C. Y. LO, *Polynomial expansions of solutions of* $u_{xx} + \varepsilon^2 u_{tt} = u_t$, Z. Rein. Angew. Math., to appear.

[4] M. H. PROTTER AND H. F. WEINBERGER, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

[5] P. C. ROSENBLOOM AND D. V. WIDDER, *Expansions in terms of heat polynomials and associated functions*, Trans. Amer. Math. Soc., 92 (1959), pp. 220–266.

[6] A. N. TYCHONOV AND A. A. SAMERSKI, *Partial Differential Equations of Mathematical Physics*, Holden-Day, San Francisco, Calif., 1967.

[7] D. V. WIDDER, *Series expansions of solutions of the heat equation in n dimensions*, Ann. Mat. Pura Appl. (4), 55 (1961), pp. 389–409.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

VERTICAL ASYMPTOTES AND BOUNDS FOR CERTAIN SOLUTIONS OF A CLASS OF SECOND ORDER DIFFERENTIAL EQUATIONS*

S. B. ELIASON†

Abstract. In papers of Wong and Hille results appear concerning the existence of vertical asymptotes of certain solutions of second order nonlinear differential equations. Specifically, Hille considers the Thomas–Fermi equation. Results here, which in part involve methods of these authors, provide upper and lower bounds for solutions of the initial value problem

$$y'' = p(x)y^{\gamma}, \quad y(a) > 0, \quad y'(a) = 0,$$

as well as bounds for the first vertical asymptote $b > a$ of such solutions. The coefficient function p is positive and continuous on an appropriate interval under consideration and $\gamma \geqq 1$. The bounds are given in terms of integral functionals involving the coefficient function p and solutions, computable in terms of the incomplete beta-function, of a similar initial value problem where the coefficient function is constant. The results are applicable in their most complete form when the coefficient function is either monotone increasing or monotone decreasing. Specific results are obtained for certain special equations including the Thomas–Fermi equation.

1. Introduction. We consider here certain real-valued solutions of the second order differential equation

$$y'' = p(x)y^{\gamma}, \tag{1.1}$$

where throughout most of our discussion we assume the following hypothesis:

(H_1) the real-valued function p is continuous on $[a, b)$ and positive on (a, b), where $a < b$ are real numbers (or $b = +\infty$), and $\gamma \geqq 1$ is a real number.

The author [3] provides a result which may be applied to (1.1). It states that if a solution y of (1.1) satisfies

$$y(a) > 0, \quad y'(a) = 0 \tag{1.2}$$

and

$$\lim_{x \to b-} y(x) = +\infty, \tag{1.3}$$

where $\gamma > 1$ and $b < +\infty$, then

$$\gamma[2/(\gamma - 1)]^2 \leqq (b - a)[y(a)]^{\gamma - 1} \int_a^b p\,dx. \tag{1.4}$$

It is clear that this result can be extended to apply if negative-valued solutions are considered. For this, in (1.1), y^{γ} needs to be changed to $(\operatorname{sgn} y)|y|^{\gamma}$ and absolute values need to be inserted in appropriate places in (1.2), (1.3) and (1.4). For ease of notation we shall not take this extension into consideration in this paper.

We see that (1.4) places a lower bound on a functional involving the initial value $y(a)$, the coefficient function p and the distance from a to a vertical asymptote b of y, where $b > a$. The sharpness of this result is shown by the author [3] for

* Received by the editors June 24, 1971.

† Department of Mathematics, University of Oklahoma, Norman, Oklahoma 73069. This research was supported by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant AFOSR-71-2069.

the case of $\gamma = 3$, i.e., the constant on the left in (1.4) cannot be increased without making (1.4) invalid for some coefficient function p and corresponding solution y of (1.1), (1.2) and (1.3).

Wong [9] has a short section on the existence of vertical asymptotes of certain solutions of equations more general than (1.1). Hille [6] considers the same question for the Thomas-Fermi equation

$$y'' = x^{-1/2} y^{3/2}, \tag{1.5}$$

which is a special case of (1.1) under (H_1) when $0 < a$. Both Wong and Hille show, in the setting of their equations, that there are solutions possessing vertical asymptotes in an interval. They do not restrict themselves to the initial conditions as in (1.2), but allow the initial slope to vary. The main result of § 4 of Hille [6] is the following theorem.

THEOREM 1.1 (Hille). *Let y be a solution of* (1.5) *defined by initial conditions of the form*

$$y(a) = c_0, \quad y'(a) = c_1, \tag{1.6}$$

$$0 \leqq c_0, \quad 0 \leqq c_1, \quad c_0 + c_1 > 0.$$

Then there exists a number $a < b < +\infty$ and $y(x) \to +\infty$ as $x \to b-$. Further for $b - x$ small,

$$y(x) < 400b(b-x)^{-4}. \tag{1.7}$$

If in particular

$$5c_1^2 \geqq 4a^{-1/2} c_0^{5/2}, \tag{1.8}$$

then

$$b < [a^{3/4} + (3/2)5^{1/2} b_0^{-1/4}]^{4/3} \tag{1.9}$$

and (1.7) *holds for all $a < x < b$.*

In the proof of the above theorem Hille also establishes a lower bound on the solution which, unlike the bound in (1.7), does not depend on a previous knowledge of the value of the vertical asymptote b.

The purpose of this paper is to use solutions of special equations having constant coefficients to provide both upper and lower bounds for solutions of (1.1) and (1.2), provided the coefficient function p satisfies certain properties to be described, and these bounds will be independent of a previous knowledge of the value of a vertical asymptote. Also we shall obtain upper and lower bounds on the value of the vertical asymptote (possibly $= +\infty$).

We shall use certain methods found in Wong [9] and Hille [6] as well as others motivated by results in Nehari [7], Barnes [2] and the author [4], [5], in which differential equations of a different type are considered, but which do consider integral functionals similar to those we consider here.

2. The constant coefficient equation. For comparison purposes let u_γ be the solution of the initial value problem

$$u'' = u^\gamma, \quad u(0) = 1, \quad u'(0) = 0, \tag{2.1}$$

where $\gamma \geqq 1$.

Clearly for $\gamma = 1$ we have $u_\gamma(x) = \cosh x$. Thus u_γ has no finite vertical asymptote, or we may say it has a vertical asymptote at $z(1) = +\infty$.

When $\gamma > 1$, $u_\gamma(x)$ may be expressed in terms of the incomplete beta function. In fact, by multiplying both sides of the differential equation in (2.1) by u' and integrating we obtain

$$u_\gamma'^2(x) = [2/(\gamma + 1)][u_\gamma^{\gamma+1}(x) - u_\gamma^{\gamma+1}(0)].$$

Then by taking square roots, dividing, noting that the improper integral exists for $x > 0$ as long as u_γ is continuous on $[0, x]$, we have for all such x that

$$x = [(\gamma + 1)/2]^{1/2} \int_1^{u_\gamma(x)} (u^{\gamma+1} - 1)^{-1/2}\, du. \tag{2.2}$$

Now clearly u_γ is increasing on $[0, x]$ as long as it is continuous and since the improper integral

$$\int_1^\infty (u^{\gamma+1} - 1)^{-1/2}\, du$$

converges there is a least positive real $z(\gamma)$, called the *first vertical asymptote* of u_γ, such that $u_\gamma(x) \to +\infty$ as $x \to z(\gamma)-$. Thus u_γ has an inverse on the interval $[0, z(\gamma))$ and its values may be computed as

$$\begin{aligned} u_\gamma^{-1}(x) &= [(\gamma + 1)/2]^{1/2} \int_1^x (u^{\gamma+1} - 1)^{-1/2}\, du \\ &= [2(\gamma + 1)]^{-1/2} \int_{x^{-(\gamma+1)}}^1 v^{-1+(\gamma-1)/2(\gamma+1)}(1 - v)^{-1+1/2}\, dv \\ &= [2(\gamma + 1)]^{-1/2} B_{1-x^{-(\gamma+1)}}(1/2, (\gamma - 1)/2(\gamma + 1)), \end{aligned} \tag{2.3}$$

under the substitution $u = v^{-1/(\gamma+1)}$. Now $B_x(z, w)$ is the incomplete beta function as defined by Abramowitz and Stegun [1] and Pearson [8] and its values may also be obtained from these sources for various x, z and w. Also from Abramowitz and Stegun [1] it follows that $B_1(z, w)$ is the complete beta function which in standard notation is given by

$$B(z, w) = \Gamma(z)\Gamma(w)/\Gamma(z + w),$$

where Γ is the complete gamma function. Thus $z(\gamma)$, $\gamma > 1$, may be computed by using (2.3), letting $x \to \infty$, as

$$z(\gamma) = [2(\gamma + 1)]^{-1/2}\Gamma(1/2)\Gamma((\gamma - 1)/2(\gamma + 1))/\Gamma((1/2) + (\gamma - 1)/2(\gamma + 1)). \tag{2.4}$$

3. The first comparison of solutions and asymptotes. To begin, we introduce some notation which will allow us to modify methods used by Wong [9].

Assume p satisfies (H_1) and define p_* and p^* on $[a, b)$ by

$$p_*(x) = \inf\{p(t) : t \in [a, x]\}$$

$$\text{and} \tag{3.1}$$

$$p^*(x) = \sup\{p(t) : t \in [a, x]\}.$$

Now with y being a solution of (1.1) and (1.2), as long as y is continuous on $[a, x]$, $a < x < b$, we may multiply (1.1) by y' and integrate to obtain

$$y'^2(x) = 2 \int_a^x p(t) y^\gamma(t) y'(t)\, dt. \tag{3.2}$$

Now since $y(t) > 0$ and $y'(t) > 0$ on (a, x) we have

$$\begin{aligned} &[2/(\gamma + 1)] p_*(x)[y^{\gamma+1}(x) - y^{\gamma+1}(a)] \leqq y'^2(x) \\ &\leqq [2/(\gamma + 1)] p^*(x)[y^{\gamma+1}(x) - y^{\gamma+1}(a)], \end{aligned} \tag{3.3}$$

where the inequalities are both strict unless $p(t) \equiv p(a)$ on $[a, x]$. We may now take square roots, divide by $[2/(\gamma + 1)]^{1/2}[y^{\gamma+1}(x) - y^{\gamma+1}(a)]^{1/2}$ and integrate. By making use of the fact that

$$\begin{aligned} &[(\gamma + 1)/2]^{1/2} \int_a^x y'(t)[y^{\gamma+1}(t) - y^{\gamma+1}(a)]^{-1/2}\, dt \\ &= [(\gamma + 1)/2]^{1/2}[y(a)]^{-(\gamma-1)/2} \int_1^{y(x)/y(a)} (u^{\gamma+1} - 1)^{-1/2}\, du \\ &= [y(a)]^{-(\gamma-1)/2} u_\gamma^{-1}(y(x)/y(a)) \end{aligned} \tag{3.4}$$

(see § 2 and note that (3.4) is valid for all $\gamma \geqq 1$), it follows from (3.3), (3.4) and the remarks between that

$$\begin{aligned} &[y(a)]^{(\gamma-1)/2} \int_a^x p_*^{1/2}\, dt \leqq u_\gamma^{-1}(y(x)/y(a)) \\ &\leqq [y(a)]^{(\gamma-1)/2} \int_a^x p^{*1/2}\, dt. \end{aligned} \tag{3.5}$$

As in (3.3), the inequalities in (3.5) are strict unless $p(t) \equiv p(a)$ on $[a, x]$. Now since u_γ is increasing we may apply it to all quantities in (3.5). This yields a lower bound on $y(x)$ on $[a, b)$ for as long as y is continuous on $[a, x]$. It yields an upper bound on $y(x)$ as long as

$$[y(a)]^{(\gamma-1)/2} \int_a^x p^{*1/2}\, dt < z(\gamma), \qquad a < x < b. \tag{3.6}$$

From (3.5) it also follows that if $b > a$ is the first vertical asymptote of y, then

$$[y(a)]^{(\gamma-1)/2} \int_a^b p_*^{1/2}\, dt \leqq z(\gamma) \leqq [y(a)]^{(\gamma-1)/2} \int_a^b p^{*1/2}\, dt, \tag{3.7}$$

where (3.7) is valid even in the extended sense of $b = +\infty$ and also in the case of $\gamma = 1$ where $z(1) = +\infty$, which is of little interest since all solutions of (1.1) and (1.2) extend throughout $[a, b)$. Now for $\gamma > 1$ the inequalities in (3.7) are both strict unless

$$p(x) \equiv z^2(\gamma)/(b - a)^2[y(a)]^{\gamma-1} \quad \text{on} \quad [a, b). \tag{3.8}$$

For the special cases when p is monotone increasing (decreasing), on $[a, x]$, $a < x < b$, then $p^*(t) \equiv p(t)$ $(p_*(t) \equiv p(t))$ and $p_*(t) \equiv p(a)$ $(p^*(t) \equiv p(a))$ for

$t \in [a, x)$. Respective of these cases the second and first inequalities of (3.5) become

$$y(x) \leqq y(a)u_\gamma\left([y(a)]^{(\gamma-1)/2}\int_a^x p^{1/2}\,dt\right), \tag{3.9}$$

provided (3.6) holds with $p^* = p$, and

$$y(a)u_\gamma\left([y(a)]^{(\gamma-1)/2}\int_a^x p^{1/2}\,dt\right) \leqq y(x). \tag{3.9'}$$

Also, respective of the cases of p being monotone increasing (decreasing) on $[a, b)$, (3.7) yields

$$z(\gamma) \leqq [y(a)]^{(\gamma-1)/2}\int_a^b p^{1/2}\,dt, \tag{3.10}$$

which implies by the Schwarz inequality that

$$z^2(\gamma) \leqq (b-a)[y(a)]^{\gamma-1}\int_a^b p\,dt, \tag{3.10^2}$$

and

$$[y(a)]^{(\gamma-1)/2}\int_a^b p^{1/2}\,dt \leqq z(\gamma). \tag{3.10'}$$

Here we pause to notice that when $\gamma > 1$, (3.10^2) places a lower bound on the functional which appeared in (1.4). Thus when p is monotone increasing on $[a, b)$, since $z^2(\gamma) > \gamma[2/(\gamma-1)]^2$, (3.10^2) sharpens (1.4). Due to the Schwarz inequality, unless p is identically constant, (3.10) is sharper than (3.10^2) for each fixed p.

We summarize the above results in the following theorem.

THEOREM 3.1. *Let p satisfy* (H_1) *and y be a solution of* (1.1) *and* (1.2). *Then as long as y is continuous on $[a, x]$, $a < x < b$,* (3.5) *holds, where the inequalities are strict unless $p(t) \equiv p(a)$ on $[a, x]$. Also* (3.7) *becomes a necessary condition for y to have its first vertical asymptote at $b > a$, where when $\gamma > 1$ these inequalities are strict unless* (3.8) *holds. In the special case of p being monotone increasing on $[a, x]$, $a < x < b$,* (3.5) *yields* (3.9) *provided* (3.6) *holds where $p^* = p$; and in turn* (3.10) *and* (3.10^2) *are valid from* (3.7) *if p is monotone increasing on $[a, b)$. Finally the case of p being monotone decreasing on $[a, x]$, $a < x < b$, and $[a, b)$ respectively yields* (3.9′) *and* (3.10′) *from* (3.5) *and* (3.7).

4. The second comparison of solutions and asymptotes. We again begin by introducing notation related to a concept used by Barnes [2] and the author [4] and [5]. From this, for certain classes of coefficient functions, we shall be able to obtain inequalities going in reverse directions from those in § 3. The new inequalities here involve the coefficient function by an averaging process. We shall be able to apply the inequalities of both sections if, for example, the coefficient function is either monotone increasing or monotone decreasing.

DEFINITION 4.1. Let p satisfy (H_1) and define P_a on $[a, b)$ by

$$P_a(x) = \begin{cases} (x-a)^{-1}\displaystyle\int_a^x p\,dt & \text{if} \quad a < x < b, \\ p(a) & \text{if} \quad x = a. \end{cases} \tag{4.1}$$

Then p is said to be monotone increasing (decreasing) on the average from a on (a, b) if P_a is monotone increasing (decreasing) on (a, b).

Remark. Even in the case of the Thomas-Fermi equation when $a = 0$ where $p(x) = x^{-1/2}$ it follows that the corresponding P_0 exists on $(0, \infty)$. Due to later use, we also point out that in this case $P_0^{1/2}$ is integrable on $[0, x]$ for each $x > 0$.

It is interesting to note that p satisfying (H_1) and being monotone increasing (decreasing) on (a, b) implies the same of P_a. On the other hand, let p be defined by $p(x) = x(4 - x)$. Then p is not monotone increasing on $(0, 3)$ whereas the corresponding P_0 is.

Now let y be a solution of (1.1) and (1.2) where (H_1) is assumed to hold. We may continue from (3.2) to establish for $a < x < b$, as long as y is continuous on $[a, x]$,

$$
\begin{aligned}
y'^2(x) &= 2\int_a^x \left(\int_a^t p\,ds\right)' y^\gamma(t)y'(t)\,dt \\
&= 2y^\gamma(x)y'(x)\int_a^x p\,ds - 2\int_a^x (t-a)^{-1}\left(\int_a^t p\,ds\right)(t-a)[y^\gamma(t)y'(t)]'\,dt,
\end{aligned} \tag{4.2}
$$

by an integration by parts and division and multiplication of the one integrand by $(t - a)$.

We have $[y^\gamma(t)y'(t)]' > 0$ on (a, x) as long as y is continuous on $[a, x]$, $a < x < b$. Thus respective of whether P_a, as given in (4.1), is monotone increasing or monotone decreasing on (a, x), we have

$$
-2P_a(x)\int_a^x (t-a)[y^\gamma(t)y'(t)]'\,dt \leqq -2\int_a^x P_a(t)(t-a)[y^\gamma(t)y'(t)]'\,dt \tag{4.3}
$$

and

$$
-2\int_a^x P_a(t)(t-a)[y^\gamma(t)y'(t)]'\,dt \geqq -2P_a(x)\int_a^x (t-a)[y^\gamma(t)y'(t)]'\,dt. \tag{4.3'}
$$

By an integration by parts we have

$$
\int_a^x (t-a)[y^\gamma(t)y'(t)]'\,dt = (x-a)y^\gamma(x)y'(x) - (\gamma+1)^{-1}[y^{\gamma+1}(x) - y^{\gamma+1}(a)]. \tag{4.4}
$$

Placing (4.4) in (4.3) and (4.3′) and in turn (4.2) yields

$$
y'^2(x) \geqq [2/(\gamma+1)]P_a(x)[y^{\gamma+1}(x) - y^{\gamma+1}(a)] \tag{4.5}
$$

and

$$
y'^2(x) \leqq [2/(\gamma+1)]P_a(x)[y^{\gamma+1}(x) - y^{\gamma+1}(a)] \tag{4.5'}
$$

respectively of whether P_a is monotone increasing or monotone decreasing on (a, x) as long as y is continuous on $[a, x]$, $a < x < b$.

Clearly (4.5) and (4.5′) correspond to the first and second inequalities of (3.3) respectively. Thus the results of § 3 from (3.3) and following may all be repeated, with p_* exchanged for P_a when P_a is monotone increasing on the appropriate interval, and p^* exchanged for P_a when P_a is monotone decreasing. According to previous usage we shall use unprimed and primed numbers to label displays respective of these conditions. Thus (3.5) yields respectively

$$[y(a)]^{(\gamma-1)/2} \int_a^x P_a^{1/2}\,dt \leqq u_\gamma^{-1}(y(x)/y(a)) \tag{4.6}$$

and

$$u_\gamma^{-1}(y(x)/y(a)) \leqq [y(a)]^{(\gamma-1)/2} \int_a^x P_a^{1/2}\,dt, \tag{4.6′}$$

provided y is continuous on $[a, x]$, $a < x < b$.

We pause to notice that the inequalities (4.3) and (4.3′) are strict in their respective situations unless $P_a(t) \equiv P_a(a) = p(a)$ on $[a, x]$. Now this is the case if and only if $p(t) \equiv p(a)$ on $[a, x]$. Thus, as in § 3, we have (4.6) and (4.6′) strict unless $p(t) \equiv p(a)$ on $[a, x]$.

Again applying u_γ to (4.6) and (4.6′) yields

$$y(a)u_\gamma\left([y(a)]^{(\gamma-1)/2} \int_a^x P_a^{1/2}\,dt\right) \leqq y(x) \tag{4.7}$$

as long as y is continuous on $[a, x]$, $a < x < b$; and

$$y(x) \leqq y(a)u_\gamma\left([y(a)]^{(\gamma-1)/2} \int_a^x P_a^{1/2}\,dt\right) \tag{4.7′}$$

as long as

$$[y(a)]^{(\gamma-1)/2} \int_a^x P_a^{1/2}\,dt < z(\gamma), \qquad a < x < b. \tag{4.8}$$

Also as in § 3, if $b > a$ is the first vertical asymptote of y, then

$$[y(a)]^{(\gamma-1)/2} \int_a^x P_a^{1/2}\,dt \leqq z(\gamma) \tag{4.9}$$

and

$$z(\gamma) \leqq [y(a)]^{(\gamma-1)/2} \int_a^x P_a^{1/2}\,dt, \tag{4.9′}$$

respective of the unprimed and primed conditions. Furthermore by previous remarks concerning strictness, when $\gamma > 1$ the inequalities (4.9) and (4.9′) are strict in their respective cases unless (3.8) holds.

We summarize these results in the following theorem.

THEOREM 4.1. *Let p satisfy* (H_1), *y be a solution of* (1.1) *and* (1.2) *and P_a be defined as in* (4.1). *Then as long as y is continuous on $[a, x]$, $a < x < b$, and respective of whether P_a is monotone increasing or monotone decreasing on $[a, x]$, we have that* (4.5) *and* (4.5′) *must hold. These in turn yield respectively* (4.6), (4.7) *and* (4.6′), (4.7′), *where* (4.8) *restricts* x *in* (4.7′). *The inequalities* (4.5), (4.5′), (4.6), (4.6′), (4.7) *and* (4.7′) *are all strict unless* $p(t) \equiv p(a)$ *on* $[a, x]$. *Also* (4.9) *and* (4.9′) *become necessary conditions for* y *to have its first vertical asymptote at* $b > a$, *where when* $\gamma > 1$ *these inequalities are strict unless* (3.8) *holds.*

5. Applications. As remarked earlier, we may apply the results of both § 3 and § 4 if for example the coefficient function in (1.1) is either monotone increasing or monotone decreasing. We first compare our results with those of Theorem 1.1 (Hille).

For the Thomas-Fermi equation (1.5) we have $p(x) = x^{-1/2}$ and by direct computation it follows that

$$\int_a^x p^{1/2}\,dt = (4/3)(x^{3/4} - a^{3/4}),$$

$$P_a(x) = 2(x^{1/2} + a^{1/2})^{-1}, \tag{5.1}$$

$$\int_a^x P_a^{1/2}\,dt = (4/3)2^{1/2}[(x^{1/2} + a^{1/2})^{1/2}(x^{1/2} - 2a^{1/2}) + (2a^{1/2})^{1/2}a^{1/2}].$$

Clearly p is decreasing and satisfies (H_1) on $[a, \infty)$ if $a > 0$. Thus (3.9′) places a lower bound on $y(x)$, where y is a solution of (1.5) and (1.2) with $a > 0$, and (4.7′) places an upper bound on $y(x)$ as long as (4.8) holds with $\gamma = 3/2$. These bounds are computable in terms of the incomplete beta function, as earlier remarked, and are not dependent on a previous knowledge of the value of the vertical asymptote.

Also (3.10′) and (4.9′) may be applied. Here we see that if $b > a$ is the first vertical asymptote, then

$$\begin{aligned} &(4/3)(b^{3/4} - a^{3/4})[y(a)]^{1/2} < z(3/2) \\ &< (4/3)2^{1/2}[(b^{1/2} + a^{1/2})^{1/2}(b^{1/2} - 2a^{1/2}) + (2a^{1/2})^{1/2}a^{1/2}][y(a)]^{1/2}. \end{aligned} \tag{5.2}$$

Now the first inequality in (5.2) may be easily solved for b and this yields

$$b < [a^{3/4} + (3/4)z(3/2)[y(a)]^{-1/4}]^{4/3}. \tag{5.3}$$

We may compare this directly with (1.9). Since

$$z(3/2) = \Gamma(1/10)\Gamma(1/2)/5^{1/2}\Gamma(3/5), \tag{5.4}$$

from (2.4), we may use approximate values from Abramowitz and Stegun [1] to show that

$$(3/4)z(3/2) > (3/2)5^{1/2}, \tag{5.5}$$

but these numbers are quite close to each other. In fact the ratio of the two is approximately 1.13. Thus Hille's bound, if it would apply (note (1.8)), would be slightly better.

We see that (5.2) yields a lower bound on b as well. This inequality is too difficult to solve algebraically unless $a = 0$.

Remark. The results of § 3 and § 4 may be applied to handle the Thomas-Fermi equation even when $a = 0$. In this case and in § 3, $p_* = p$ and $p_*^{1/2}$ is integrable (properly or improperly) on compact subintervals of $[0, \infty)$. Thus the first inequalities in (3.5) and (3.7) can still be shown to be valid. Also (3.9′) and (3.10′) extend in this case. Concerning § 4, the remark following Definition 4.1 and an inspection of the results show there is no problem in making our application here.

Thus for the case of $a = 0$, mentioned above, we solve the second inequality in (5.2) to obtain

$$b > \{[(3/4)z(3/2)]^4[2y(0)]^{-2}\}^{1/3}. \tag{5.6}$$

Our results above for the Thomas-Fermi equation obviously extend to cover the equation

$$y'' = x^\lambda y^\gamma, \tag{5.7}$$

when $\lambda < 0$ and $\gamma \geqq 1$, provided $a > 0$ when $\lambda \leqq -1$, and $a \geqq 0$ when $-1 < \lambda < 0$. This includes, for example, the Emden-Fowler equation where $\gamma > 1$ and $\lambda = 1 - \gamma$. We choose not to go through the computations in these cases. We do mention here, however, that with $p(x) = x^\lambda$, $a > 0$ and $\lambda < -2$ we have

$$\int_a^\infty p^{1/2}\,dx < +\infty \quad \text{and} \quad \int_a^\infty P_a^{1/2}\,dx = +\infty$$

so that the bounds from (3.9′) and (4.7′) etc. are not as precise as is the case with the Thomas-Fermi equation.

As a further example we do wish to provide computations for the equation (5.7) in the case of $\lambda > 0$, $\gamma \geqq 1$ and $a = 0$. Here with $p(x) = x^\lambda$ we have

$$\begin{aligned} \int_0^x p^{1/2}\,dt &= [2/(\lambda + 2)]x^{(\lambda+2)/2}, \\ P_0(x) &= (\lambda + 1)^{-1}x^\lambda, \\ \int_0^x P_0^{1/2}\,dt &= [2/(\lambda + 2)](\lambda + 1)^{-1/2}x^{(\lambda+2)/2}. \end{aligned} \tag{5.8}$$

Thus from (3.9) and (4.7) we have

$$\begin{aligned} [y(0)]u_\gamma([2/(\lambda + 2)](\lambda + 1)^{-1/2}x^{(\lambda+2)/2}[y(0)]^{(\gamma-1)/2}) &< y(x) \\ &< [y(0)]u_\gamma([2/(\lambda + 2)]x^{(\lambda+2)/2}[y(0)]^{(\gamma-1)/2}). \end{aligned} \tag{5.9}$$

We notice here that both upper and lower bounds on $y(x)$ are of the same order of growth, only different constants appear as coefficients in these expressions. When $\gamma = 1$ we have from § 2 that $u_\gamma(x) = \cosh x$, and so (5.9) provides fairly accurate bounds for y on $[0, \infty)$.

Also for $\gamma > 1$, from (3.10) and (4.9), if $b > a$ is the first vertical asymptote of a solution y of (5.7) and (1.2), where $a = 0$ and $\lambda = 0$, we have

$$\begin{aligned} [(\lambda + 2)^2 z^2(\gamma)/4y^{\gamma-1}(0)]^{1/(\lambda+2)} &< b \\ &< [(\lambda + 2)^2(\lambda + 1)z^2(\gamma)/4y^{\gamma-1}(0)]^{1/(\lambda+2)}, \end{aligned} \tag{5.10}$$

which provides fairly sharp bounds since these bounds differ by a multiplicative factor of $(\lambda + 1)^{1/(\lambda+2)}$. We note here also that this factor expression in λ tends to 1 as λ becomes very large. Thus (5.10) becomes more precise as λ becomes very large.

6. Concluding comments. The methods of this paper are quite dependent on the fact that the solution of (1.1) satisfy the initial conditions (1.2). There are places, however, where initial slopes, other than zero, may be considered. The first inequalities in (3.3), (3.4) and (3.7) are still valid if $y'(a) > 0$ and $y(a) > 0$. Consequently in this case and when p is monotone decreasing, (3.9′) and (3.10′) are valid. Due to the methods of our proofs we are not able to draw many conclusions for the case of $y'(a) < 0$ nor for the boundary conditions

$$y(a) = 0, \quad y'(a) > 0. \tag{6.1}$$

The boundary conditions (6.1) are certainly of interest and hopefully methods will be obtained to handle them.

As a brief comment we note that the roles of a and b may be interchanged and in so doing the functional P_a in (4.1) needs changing to

$$P_b(x) = \begin{cases} (b - x)^{-1} \displaystyle\int_x^b p\,dt & \text{if} \quad a < x < b, \\ p(b) & \text{if} \quad x = b, \end{cases}$$

with corresponding changes made in (H_1) and the various results.

Next, the result (1.4) of the author [3] is valid for equations more general than (1.1). The results of § 3 and § 4 will also apply to certain solutions of the equation

$$y'' = p(x)f(y) \tag{6.2}$$

provided (H_1) is satisfied (the statement $\gamma \geqq 1$ does not apply), $f:[M, \infty) \to R$ is differentiable with $f(y) > 0$ and $f'(y) > 0$ on (M, ∞) for some constant $M > 0$, and provided the results of § 2 run through for a certain computable solution of a specific initial value problem having a constant coefficient differential equation.

As an example, $y(x) = \sec x$ yields y a solution of

$$y'' = y(2y^2 - 1), \quad y(0) = 1, \quad y'(0) = 0. \tag{6.3}$$

In this case the computations of § 2 may be repeated and (2.3) becomes

$$\operatorname{arc\,sec} x = \int_1^x u^{-1}(u^2 - 1)^{-1/2}\,du. \tag{6.4}$$

Thus in § 3 we may consider solutions of (6.2) and (1.2) where $f(y) = y(2y^2 - 1)$, p satisfies (H_1) and where, due to later restrictions, we must consider $y(a) \geqq 1 \equiv M$. In this case (3.2) becomes

$$y'^2(x) = 2\int_a^x p(t)[y(t)(2y^2(t) - 1)]y'(t)\,dt, \tag{6.5}$$

(3.3) becomes

$$p_*(x)[y^4(x) - y^2(x)] \leqq y'^2(x) \leqq p^*(x)[y^4(x) - y^2(x)], \tag{6.6}$$

(3.4) becomes

$$\int_a^x [y(t)(y^2(t) - 1)^{1/2}]^{-1} y'(t)\,dt = \operatorname{arc\,sec} y(x) - \operatorname{arc\,sec} y(a) \tag{6.7}$$

and finally, from (6.6) and (6.7), (3.5) becomes

$$\int_a^x p_*^{1/2}\,dt + \operatorname{arc\,sec} y(a) \leqq \operatorname{arc\,sec} y(x) \leqq \int_a^x p^{*1/2}\,dt + \operatorname{arc\,sec} y(a), \tag{6.8}$$

for as long as y is continuous on $[a, x]$, $a < x < b$.

Similarly by direct inspection and some computation, with y as given above, where we also note $[f(y(t))y'(t)]' > 0$ on (a, x), the results (4.2) through (4.5′) and others of § 4 all extend to cover our present example. Thus the results of § 5 can also be modified to handle this situation.

REFERENCES

[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards Appl. Math. Series #55, U.S. Govt. Printing Office, Washington, D.C., 1965.

[2] D. C. Barnes, *Some isoperimetric inequalities for the eigenvalues of vibrating strings*, Pacific J. Math., 29 (1969), pp. 43–61.

[3] S. B. Eliason, *On the distance between vertical asymptotes of solutions of a second order differential equation*, J. Math. Anal. Appl., 35 (1971), pp. 148–156.

[4] ———, *Comparison theorems for second order nonlinear differential equations*, Quart. Appl. Math., 24 (1971), pp. 391–402.

[5] ———, *Comparison theorems related to nonlinear eigenvalue problems*, SIAM J. Appl. Math., 21 (1971), pp. 552–564.

[6] E. Hille, *Some aspects of the Thomas-Fermi equation*, J. Analyse Math., 23 (1970), pp. 147–170.

[7] Z. Nehari, *Oscillation criteria for second-order linear differential equations*, Trans. Amer. Math. Soc., 85 (1957), pp. 428–445.

[8] K. Pearson, *The Incomplete Beta-Function*, "Biometrika" Office, The University Press, Cambridge, 1948.

[9] P. K. Wong, *Existence and asymptotic behavior of proper solutions of a class of second-order nonlinear differential equations*, Pacific J. Math., 13 (1963), pp. 737–760.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

FUNCTIONS WHOSE FOURIER TRANSFORMS DECAY AT INFINITY: AN EXTENSION OF THE RIEMANN-LEBESGUE LEMMA*

N. BLEISTEIN†, R. A. HANDELSMAN‡ AND J. S. LEW§

Abstract. An extension of the Riemann–Lebesgue lemma is stated and proved. We define the space LL of all complex-valued locally integrable functions on $[0, +\infty)$, and the space RL of all functions f in LL such that

$$F(\omega) = \lim_{u \to +\infty} \int_0^u \exp(i\omega t) f(t)\, dt$$

exists for all sufficiently large ω and such that $\lim_{\omega \to +\infty} F(\omega) = 0$. First we consider all functions f in LL with asymptotic expansions, as $t \to +\infty$, of the form

$$f(t) \sim \exp[i\gamma t^\nu] \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} c_{mn} t^{-r(m)} (\log t)^n.$$

Here γ and ν are real, $N(m)$ is finite for each m and $0 < \operatorname{Re}(r(0)) \leqq \operatorname{Re}(r(1)) \leqq \cdots$. It is shown that RL contains all such functions except in the anomalous case $\gamma < 0 < 1 < \nu < 2$, $\operatorname{Re}(r(0)) + \nu/2 \leqq 1$. Then we show that RL contains all functions in LL whose amplitude and phase satisfy any of three qualitative sets of assumptions. These later results collectively generalize the previous assertion.

1. Introduction. If $f(t)$ belongs to the class $L^1[0, +\infty)$, then, as is well known, its Fourier transform

$$F(\omega) = \int_0^\infty \exp(i\omega t) f(t)\, dt \tag{1.1}$$

exists for all real ω and, by the Riemann-Lebesgue lemma (see Titchmarsh (1948, p. 11)), satisfies

$$\lim_{\omega \to +\infty} F(\omega) = 0. \tag{1.2}$$

On the other hand, it is also known that $F(\omega)$ may exist and satisfy (1.2) even when $f(t)$ is not in $L^1[0, +\infty)$. In particular, (1.2) holds whenever (1.1) converges uniformly in ω for all sufficiently large ω (see Doetsch (1950, p. 171)). The class of functions for which this is true includes $L^1[0, +\infty)$ but, in fact, is much broader. For example, it includes the function $f(t) = t^{-r}$ for $0 < r < 1$, which is not in $L^1[0, \infty)$.

Unfortunately this more powerful criterion is not always easy to apply. For example, it would be rather difficult to test the uniform convergence of (1.1) for a function such as $f(t) = t^{-1/4} \exp(-it^{3/2})$. Thus, motivated by our examples, we shall introduce a space RL consisting essentially of all functions which satisfy (1.2) and shall seek to describe a wider class of functions than $L^1[0, +\infty)$ which

---

* Received by the editors February 8, 1971, and in revised form November 4, 1971. The research of N. Bleistein and R. A. Handelsman was supported in part by the Air Force Office of Scientific Research under Grant AFOSR 70-1923 and by the National Science Foundation.

† Department of Mathematics, University of Denver, Denver, Colorado 80210.

‡ Department of Mathematics, University of Illinois at Chicago Circle, Chicago, Illinois 60680.

§ Mathematical Sciences Department, IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598.

belong to RL. Within this wider class we shall seek in particular to include functions which are integrable over finite intervals but which lie outside $L^1[0, +\infty)$ because they decay too slowly near $+\infty$. Indeed, in the sections below we obtain a family which we believe is large enough to include most functions arising in practice. In doing this we find that, in general, the resulting transforms (1.1) are improper Lebesgue-Stieltjes integrals and that these integrals need not converge for all real ω. We shall prove, however, for this family that (1.1) exists for all sufficiently large ω, so that (1.2) can hold nevertheless.

In § 2 we define the space RL more precisely, recall some useful inequalities for functions of bounded variation, and prove the convergence as stated of the relevant improper integrals. In § 3, we consider the class of functions locally integrable over finite intervals and having, as $t \to +\infty$, the asymptotic form

$$f(t) \sim \exp[i\gamma t^{\nu}] \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} c_{mn} t^{-r(m)} (\log t)^n. \tag{1.3}$$

By independent arguments we establish (1.2) for this class except for the range of parameters $\gamma < 0 < 1 < \nu < 2$, $\operatorname{Re}(r(0)) + \nu/2 \leqq 1$. We observe here as a special case that the Airy function $\operatorname{Ai}(-t)$ has an expansion of the form (1.3) with $\nu = \frac{3}{2}$ and $r(0) = \frac{1}{4}$. Hence this function satisfies $\operatorname{Re}(r(0)) + \nu/2 = 1$, and its Fourier transform, $2\pi^{-1}\cos(\omega^3/3)$, neither grows nor decays at ∞. In other cases, as we shall show, $F(\omega)$ can even increase as $\omega \to +\infty$. Finally, in § 4 we obtain additional theorems which extend the preceding result to more qualitatively described functions. Moreover, the results of § 4 include functions not precisely of the form (1.3) while those of § 3 include most functions of practical importance and yield sharper estimates, when they apply, of $F(\omega)$ as $\omega \to +\infty$.

2. Preliminary results. In this section we shall obtain results, some of them well known, which will be useful in our later discussions. Throughout, we let R denote the real numbers and let C denote the complex numbers. Thus if a is in R, then $R^{[a,+\infty)}$ and $C^{[a,+\infty)}$ are respectively the sets of all functions from [a, $+\infty$) into R and C. For any f in $R^{[a,+\infty)}$ we let $f\uparrow$ or $f\downarrow$ denote respectively that f is nondecreasing and nonincreasing. Moreover, we call f monotone if either $f\uparrow$ or $f\downarrow$.

For any f in $C^{[a,+\infty)}$ we write

$$f(+\infty) = \lim_{t \to +\infty} f(t) \tag{2.1}$$

when this limit, finite or infinite, is well-defined. Indeed, we note that $f(+\infty)$ exists whenever f is monotone on some $[b, +\infty)$. This in turn implies that $f(+\infty)$ exists whenever f is a sum of monotone functions, at most one of which is unbounded, and hence whenever f has bounded variation on some $[b, +\infty)$ since any such f is the difference of two bounded nondecreasing functions (see Widder (1941, p. 6)).

For any f in $C^{[0,+\infty)}$ and any a in $[0, +\infty)$ we let $s_a(f)$ be the supremum of $|f|$ on $[a, +\infty)$ and we let $v_{a,\infty}(f)$ be the variation of f on $[a, +\infty)$, that is, the supremum of

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \tag{2.2}$$

for all increasing sequences $(t_0, \cdots, t_n)$ in $[a, +\infty)$. We permit $s_a(f)$ and $v_{a,\infty}(f)$ to assume the value $+\infty$, so that both are defined for all f. Relevant properties of $v_{a,\infty}(f)$ are reviewed in the following lemma.

LEMMA 1. *For any a, b, in $[0, +\infty)$, c in C, f, g in $C^{[0,+\infty)}$:*

(i) $s_a(f) \leqq |f(a)| + v_{a,\infty}(f)$,
(ii) $v_{a,\infty}(c) = 0$,
(iii) $v_{a,\infty}(cf) = |c| v_{a,\infty}(f)$,
(iv) $v_{a,\infty}(f + g) \leqq v_{a,\infty}(f) + v_{a,\infty}(g)$,
(v) $v_{a,\infty}(fg) \leqq s_a(f) v_{a,\infty}(g) + v_{a,\infty}(f) s_a(g)$,
(vi) *if f is nonnegative and monotone on $[a, +\infty)$, then $v_{a,\infty}(f) \leqq s_a(f)$,*
(vii) *if $0 < b \leqq f$ on $[a, +\infty)$, then $v_{a,\infty}(1/f) \leqq v_{a,\infty}(f)/b^2$,*
(viii) *if $0 \leqq f \downarrow$ and $g = g_1 + g_2$ with $0 < b \leqq g_1 \uparrow$ and $0 \leqq g_2$, then*

$$v_{a,\infty}(f/g) \leqq b^{-1} f(a)\{1 + b^{-1}[s_a(g_2) + v_{a,\infty}(g_2)]\}. \tag{2.3}$$

Proof. Properties (i)–(vii) are obvious from our definitions and indeed are well known. To prove (viii) we note that $0 \leqq g_2/g_1$ and hence that

$$\begin{aligned} v_{a,\infty}[(1 + g_2/g_1)^{-1}] &\leqq v_{a,\infty}(g_2/g_1) \leqq v_{a,\infty}(1/g_1) s_a(g_2) + s_a(1/g_1) v_{a,\infty}(g_2) \\ &\leqq b^{-1}[s_a(g_2) + v_{a,\infty}(g_2)] \end{aligned} \tag{2.4}$$

by properties (iv)–(vii). Thus the identity

$$f/g = (f/g_1)(1 + g_2/g_1)^{-1} \tag{2.5}$$

yields (2.3) by property (v).

We shall be concerned primarily with functions on $[0, +\infty)$ and, since this interval is understood throughout, we let LL denote the set of all functions in $C^{[0,+\infty)}$ which are locally integrable and let L denote the set of all functions in LL which are absolutely integrable. Moreover, we let B denote the set of all functions in LL with $f(+\infty) = \lim_{t \to +\infty} f(t) = 0$ and with $v_{a,\infty}(f) < +\infty$ for some nonnegative a. Then B, L and LL are complex vector spaces, and B has a useful decomposition property, established in the following lemma.

LEMMA 2. *If f is in B, then $f = \sum_{i=0}^{4} c_i f_i$, where f_0 is in L, $c_0, \cdots, c_4$ are in C and $f_1, \cdots, f_4$ are nonnegative nonincreasing functions with $f_i(+\infty) = 0$.*

Proof. If we choose $a \geqq 0$ so that $v_{a,\infty}(f) < +\infty$ and let

$$f_0(t) = \begin{cases} f(t) & \text{on} \quad [0, a), \\ 0 & \text{on} \quad [a, +\infty), \end{cases} \tag{2.6}$$

then $\operatorname{Re}(f - f_0)$ and $\operatorname{Im}(f - f_0)$ have bounded variation on $[0, \infty)$ so that

$$\operatorname{Re}(f - f_0) = g_1 - g_2, \quad \operatorname{Im}(f - f_0) = g_3 - g_4, \tag{2.7}$$

where $g_1, \cdots, g_4$ are nonnegative and nondecreasing (Widder (1941, p. 6)). Moreover, $g_i(t) = 0$ on $[0, a)$ by construction and $g_i(+\infty)$ is finite for $i = 1, 2, 3, 4$, while $g_1(+\infty) = g_2(+\infty)$ and $g_3(+\infty) = g_4(+\infty)$ by our assumptions on f. Thus f has the desired form with $f_i = g_i(+\infty) - g_i$ for $i = 1, \cdots, 4$.

For any f in LL we now define

$$F(\omega) = \lim_{u \to +\infty} \int_0^u \exp(i\omega t) f(t)\, dt \tag{2.8}$$

when this limit exists. Motivated by the Riemann–Lebesgue lemma we let RL be the set of all f in LL such that $F(\omega)$ exists for sufficiently large ω and vanishes as $\omega \to +\infty$. Clearly L is a subspace of RL and RL is a subspace of LL. We note that if g is in RL and f is in LL with f a translate of g by some real a, then f is in RL since $F(\omega) = \exp(i\omega a)G(\omega)$. Also if g is in RL and f is in LL with f different from g only on some $[0, a)$, then f is in RL since $f - g$ is in L.

To discuss the behavior of $F(\omega)$ near $-\infty$ we need only observe that $\lim_{\omega \to -\infty} F(\omega) = 0$ if and only if RL contains the complex conjugate of f. To discuss the behavior of

$$\int_{-\infty}^{0} \exp(i\omega t) g(t)\, dt = \int_{0}^{+\infty} \exp(-i\omega t) g(-t)\, dt \tag{2.9}$$

near $\pm\infty$, where g is locally integrable on $(-\infty, 0]$, we need only determine whether or not RL contains $f(t) = g(-t)$ and its complex conjugate. Finally to discuss the asymptotic behavior of Fourier sine and cosine transforms we need only combine these remarks suitably, so that our problem is simply to find criteria under which functions f lie in RL.

We begin with a preliminary result which includes the example $f(t) = t^{-r}$. We then establish the existence of $F(\omega)$ for sufficiently large ω and for a class of functions including all of those treated hereafter.

THEOREM 1. *If f is in $B + L$, the vector sum of B and L, then $F(\omega)$ exists for all $\omega \neq 0$ and vanishes as $\omega \to \pm\infty$, so that $B + L \subset RL$.*

Proof. If f is in L, then this statement is the ordinary Riemann-Lebesgue lemma, so that without loss of generality we may suppose that f is in B. Thus Lemma 2 implies that we may suppose $0 \leqq f\downarrow$ and $f(+\infty) = 0$. If $\omega \neq 0$, then such an f satisfies

$$\int_0^u \exp(i\omega t) f(t)\, dt = (i\omega)^{-1}\left[\exp(i\omega u) f(u) - f(0) - \int_0^u \exp(i\omega t)\, df(t)\right] \tag{2.10}$$

through Riemann-Stieltjes integration by parts (Widder (1941, p. 7)). Thus

$$|F(\omega)| \leqq \omega^{-1}[f(0) + v_{0,\infty}(f)] = O(\omega^{-1}) \tag{2.11}$$

as $\omega \to \pm\infty$.

THEOREM 2. *Let $f(t) = g(t)\exp(ip(t))$ with g in B and $p = p_0 + p_1 + p_2$. Let p_0, p_1, p_2 be functions in $R^{[0,+\infty)}$ with p_0 and p_2' in B, p_1' in LL and eventually monotone. Then $F(\omega)$ exists for sufficiently large ω.*

Proof. If we choose a so that $v_{a,\infty}(g)$ and $v_{a,\infty}(p_0) < +\infty$, then we find by computation that $v_{a,\infty}(\exp(ip_0)) < +\infty$, and hence by Lemma 1.5 that $v_{a,\infty}[g\exp(ip_0)] < +\infty$. Also we note that $\exp(ip_0)$ is bounded and measurable on $[0, a)$, so that $g\exp(ip_0)$ is absolutely integrable on $[0, a)$. Thus we may assume $p_0 = 0$, since $g\exp(ip_0)$ is in B, and we may assume $0 \leqq g\downarrow$, $g(+\infty) = 0$ through the decomposition afforded by Lemma 2. We now choose a so that p_1' is monotone on $[a, +\infty)$, $v_{a,\infty}(p_2') < +\infty$, and remark that by hypothesis $p_1'(+\infty)$, $p_2'(+\infty)$ are well-defined.

If $p_1'(+\infty)$ is finite, then $v_{a,\infty}(p')$ is finite, and if

$$\omega \geqq b + p_1'(+\infty) + s_a[p' - p_1'(+\infty)], \qquad b > 0, \tag{2.12}$$

then $v_{a,\infty}[g/(\omega + p')] = O(b^{-1})$ by Lemma 1.8. If $p_1'(+\infty)$ is infinite, then for each ω there exists $a(b,\omega)$ such that $\omega + p'$ has one sign and

$$|\omega + p'| \geqq b + s_{a(b,\omega)}(p_2'), \qquad b > 0, \tag{2.13}$$

on $[a(b,\omega) + \infty)$, whence $v_{a(b,\omega),\infty}[g/(\omega + p')] = O(b^{-1})$ by Lemma 1.8. In either case for each $\omega \geqq$ some ω_0 there exists $u(\omega)$ such that $v_{u,\infty}[g/(\omega + p')] < +\infty$; and for any $v \geqq u$,

$$\begin{aligned} \int_u^v \exp(i\omega t) f(t)\,dt &= [\exp(i\omega t + ip(t))g(t)/(i\omega + ip'(t))]_u^v \\ &\quad - \int_u^v \exp(i\omega t + ip(t))d[g(t)/(i\omega + ip'(t))] \end{aligned} \tag{2.14}$$

through Riemann-Stieltjes integration by parts. Since $g(+\infty) = 0$ this shows the convergence of

$$F(\omega) = \lim_{v \to +\infty} \left(\int_0^u + \int_u^v \right) \exp(i\omega t) f(t)\,dt, \qquad \omega \geqq \omega_0. \tag{2.15}$$

3. Asymptotic *RL* criteria. In this section we study the behavior of transforms $F(\omega)$ for functions $f(t)$ with asymptotic form (1.3), and we prove a theorem which not only motivates our later results but also includes many practical cases. To facilitate the presentation of this theorem we first treat the Fourier transform of a single term in (1.3). That is, we consider the special class of integrals

$$J^n(a, r, \omega) = \int_a^\infty \exp[i\omega t + i\gamma t^\nu] t^{-r} (\log t)^n\,dt, \tag{3.1}$$

where n is a nonnegative integer, a is a nonnegative real number, r is a suitable complex number, γ and ν are arbitrary real numbers. If a is positive, then these transforms exist by Theorem 2 for $-\infty < \nu < +\infty$, $0 < \operatorname{Re}(r)$; but if $a = 0$, then they may not exist unless $0 \leqq \nu$, $0 < \operatorname{Re}(r) < 1$. We must therefore distinguish these cases in the following analysis, which obtains order estimates near $+\infty$ for the transforms (3.1) and thereby includes most of the argument for the main theorem of this section.

We might point out that the function defined by (3.1) is related to an integral originally studied by Faxén (1921), and subsequently encountered by various authors (Abramowitz and Stegun (1964, p. 1002)). This integral may be written in the normalized form

$$Fi(\alpha, \beta; x) = \int_0^\infty \exp(-t + xt^\alpha) t^{\beta - 1}\,dt.$$

Indeed, the integral (3.1) with $a = 0$ is essentially $(\partial/\partial\beta)^m Fi(\alpha, \beta; x)$ for some complex x. Furthermore, we note that (3.1) is related to the stable density functions of probability theory (Feller (1966, p. 548)).

LEMMA 3. *Let a and* $\operatorname{Re}(r)$ *be positive and define*

$$J(a, \omega) = \begin{cases} \omega^{-n-1} & \text{if } \ a = 1, \\ \omega^{-1} & \text{if } \ 0 < a \neq 1. \end{cases} \tag{3.2}$$

Case (i): *If* (γ, ν) *is not in* $(-\infty, 0) \times (1, +\infty)$, *then, as* $\omega \to +\infty$,

$$J^n(a, r, \omega) = O[J(a, \omega)]. \tag{3.3}$$

Case (ii): *If* (γ, ν) *is in* $(-\infty, 0) \times (1, +\infty)$, *then, as* $\omega \to +\infty$,

$$J^n(a, r, \omega) = O[J(a, \omega) + (\log \omega)^n \omega^{(1 - \mathrm{Re}(r) - \nu/2)/(\nu - 1)}]. \tag{3.4}$$

Proof. In Case (i) we integrate (3.1) by parts, taking some care about the upper limit of integration, that is,

$$\begin{aligned} J^n(a, r, \omega) = {} & \lim_{A \to \infty} \left[\frac{\exp(i\omega t + irt^\nu)}{i\omega + i\nu\gamma t^{\nu-1}} t^{-r} (\log t)^n \right]_a^A \\ & - \int_a^\infty \exp(i\omega t + i\gamma t^\nu) \frac{d}{dt} \left\{ \frac{t^{-r} (\log t)^n}{i\omega + i\nu\gamma t^{\nu-1}} \right\} dt. \end{aligned} \tag{3.5}$$

Here we may let $A \to \infty$ in the first line and obtain an estimate $O(\omega^{-1})$. By carrying out the explicit differentiation in the second line we then obtain ω^{-1} multiplying the Fourier transform of an L^1-function. Thus the estimate $O(\omega^{-1})$ holds in this case. For $a = 1$, we must integrate by parts n times before obtaining an explicit nonzero amplitude from the endpoint a. Hence, we have the result (3.3).

From the explicit result (3.5) we see that this method fails in the range of Case (ii), because for any positive a and large enough ω the exponent $i\omega t + i\gamma t^\nu$ has a stationary point in the domain of integration and hence the denominator $i\omega + i\nu\gamma t^{\nu-1}$ vanishes on this domain. We would like here to obtain our estimate by the method of stationary phase, but that method in rigorous form requires an L^1 amplitude, which we do not have at hand. Therefore, we must essentially rederive and extend the method to include the present case, in which (3.1) converges only conditionally.

To approximate (3.1) when (γ, ν) is in $(-\infty, 0) \times (1, +\infty)$ we set

$$q(t) = \omega t + \gamma t^\nu, \quad \beta = (\nu - 1)^{-1} \tag{3.6}$$

and use the method of steepest descent (see Erdélyi (1956, pp. 39–41)). Under the assumptions for (3.4) the function $q'(t) = dq/dt$ has simple zeros at

$$t_m = |\omega/\gamma\nu|^\beta \exp(2\pi i\beta m), \qquad m = 0, \pm 1, \pm 2, \cdots, \tag{3.7}$$

so that $\exp(iq(t))$ has simple saddle points at these t_m. Since a is fixed, t_0 increases with increasing ω, and since we are concerned with the limit $\omega \to +\infty$, we can without loss of generality assume that $a < t_0$.

A path C_a of steepest descent leaves any a in $[0, t_0)$ with the direction $\arg t = \pi/2$, and ends at ∞ with the direction $\arg t = 3\pi/2\nu$. Two paths of steepest descent C_+ and C_- leave the saddle point with the respective directions $\arg t = 3\pi/4$ and $\arg t = -\pi/4$ and end at ∞ with the respective directions $\arg t = 3\pi/2\nu$ and $\arg t = -\pi/2\nu$. The original contour may now be deformed into the path C_a from a to $\infty \exp(3\pi i/2\nu)$, the path $-C_+$ from $\infty \exp(3\pi i/2\nu)$ to t_0 and the path C_- from t_0 to $\infty \exp(-\pi/2\nu)$.

We find by Watson's lemma that the integral along C_a is $O[J(a, \omega)]$ as $\omega \to +\infty$. To discuss the integrals along the contours $C_\pm$ we introduce a new variable of integration u on the contour $C_- - C_+$, defined by

$$t = \omega^\beta u, \quad \beta = (\nu - 1)^{-1}, \tag{3.8}$$

so that we can write

$$J^n(a, r, \omega) = O[J(a, \omega)] + \omega^{\beta(1-r)} \int_{C_-^* - C_+^*} u^{-r} \exp[i\omega^{\beta\nu}(u + \gamma u^\nu)] \cdot (\log u + \beta \log \omega)^n \, du. \tag{3.9}$$

The mapping (3.8) carries the saddle point $t = t_0$ into $u_0 = |\gamma\nu|^{-\beta}$ which, we note, is independent of ω. Moreover, the general description of the image contour $C_-^* - C_+^*$ is otherwise unaltered from that of $C_- - C_+$. We then find by the method of steepest descent that the integral in (3.9) yields the second term of (3.4).

Remark. If we let $a = 0 < \mathrm{Re}(r) < 1$ and repeat the preceding argument, then we can recover (3.3) and (3.4) with $J(a, \omega) = \omega^{r-1}(\log \omega)^n$, but we shall not need this result hereafter.

If we let

$$f(t) = \begin{cases} 0 & \text{on} \quad [0, a), \\ \exp(i\gamma t^\nu) t^{-r} (\log t)^n & \text{on} \quad [a, +\infty), \end{cases} \tag{3.10}$$

with γ, ν, a, r, n, as in Lemma 3, then we have f in RL by this last lemma unless

$$\gamma < 0 < 1 < \nu \quad \text{and} \quad \mathrm{Re}(r) + \nu/2 \leqq 1. \tag{3.11}$$

Furthermore we cannot achieve (3.11) unless $1 < \nu < 2$, since we have assumed $0 < \mathrm{Re}(r)$ in Lemma 3. These last remarks now yield the following theorem, which give criteria for RL sufficient in most practical cases.

THEOREM 3. *Let f be in LL and, as $t \to \infty$,*

$$f(t) - g(t) \sim \exp(i\gamma t^\nu) \sum_{m=0}^{\infty} \sum_{n=0}^{N(m)} c_{mn} t^{-r(m)} (\log t)^n \tag{3.12}$$

for some g in $B + L$. Here $N(m)$ is finite for each m, γ and ν are arbitrary real numbers, the c_{mn} are arbitrary complex numbers, and the $r(m)$ are complex numbers such that $0 < \mathrm{Re}(r(m)) \uparrow +\infty$ as $m \to \infty$. Then f is in RL unless

$$\gamma < 0 < 1 < \nu < 2 \quad \textit{and} \quad \mathrm{Re}(r(0)) + \nu/2 \leqq 1. \tag{3.13}$$

Proof. If $f_2 = f - f_1$, where

$$f_1(t) = \begin{cases} 0 & \text{on} \quad [0, 1), \\ \exp(i\gamma t^\nu) \displaystyle\sum_{0 < \mathrm{Re}(r(m)) \leqq 1} \sum_{n=0}^{N(m)} c_{mn} t^{-r(m)} (\log t)^n & \text{on} \quad [1, +\infty), \end{cases} \tag{3.14}$$

then f_2 is in $B + L$ and f_1 is a finite sum. A typical term of this sum has the form (3.10) so that its Fourier transform has the form $c_{mn}J^n(1, r(m); \omega)$. Thus f_1 is in RL unless $\mathrm{Re}(r(m)) + \nu/2 \leqq 1$ for some m, which implies (3.13) since $\mathrm{Re}(r(m))$ increases with m.

To understand both Theorem 3 and the anomaly (3.13) we return to Lemma 3 and observe that the term $O[J(a, \omega)]$, which decays as $\omega \to \infty$, is the contribution from the critical point $t = a$, while the second term in (3.4), which yields the anomaly, is the contribution from the critical point $t = +\infty$. The magnitude of this second contribution is governed by two opposing mechanisms. Near $+\infty$ we have that on the one hand the rapid oscillation in (3.1) of the factor $\exp[i\omega t + i\gamma t^\nu]$ tends to decrease this contribution through cancellation effects, while on the other hand the dependence on r of the factor $t^{-r}(\log t)^n$ tends to increase this contribution as $\mathrm{Re}(r)$ decreases. The phase $q(t)$ of the oscillation is given by (3.6) and the resulting cancellation in (3.1) is itself diminished near any real point where $q(t)$ is stationary, that is, where $q'(t) = 0$.

No such stationary point exists when $\nu = 1$, but otherwise a stationary point occurs on $[0, +\infty)$ if and only if $\gamma\nu < 0$, and is then given by

$$t_0 = |\omega/\gamma\nu|^\beta, \quad \beta = (\nu - 1)^{-1}. \tag{3.15}$$

If $\nu < 1$, then $t_0 \to 0$ as $\omega \to \infty$, so that t_0 does not affect the contribution from $t = +\infty$; but if $\nu > 1$, then $t_0 \to \infty$ as $\omega \to \infty$ so that the contribution from the saddle point $t = t_0$ tends to increase the contribution from $t = \infty$. If $\nu \geqq 2$, then the oscillation of $\exp(iq(t))$ again becomes rapid enough near $+\infty$ to cause decay of the total contribution from $t = \infty$. However, if $1 < \nu < 2$, then the oscillation is not rapid enough, when $\mathrm{Re}(r)$ is small, to cause decay. Indeed Lemma 3, which is correct in all cases, shows that this contribution is $O(1)$ when $\mathrm{Re}(r) + \nu/2 = 1$ and even grows when $\mathrm{Re}(r) + \nu/2 < 1$.

4. Qualitative *RL* criteria. In the preceding section we have extended the Riemann-Lebesgue lemma to a class of functions in LL with a given asymptotic form near $+\infty$. As we have emphasized, this result should suffice for most applications but nevertheless can be further extended in various ways. Indeed, on the one hand, we might give up the demand that f be absolutely integrable near $0+$ and allow f to have, as $t \to 0+$, an asymptotic form given by

$$f(t) \sim \exp(i\gamma t^{-\nu})\, d[t^{-r}(\log t)^n] \tag{4.1}$$

with $0 < \nu$ and $r < \min(2, 1 + \nu/2)$. Naturally, under assumption (4.1) the contribution to $F(\omega)$ from the critical point at $t = 0$ will no longer be $O[\omega^{r-1}(\log \omega)^n]$ as noted after Lemma 3, but will be derived again by arguments like those of Lemma 3, involving Watson's lemma and the method of steepest descent.

On the other hand, one might generalize the expansion (3.12) near $+\infty$ and allow f to have an asymptotic form

$$f(t) \sim \exp\left[i\gamma t^\nu \sum_{l=0}^{l<\nu/\delta} b_l t^{-\delta l} \right] \sum_{m,n=0}^{\infty} c_{mn} t^{-r(m)} (\log t)^{s(n)} \tag{4.2}$$

as $t \to \infty$. Here $\delta > 0$, $b_0 = 1$, all b_l are real and $s(n)$ may take nonintegral or even negative values, while the other parameters are still restricted as in the form (3.12). Some results of this type may also be proved similarly to Theorem 3, but these may also be subsumed under a further extension in which f obeys only qualitative assumptions near ∞. This extension includes Lemma 3 for all values of (γ, ν, a, r, n) but requires different hypotheses in different ranges of these parameters so that it will now be presented as three theorems. For example, the first of these theorems treats functions with behavior somewhat similar to (3.12) but with (γ, ν) not in $(-\infty, 0) \times (1, \infty)$.

THEOREM 4. *Let* $f(t) = g(t)\exp[ip(t)]$ *with* g *in* B *and* $p = p_0 + p_1 + p_2$. *Let* p_0, p_1 *and* p_2 *be functions in* $R^{[0,+\infty)}$ *with* p_0, p_2' *in* B, p_1' *in* LL *and eventually nondecreasing. Then* f *is in* RL.

Proof. As in Theorem 2, we can absorb p_0, decompose g, and assume $p_0 = 0, 0 \leqq g\downarrow$ and $g(+\infty) = 0$ without loss of generality. Also through Theorem 2, we know that $F(\omega)$ exists as an improper integral for sufficiently large ω. By the remarks following (2.8) we need only show that for some fixed $a \geqq 0$,

$$\int_a^\infty \exp(i\omega t) f(t)\,dt \to 0 \tag{4.3}$$

as $\omega \to +\infty$. We can choose a so that $v_{a,\infty}(p_2') < +\infty$, $p_1'\uparrow$ on $[a, +\infty)$, and we can choose ω_1, ω_2 so that $0 \leqq \omega_1 + p_1', \ \omega_2 + p_2'$ on $[a, +\infty)$. If $b > 0$ and $\omega = b + \omega_1 + \omega_2$, then by Lemma 1.8, $v_{a,\infty}[g/(\omega + p')] = O(b^{-1})$, and through Riemann-Stieltjes integration by parts (see Widder (1941, p. 7)),

$$\begin{aligned}\int_a^\infty \exp(i\omega t) f(t)\,dt &= -\exp(i\omega a + ip'(a))g(a)/(\omega + p'(a)) \\ &\quad - \int_a^\infty \exp(i\omega t + ip'(t))d[g/(\omega + p'(t))] = O(b^{-1})\end{aligned} \tag{4.4}$$

as $\omega \to +\infty$. This yields (4.3) since $O(b^{-1}) = O(\omega^{-1})$.

We now obtain qualitative criteria which correspond to (3.4) and in particular to the part of Lemma 3 for which we invoked the method of steepest descent. First we generalize in Theorem 5 the more amenable case within (3.4) in which $\gamma < 0 < 2 < \nu$, and then we generalize in Theorem 6 the anomalous case in which $\gamma < 0 < 1 < \nu \leqq 2$. These results use, and thus follow, a basic lemma of Titchmarsh (1948, p. 22) which in turn extends an earlier theorem of Landau (1927, p. 413). The proofs are suggested by, but are apparently not contained in, the work of Titchmarsh.

LEMMA 4. *On any interval* $[a, b]$ *let* f, g, h *be real-valued functions with* $0 \leqq f\downarrow$, g/h' *monotone,* h *absolutely continuous. Then*

$$\left|\int_a^b f(t)g(t)\exp[ih(t)]\,dt\right| \leqq 2\sqrt{2}f(a)\max\{|g(t)/h'(t)|\,; a \leqq t \leqq b\}. \tag{4.5}$$

Proof. This result is obtained by Titchmarsh (1948, p. 22) with $\sin h$ or $\cos h$ in place of $\exp(ih)$ on the left side and 2 in place of $2\sqrt{2}$ on the right side.

THEOREM 5. *Let* $f(t) = g(t)\exp[ip_0(t) - iq(t)]$ *with* g *in* B, p_0 *in* $B \cap R^{[0,+\infty)}$ *and* q *in* $LL \cap R^{[0,+\infty)}$. *Let* q *be in* C^2 *on some* $[a, +\infty)$ *with* $q''(t)\uparrow$ *and* $q''(+\infty) = +\infty$. *Then* f *is in* RL.

Proof. As in Theorem 2, we know that $F(\omega)$ exists for sufficiently large ω. We can choose a so large that $0 < q(t), q'(t), q''(t)$ on $[a, +\infty)$ and then we can assume $a = 0$ by the remarks following (2.8). Thus on $[0, +\infty)$ after this reduction, $q'(t)$ is continuous and strictly increasing with $q'(+\infty) = +\infty$ so that for all sufficiently large ω, the equation $q'(t) = \omega$ has a unique solution $t(\omega)$ with $t(\omega)\uparrow$ and $t(+\infty) = +\infty$. Hence we note that, as $\omega \to +\infty$,

$$\left|\int_{t(\omega)-1}^{t(\omega)+1} \exp(i\omega t) f(t)\,dt\right| = \left|\int_{t(\omega)-1}^{t(\omega)+1} g(t)\exp[i\omega t - iq(t)]\,dt\right| \leqq 2g[t(\omega) - 1] \to 0. \tag{4.6}$$

If $h(t) = \omega t - q(t)$ so that $h'(t) = \omega - q'(t)$, then $0 < 1/h'\uparrow$ on $[0, t(\omega) - 1)$ and $0 < -1/h'\downarrow$ on $[t(\omega) + 1, +\infty)$. If η denotes a number in $(0, 1)$ which need not be the same at each appearance, then by Lemma 4 and the mean value theorem,

$$\begin{aligned}\left|\int_0^{t(\omega)-1} q(t)\exp[ih(t)]\,dt\right| &= g(0)O(1/h'[t(\omega) - 1]) \\ &= O(q'[t(\omega)] - q'[t(\omega) - 1])^{-1} \\ &= O(1/q''[t(\omega) - \eta]) = O(1/q''[t(\omega) - 1]) \to 0\end{aligned} \tag{4.7}$$

as $\omega \to +\infty$. Also,

$$\begin{aligned}\left|\int_{t(\omega)+1}^{\infty} g(t)\exp[ih(t)]\,dt\right| &= g[t(\omega) + 1]O(-1/h'[t(\omega) + 1]) \\ &= O(q'[t(\omega) + 1] - q'[t(\omega)])^{-1} \\ &= O(1/q''[t(\omega) + \eta]) \\ &= O(1/q''[t(\omega)]) \to 0\end{aligned} \tag{4.8}$$

as $\omega \to +\infty$. We obtain the desired result by combining (4.6)–(4.8).

THEOREM 6. *Let* $f(t) = g(t)\exp[-iq(t)]$ *with* g *and* q *in* LL. *Then* f *is in* RL *whenever the following hold on some* $[a, \infty)$: (i) g *is nonincreasing with* $g(+\infty) = 0$; (ii) q *is* C^2 *with* $0 < q', q''$; (iii) $q''(t)\downarrow$ *but* $tq''(t)\uparrow$ *with limit* $+\infty$; (iv) *there exists* δ *in* $(0, 1)$ *for which* $g^2(\delta t)/q''(t)\downarrow$ *with limit* 0.

Proof. We can choose a so large that $0 < q(t)$ also on $[a, +\infty)$, and then we can assume $a = 0$ by the remarks following (2.8). We verify by Theorem 2 that $F(\omega)$ exists for sufficiently large ω; we define, as in Theorem 5, a function $t(\omega)$ such that $q'[t(\omega)] = \omega$; and we show as before that $t(\omega)\uparrow$ with $t(+\infty) = +\infty$. This last assertion follows from $q'(+\infty) = +\infty$, which in turn follows from $tq''(t)\uparrow$.

If $d(\omega) = q''(t(\omega))^{-1/2}$, then by assumptions (iii) and (iv):

$$\begin{aligned}&d(\omega)g[\delta t(\omega)]\downarrow 0, \\ &d(\omega)/t(\omega) = [t^{-1/2}(tq''(t))^{-1/2}]_{t=t(\omega)} \downarrow 0\end{aligned} \tag{4.9}$$

as $\omega \to +\infty$, so that $\delta t(\omega) < t(\omega) - d(\omega)$ for sufficiently large ω. Hence we note that

$$\begin{aligned}\left|\int_{t(\omega)-d(\omega)}^{t(\omega)+d(\omega)} g(t)\exp[i\omega t - iq(t)]\,dt\right| &\leqq 2d(\omega)g[t(\omega) - d(\omega)] \\ &\leqq 2d(\omega)g[\delta t(\omega)] \downarrow 0\end{aligned} \tag{4.10}$$

as $\omega \to +\infty$.

If $h(t) = \omega t - q(t)$, so that $h'(t) = \omega - q'(t)$, then $0 < 1/h'\uparrow$ on $[0, t(\omega) - d(\omega))$ and $0 < -1/h'\downarrow$ on $[t(\omega) + d(\omega), +\infty)$. If η denotes a number in $(0, 1)$ which need not be the same at each appearance, then by Lemma 4 and the mean value theorem,

$$\begin{aligned}\left|\int_0^{\delta t(\omega)} g(t)\exp[ih(t)]\,dt\right| &= O(q(0)/h'[\delta t(\omega)]) = O(q'[t(\omega)] - q'[\delta t(\omega)])^{-1} \\ &= O(1/t(\omega)q''[(\delta + \eta - \delta\eta)t(\omega)]) \\ &= O(1/t(\omega)q''[t(\omega)]) \to 0,\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
\left|\int_{\delta t(\omega)}^{t(\omega)-d(\omega)} g(t)\exp[ih(t)]\,dt\right| &= O(g[\delta t(\omega)]/h'[t(\omega)-d(\omega)]) \\
&= g[\delta t(\omega)]O(q'[t(\omega)]-q'[t(\omega)-d(\omega)])^{-1} \\
&= O(g[\delta t(\omega)]/d(\omega)q''[t(\omega)-\eta d(\omega)]) \\
&= O(g[\delta t(\omega)]d(\omega)) \to 0,
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
\left|\int_{t(\omega)+d(\omega)}^{\infty} g(t)\exp[ih(t)]\,dt\right| &= O(g[t(\omega)+d(\omega)]/h'[t(\omega)+d(\omega)]) \\
&= O(g[t(\omega)+d(\omega)]/d(\omega)q''[t(\omega)+\eta d(\omega)]) \\
&= O\left[\frac{(t(\omega)q''[t(\omega)])^{1/2}g[t(\omega)+d(\omega)]}{t(\omega)^{1/2}q''[t(\omega)+\eta d(\omega)]}\right] \\
&= \left[\frac{t(\omega)+\eta d(\omega)}{t(\omega)}\right]^{1/2} O\left[\frac{g[t(\omega)+d(\omega)]}{q''[t(\omega)+\eta d(\omega)]^{1/2}}\right] \\
&= O(g[\delta t(\omega)+\delta d(\omega)]/q''[t(\omega)+d(\omega)]^{1/2}) \to 0
\end{aligned}
\tag{4.13}
$$

as $\omega \to +\infty$. We obtain the desired result by combining (4.10)–(4.13).

The stationary point (3.15) which we discussed in § 3 satisfies $q'(t) = 0$ for the given phase $q(t)$, so that it corresponds to the point $t(\omega)$ in the proof of Theorems 5 and 6. The domain (3.13) which we proscribed in the anomalous case relates the amplitude and phase of the given $f(t)$, so that it corresponds to assumption (iv) in the statement of Theorem 6. Thus an assumption like (iv) is clearly needed in Theorem 6, so that a function p_0 in B cannot always be added to the phase and then absorbed as in Theorem 5. However, if $g \exp[ip]$ is in RL for some g and real-valued p in LL, where g is eventually bounded, then $(g + \Delta g)\exp[i(p + \Delta p)]$ is in RL for any Δg and real-valued Δp in L, since the difference Δf can be shown to lie in L. Through this remark we can construct still more general functions f in RL, whose amplitude and phase need not be continuous, or have bounded variation.

REFERENCES

M. Abramowitz and I. A. Stegun (1964), *Handbook of Mathematical Functions*, U.S. Government Printing Office, Washington, D.C.

G. Doetsch (1950), *Handbuch der Laplace-Transformation, Band I, Theorie der Laplace-Transformation*, Verlag Birkhauser, Basel.

A. Erdélyi (1956), *Asymptotic Expansions*, Dover, New York.

H. Faxén (1921), *Expansion in series of the integral* $\int_y^\infty \exp[-x(t \pm t^{-n})]t^2\,dt$, Ark. Mat. Astr. Fys., 15, no. 13, pp. 1–57.

W. Feller (1966), *An Introduction to Probability Theory and Its Applications*, John Wiley, New York.

E. Landau (1927), *Vorlesungen über Zahlentheorie*, Chelsea, New York, 1955.

E. C. Titchmarsh (1948), *Introduction to the Theory of Fourier Integrals*, 2nd ed., Oxford University Press, London.

D. V. Widder (1941), *The Laplace Transform*, Princeton University Press, Princeton, N.J.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

REMARKS ON THE EXISTENCE THEORY FOR MULTIPLE SOLUTIONS OF A SINGULAR PERTURBATION PROBLEM*

SEYMOUR V. PARTER†

Abstract. The existence and nonexistence of solutions of $\varepsilon y'' + y' = g(x, y)$, $y'(0) - ay(0) = A$, $y'(1) + by(1) = B$ is discussed. Iterative methods may be used to obtain "every other" solution.

1. Introduction.

Consider the singular perturbation problem

$$L_\varepsilon[y] \equiv \varepsilon y'' + y' = g(x, y), \qquad 0 < x < 1, \tag{1.1}$$

$$y'(0) - ay(0) = A, \qquad a \geqq 0, \tag{1.2}$$

$$y'(1) + by(1) = B, \qquad b \geqq 0. \tag{1.3}$$

In [2] D. S. Cohen showed that this problem can have several distinct "asymptotic solutions" for all sufficiently small $\varepsilon > 0$. An asymptotic solution is a function $y(x)$ which satisfies (1.1), (1.2) but only satisfies (1.3) to within $O(\varepsilon)$.

In [8] H. B. Keller extended Cohen's results by weakening the conditions on $g(x, y)$ and considering more general boundary conditions

$$f(u'(1), u(1)) = 0 \tag{1.3a}$$

in place of (1.3). Moreover, under some additional conditions Keller showed the existence of exact solutions of (1.1), (1.2), (1.3a) near the asymptotic solutions discussed by Cohen.

The results of Cohen [2] and Keller [8] are based on the "shooting method". Our analysis is based on the theory of the modified boundary value problem in which the boundary condition (1.3) is replaced by

$$y(1) = \alpha, \tag{1.3'}$$

where α is a specified real number.

In this way we provide further insights into the structure of these problems. Moreover, Theorems 3.2, 3.3 and 3.4 deal with cases not treated by Keller [8]. In particular, Theorem 3.4 is concerned with a case in which there are asymptotic solutions and there is no exact solution nearby.

Finally, in § 4 we discuss iterative methods for obtaining "every other" solution. This iteration scheme is particularly interesting because each iterate $V_n(x)$, except $V_0(x)$, is the solution of a linear boundary value problem, but $V_0(x)$ is the solution of a nonlinear boundary value problem (1.1), (1.2), (1.3′).

While we shall restrict ourselves to the original boundary conditions (1.3) the extension to boundary conditions of the more general form (1.3a) is relatively easy.

---

* Received by the editors June 7, 1971, and in revised form September 29, 1971.

† Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706. This work was sponsored by the Officer of Naval Research under Contract N 00014-67-A-0128-0004.

However, let us first reformulate our hypotheses:

(H1) $$g(x, y) \in C^1\{[0, 1] \times \mathscr{R}^1\},$$

(H2) $$|g(x, y)| \leqq M, \quad (x, y) \in \{[0, 1] \times \mathscr{R}^1\}.$$

Let

(H3) $$H(\alpha) = g(1, \alpha) + b\alpha - B$$

have exactly J roots $\alpha_1, \alpha_2, \cdots, \alpha_J$.

2. Preliminary results. In this section we collect some basic facts about quasi-linear boundary value problems (1.1), (1.2), (1.3′) and (1.1), (1.2), (1.3). Many of these results are well known, if not readily accessible in the literature. Most of these results are based on the maximum principle (see [1], [4], [5], [9], [10]). Our first result is a basic a priori estimate.

LEMMA 2.1. *Let* $\varepsilon > 0$ *be fixed. Let* $\varphi(x) \in C^2[0, 1]$ *satisfy*

(2.1) $$|L_\varepsilon[\varphi]| \leqq M,$$

(2.2) $$\varphi'(0) - a\varphi(0) = A,$$

(2.3) $$\varphi(1) = \alpha.$$

Then

(2.4) $$|\varphi(x)| \leqq |\alpha| + 2M + 2\varepsilon[(1 + a)M + |A| + |\alpha|a] = K_1(\alpha),$$

(2.5) $$|\varphi'(x)| \leqq aK_1(\alpha) + |A| + M = K_2(\alpha).$$

Proof. Let $V_0(x, \varepsilon, \alpha)$ and $U_0(x, \varepsilon, \alpha)$ be given by

(2.6a) $$V_0(x, \varepsilon, \alpha) = K + K_0 e^{-x/\varepsilon} + Mx,$$

(2.6b) $$U_0(x, \varepsilon, \alpha) = K_1 + K_2 e^{-x/\varepsilon} - Mx,$$

where

$$K_0 = \frac{\varepsilon[(1 + a)M - (A + \alpha a)]}{1 + \varepsilon a(1 - e^{-1/\varepsilon})},$$

$$K = (\alpha - M) - e^{-1/\varepsilon}K_0,$$

$$K_2 = \frac{-\varepsilon[(1 + a)M + (A + \alpha a)]}{1 + \varepsilon a(1 - e^{-1/\varepsilon})},$$

$$K_1 = (\alpha + M) - e^{-1/\varepsilon}K_2.$$

Then a direct calculation shows that $U_0(x, \varepsilon, \alpha)$ and $V_0(x, \varepsilon, \alpha)$ satisfy

$$L_\varepsilon U_0 = -M, \qquad 0 \leqq x \leqq 1,$$

$$L_\varepsilon V_0 = M, \qquad 0 \leqq x \leqq 1,$$

$$U_0'(0) - aU_0(0) = A = V_0'(0) - aV_0(0),$$

$$U_0(1) = \alpha = V_0(1).$$

And, as an immediate consequence of the maximum principle we obtain

$$V_0(x, \varepsilon, \alpha) \leqq \varphi(x) \leqq U_0(x, \varepsilon, \alpha),$$

and (2.4) follows at once.

We rewrite the basic differential equation as

$$(e^{x/\varepsilon}\varphi')' = \frac{1}{\varepsilon} e^{x/\varepsilon} L_\varepsilon[\varphi].$$

After one integration we have

$$|\varphi'(x)| \leqq |\varphi'(0)|\, e^{-x/\varepsilon} + \frac{M}{\varepsilon}\int_0^x e^{(s-x)/\varepsilon}\, ds.$$

Since

$$|\varphi'(0)| \leqq a|K_1(\alpha)| + |A|$$

we obtain (2.5).

THEOREM 2.1. *Let $\varepsilon > 0$ be fixed. For every $\alpha \in \mathscr{R}^1$ there exists a solution $Z(x, \varepsilon, \alpha)$ of* (1.1), (1.2) *and* (1.3′). *Indeed there is a maximal solution $M(x, \varepsilon, \alpha)$ and a minimal solution $m(x, \varepsilon, \alpha)$ in the sense that: if $Z(x, \varepsilon, \alpha)$ is any solution, then*

$$m(x, \varepsilon, \alpha) \leqq Z(x, \varepsilon, \alpha) \leqq M(x, \varepsilon, \alpha). \tag{2.7}$$

Moreover, $M(x, \varepsilon, \alpha)$ is monotone nondecreasing in α and continuous from the right, $m(x, \varepsilon, \alpha)$ is monotone nonincreasing in α and continuous from the left. Finally, $M'(x, \varepsilon, \alpha)$ is continuous from the right while $m'(x, \varepsilon, \alpha)$ is continuous from the left.

Proof. The existence of $m(x, \varepsilon, \alpha)$ and $M(x, \varepsilon, \alpha)$ (which may or may not be equal) follows exactly as in [1], [4], [9], [10]. We now sketch this proof. Using the basic a priori estimate (2.4) of Lemma 2.1 we may modify $g(x, y)$ for large y (see [7]) so that $g(x, y)$ may be assumed to satisfy a uniform Lipschitz condition with constant δ. Let $V_0(x, \varepsilon, \alpha)$ and $U_0(x, \varepsilon, \alpha)$ be the functions given by (2.6a) and (2.6b) respectively. Let $U_n(x, \varepsilon, \alpha)$, $v_n(x, \varepsilon, \alpha)$ satisfy

$$\begin{aligned}
&L_\varepsilon U_{n+1} - \delta U_{n+1} = g(x, U_n) - \delta U_n, && 0 \leqq x \leqq 1,\\
&L_\varepsilon v_{n+1} - \delta v_{n+1} = g(x, v_n) - \delta v_n, && 0 \leqq x \leqq 1,\\
&U'_{n+1}(0) - aU_{n+1}(0) = A = v'_{n+1}(0) - av_{n+1}(0), &&\\
&U_{n+1}(1) = \alpha = v_{n+1}(1). &&
\end{aligned}$$

A straightforward argument shows that

$$U_n(x, \varepsilon, \alpha) \searrow M(x, \varepsilon, \alpha),$$
$$v_n(x, \varepsilon, \alpha) \nearrow m(x, \varepsilon, \alpha).$$

The monotonicity of $M(x, \varepsilon, \alpha)$ and $m(x, \varepsilon, \alpha)$ (in α) follows from the same argument as the proof of the similar Theorem 2.1 of [9]. As in [9], the one-sided continuity of $M(x, \varepsilon, \alpha)$ and $m(x, \varepsilon, \alpha)$ follows from the definition of maximal and minimal solutions.

Finally, the one-sided continuity of $M'(x, \varepsilon, \alpha)$ and $m'(x, \varepsilon, \alpha)$ follows from the Green's function representation of L_ε^{-1} and the one-sided continuity of $M(x, \varepsilon, \alpha)$ and $m(x, \varepsilon, \alpha)$ respectively.

LEMMA 2.2. *Let σ be a bound for $|g_y(x, y)|$, i.e., σ is a Lipschitz constant for $g(x, y)$ as in Theorem* 2.1. *Let*

$$\bar{\varepsilon} = \frac{1}{4\sigma}. \tag{2.8}$$

Then, for all ε, $0 < \varepsilon \leqq \bar{\varepsilon}$,

$$M(x, \varepsilon, \alpha) = m(x, \varepsilon, \alpha) \tag{2.9}$$

and $M'(x, \varepsilon, \alpha) = m'(x, \varepsilon, \alpha)$ is a continuous function of α.

Proof. Let

$$W(x, \varepsilon, \alpha) = M(x, \varepsilon, \alpha) - m(x, \varepsilon, \alpha).$$

Let

$$Q(x, \varepsilon, \alpha) = e^{x/2\varepsilon} W(x, \varepsilon, \alpha).$$

Then $W(x, \varepsilon, \alpha)$ satisfies the equation

$$\varepsilon W'' + W' - g_y(x, \zeta)W = 0.$$

And $Q(x, \varepsilon, \alpha)$ satisfies the equation

$$\varepsilon Q'' - \left[\frac{1}{4\varepsilon} + g_y(x, \zeta)\right] Q = 0,$$

$$Q'(0) - \left[a + \frac{1}{2\varepsilon}\right] Q(0) = 0,$$

$$Q(1) = 0.$$

Using (2.8) and the maximum principle we see that $Q(x, \varepsilon, \alpha) \equiv 0$ and (2.9) follows at once. The continuity of $M'(x, \varepsilon, \alpha)$ follows from the one-sided continuity of Theorem 2.1.

LEMMA 2.3. *Let $\varphi(x) \in C^2[0, 1]$ satisfy*

$$\begin{aligned} &L_\varepsilon[\varphi] = g(x, \varphi), \qquad 0 \leqq x \leqq 1, \\ &|\varphi(1)| \leqq M_0, \\ &\varphi'(0) - a\varphi(0) = A. \end{aligned} \tag{2.10}$$

Then there is a constant $K_3 = K_3(M, M_0, |A|)$ depending on $M, M_0, |A|$, but not on ε, such that

$$|\varphi''(x)| \leqq \frac{K_3}{x}, \qquad 0 < x \leqq 1. \tag{2.11}$$

Proof. Let $v(x) = \varphi'(x)$. Then

$$L_\varepsilon v = g_x(x, \varphi) + g_y(x, \varphi)\varphi'.$$

Applying Lemma 2.1 we see that the right-hand side of this equation is bounded by a constant depending only on $M, M_0, |A|$. Similarly, $|v(0)|, |v(1)|$ is bounded. Thus, we may apply Theorem 2.7 of [5].

THEOREM 2.2. *Let* α_j *be a root of*

$$H(\alpha) = 0.$$

Then all solutions (1.1), (1.2), (1.3′) *with*

$$\alpha = \alpha_j$$

are asymptotic solutions of (1.1), (1.2) *and* (1.3) *in the sense of* [2].

Proof. We need merely check (1.3). We have

$$Z'(1, \varepsilon, \alpha_j) + bZ(1, \varepsilon, \alpha_j) = H(\alpha_j) + B - \varepsilon Z''(1, \varepsilon, \alpha_j);$$

hence,

$$|Z'(1, \varepsilon, \alpha_j) + bZ(1, \varepsilon, \alpha_j) - B| \leqq K_3 \cdot \varepsilon.$$

THEOREM 2.3. *Let* α *be fixed. There exists a unique function* $W(x, \alpha)$ *which satisfies*

$$W' = g(x, W), \qquad 0 < x < 1, \tag{2.12a}$$

$$W(1, \alpha) = \alpha, \tag{2.12b}$$

$$|W(x, \alpha)| \leqq K_1(\alpha). \tag{2.12c}$$

Moreover, let $Z(x, \varepsilon, \alpha)$ *be any solution of* (1.1), (1.2) *and* (1.3′). *Then*

$$\max \{|Z(x, \varepsilon, \alpha) - W(x, \alpha)| ; 0 \leqq x \leqq 1\} \to 0 \quad as \quad \varepsilon \to 0+. \tag{2.13}$$

And, for any $\delta \in (0, 1)$,

$$\begin{aligned} &\max \{|Z'(x, \varepsilon, \alpha) - W'(x, \alpha)| ; \delta \leqq x \leqq 1\} \to 0 \quad as \quad \varepsilon \to 0+, \\ &\max \{|Z''(x, \varepsilon, \alpha) - W''(x, \alpha)| ; \delta \leqq x \leqq 1\} \to 0 \quad as \quad \varepsilon \to 0+. \end{aligned} \tag{2.14}$$

Proof. Using Lemma 2.1 we may again modify $g(x, y)$ for large $|y|$ so that $g(x, y)$ satisfies a uniform Lipschitz condition. Hence the solutions of (2.12a), (2.12b), (2.12c) are unique. The theorem follows along the lines of the proof of Theorem 4.1 of [5] based on Lemma 2.1. The functions $\{Z(x, \varepsilon, \alpha)\}$ are equicontinuous and uniformly bounded. Thus a subsequence converges to a weak solution $W(x, \alpha)$ of (2.12a), (2.12b), (2.12c). However, using a theorem of Friedrichs [6] we see that $W(x, \alpha)$ is a genuine solution. The unicity of the limit function allows us to dispense with the subsequence. Finally, (2.14) is proven by using an argument of Coddington and Levinson [3] (see [5] also).

COROLLARY. *Suppose* $g(x, y) \in C^2\{[0, 1] \times \mathscr{R}\}$. *Then, for every* $\eta > 0$ *and every* $M_0 > 0$ *there exists an* $\varepsilon_0 > 0$ *such that* $0 < \varepsilon \leqq \varepsilon_0$ *implies*

$$|Z''(1, \varepsilon, \alpha) - [g_x(1, \alpha) + g_y(1, \alpha)g(1, \alpha)]| < \eta \tag{2.15}$$

for all solutions $Z(x, \varepsilon, \alpha)$ *of* (1.1), (1.2), (1.3′) *with*

$$|\alpha| \leqq M_0.$$

Proof. For each fixed α the corollary is true without assuming the additional smoothness of $g(x, y)$. An application of Theorem 2.7 of [5] as in Lemma 2.2 gives the uniform result under this additional hypothesis.

Remark. It is of interest to observe that the functions $Z(x, \varepsilon, \alpha)$ exhibit no "boundary layer" behavior near $x = 0$ while the derivatives $Z'(x, \varepsilon, \alpha)$ may do so.

3. Existence of solutions. In this section we are concerned with the existence of solutions of (1.1), (1.2) and (1.3) for small $\varepsilon > 0$. Our first result shows that if $\varepsilon > 0$ is small enough and $u(x, \varepsilon)$ is a solution, then $u(1, \varepsilon)$ must be near a zero of $H(\alpha)$.

LEMMA 3.1. *Suppose* $[x_0, x_1]$ *is a finite interval such that*

$$|H(\alpha)| \geqq H_0 > 0, \quad x_0 \leqq \alpha \leqq x_1. \tag{3.1}$$

Then there is an $\varepsilon_0 > 0$ *such that* $0 < \varepsilon \leqq \varepsilon_0$ *implies that there is no solution* $u(x, \varepsilon)$ *of* (1.1), (1.2) *and* (1.3) *such that*

$$u(1, \varepsilon) = \alpha \in [x_0, x_1].$$

Proof. Let ε_0 be so small that (using Lemma 2.2) any solution $y(x, \varepsilon)$ of (1.1), (1.2) and (1.3′) with

$$\alpha \in [x_0, x_1], \quad 0 < \varepsilon \leqq \varepsilon_0$$

satisfies

$$|\varepsilon_0 y''(1, \varepsilon)| \leqq \tfrac{1}{2} H_0.$$

Then

$$y'(1, \varepsilon) + by(1, \varepsilon) = B + H(u(1, \varepsilon)) - \varepsilon u''(1, a)$$
$$\neq B.$$

Thus, the lemma is proved.

Our first existence theorem is a new proof of a result of Keller [8].

THEOREM 3.1. *Let* α_j *be a zero of* $H(\alpha) = 0$ *and suppose* α_j *is a nodal zero. That is, there is a* $\delta > 0$ *such that, for all* δ, $0 < \delta \leqq \delta_0$,

$$H(\alpha_j + \delta) \cdot H(\alpha_j - \delta) < 0. \tag{3.2}$$

Then there is an $\varepsilon_0 > 0$ *such that*

$$0 < \varepsilon \leqq \varepsilon_0$$

implies that there is at least one solution $u(x, \varepsilon)$ *of* (1.1), (1.2) *and* (1.3) *which also satisfies*

$$\alpha_j - \delta_0 \leqq u(1, \varepsilon) \leqq \alpha_j + \delta_0. \tag{3.3}$$

Proof. Let $Z(x, \varepsilon, \alpha_j + \delta_0)$ and $Z(x, \varepsilon, \alpha_j - \delta_0)$ be solutions of (1.1), (1.2) and (1.3′) with

$$\alpha = \alpha_j \pm \delta_0.$$

Then, applying Lemma 2.3 we see that: if ε is small enough, then

$$\begin{aligned}&[Z'(1, \varepsilon, \alpha_j + \delta_0) + bZ(1, \varepsilon, \alpha_j + \delta_0) - B] \times [Z'(1, \varepsilon, \alpha_j - \delta_0)\\ &\quad + bZ(1, \varepsilon, \alpha_j - \delta_0) - B] = [H(\alpha_j + \delta_0) - \varepsilon Z''(1, \varepsilon, \alpha_j + \delta_0)]\\ &\quad \times [H(\alpha_j - \delta_0) - \varepsilon Z''(1, \varepsilon, \alpha_j - \delta_0)] < 0.\end{aligned} \tag{3.4}$$

Moreover, using Lemma 2.2, if ε is small enough $Z(x, \varepsilon, \alpha)$ is unique and

$$Z'(1, \varepsilon, \alpha) + bZ(1, \varepsilon, \alpha)$$

is a continuous function of α as α ranges over $[\alpha_j - \delta_0, \alpha_j + \delta_0]$. The theorem follows at once.

THEOREM 3.2. *Suppose* α_j *is a zero of* $H(\alpha) = 0$ *and there is a* $\delta_0 > 0$ *such that*

$$H(\alpha_j \pm \delta) > 0, \quad 0 < \delta \leqq \delta_0, \tag{3.5}$$

and

$$g_x(1, \alpha_j) + g_y(1, \alpha_j)g(1, \alpha_j) > 0. \tag{3.6}$$

Then, there exists an $\varepsilon_0 > 0$ *such that, for every* ε, $0 < \varepsilon \leqq \varepsilon_0$, *there is at least one solution* $u(x, \varepsilon)$ *of* (1.1), (1.2) *and* (1.3) *which also satisfies*

$$\alpha_j \leqq u(1, \varepsilon) \leqq \alpha_j + \delta_0 \tag{3.7a}$$

and at least one solution $u(x, \varepsilon)$ *of* (1.1), (1.2), (1.3) *which also satisfies*

$$\alpha_j - \delta_0 \leqq u(1, \varepsilon) \leqq \alpha_j. \tag{3.7b}$$

Proof. The proof is based on the same argument as in Theorem 3.1. We merely observe that if $\varepsilon > 0$ is small enough, then

$$Z'(1, \varepsilon, \alpha_j - \delta_0) + bZ(1, \varepsilon, \alpha_j - \delta_0) - B > 0,$$
$$Z'(1, \varepsilon, \alpha_j) + bZ'(1, \varepsilon, \alpha_j) - B = -\varepsilon Z''(1, \varepsilon, \alpha_j) < 0,$$
$$Z'(1, \varepsilon, \alpha_j + \delta_0) + bZ'(1, \varepsilon, \alpha_j + \delta_0) - B > 0.$$

In a completely analogous way we obtain the next result.

THEOREM 3.3. *Suppose* α_j *is a zero of* $H(\alpha) = 0$ *and there is a* $\delta_0 > 0$ *such that*

$$H(\alpha_j \pm \delta) < 0, \quad 0 < \delta \leqq \delta_0, \tag{3.8}$$

and

$$g_x(1, \alpha_j) + g_y(1, \alpha_j)g(1, \alpha_j) < 0. \tag{3.9}$$

Then there is an $\varepsilon_0 > 0$ *such that, for every* ε, $0 < \varepsilon \leqq \varepsilon_0$, *there is at least one solution* $u(x, \varepsilon)$ *of* (1.1), (1.2), (1.3) *which also satisfies*

$$\alpha_j \leqq u(1, \varepsilon) \leqq \alpha_j + \delta_0 \tag{3.10a}$$

and at least one solution $u(x, \varepsilon)$ *of* (1.1), (1.2), (1.3) *which also satisfies*

$$\alpha_j - \delta_0 \leqq u(1, \varepsilon) \leqq \alpha_j. \tag{3.10b}$$

Of equal interest are nonexistence theorems.

THEOREM 3.4. *Suppose* α_j *is a zero of* $H(\alpha) = 0$ *and there is a* $\delta_0 > 0$ *such that*

$$H(\alpha_j \pm \delta) \geqq 0, \quad 0 < \delta \leqq \delta_0. \tag{3.11}$$

Suppose $g(x, y) \in C^2\{[0, 1] \times \mathscr{R}^1\}$ *and*

$$g_x(1, \alpha_j) + g_y(1, \alpha_j) \cdot g(1, \alpha_j) < 0. \tag{3.12}$$

Then, there exist an ε_0 *and a* δ_1 *such that, for all* $Z(x, \varepsilon, \alpha)$ *with* $0 < \varepsilon \leqq \varepsilon_0$ *and*

$\alpha \in [\alpha_j - \delta_1, \alpha_j + \delta_1]$,

$$Z'(1, \varepsilon, \alpha) + bZ(1, \varepsilon, \alpha) > B. \tag{3.13}$$

Proof. Applying the corollary to Theorem 2.3 there exist an ε_0 and a δ_1 such that

$$\varepsilon Z''(1, \varepsilon, \alpha) < 0$$

for all ε, $0 < \varepsilon \leqq \varepsilon_0$ and all $\alpha \in [\alpha_j - \delta_1, \alpha_j + \delta_1]$. Thus

$$Z'(1, \varepsilon, \alpha) + bZ(1, \varepsilon, \alpha) = B + H(\alpha) - \varepsilon Z''(1, \varepsilon, \alpha) > B, \tag{3.14}$$

and the theorem is proved.

In the same way we obtain a nonexistence theorem when the inequalities (3.11) and (3.12) are reversed.

4. Iterative methods. The shooting methods of [2], [8] can be used to obtain iterative methods in which the successive iterates are solutions of certain initial value problems. In this section we discuss iterative methods in which the successive iterates are solutions of certain boundary value problems.

LEMMA 4.1. *Let $\sigma > 0$ be a constant. Suppose $\varphi(x) \in C^2[0, 1]$ and*

$$\begin{aligned} &L_\varepsilon[\varphi] - \sigma\varphi \geqq 0, \qquad 0 \leqq x \leqq 1,\\ &\varphi'(0) - a\varphi(0) \geqq 0,\\ &\varphi'(1) + b\varphi(1) \leqq 0. \end{aligned} \tag{4.1}$$

Then

$$\varphi(x) \leqq 0. \tag{4.2}$$

Proof. Applying the maximum principle we see that $\varphi(x)$ cannot possess an interior positive maximum. Suppose $\varphi(x)$ assumes a positive maximum at $x = 0$. Then $\varphi'(0) < 0$ which contradicts the boundary condition at $x = 0$. On the other hand, if $\varphi(x)$ assumes a positive maximum at $x = 1$, we have $\varphi'(1) > 0$ which contradicts the boundary condition at $x = 1$. In either case,

$$\max \varphi(x) \leqq 0.$$

LEMMA 4.2. *Suppose $g(x, y)$ satisfies a uniform Lipschitz condition with constant σ. Let $V(x)$ satisfy*

$$\begin{aligned} &L_\varepsilon[V] \geqq g(x, V), \qquad 0 \leqq x \leqq 1,\\ &V'(0) - aV(0) \geqq A,\\ &V'(1) + bV(1) \leqq B. \end{aligned} \tag{4.3}$$

Let $U(x)$ satisfy

$$V(x) \leqq U(x), \qquad 0 \leqq x \leqq 1, \tag{4.4a}$$

and

$$\begin{aligned} &L_\varepsilon[U] \leqq g(x, U), \qquad 0 \leqq x \leqq 1,\\ &U'(0) - aU(0) \leqq A,\\ &U'(1) + bU(1) \geqq B. \end{aligned} \tag{4.4b}$$

Let V_1 be the unique solution of the linear boundary value problem

$$(4.5)\qquad \begin{aligned} &(L_\varepsilon - \sigma)V_1 = g(t, V) - \sigma V, && 0 \leqq x \leqq 1,\\ &V_1'(0) - aV_1(0) = A,\\ &V_1'(1) + bV_1(1) = B. \end{aligned}$$

Then

$$(4.6)\qquad L_\varepsilon V_1 \geqq g(x, V_1), \qquad 0 \leqq x \leqq 1,$$

and

$$(4.7)\qquad V(x) \leqq V_1(x) \leqq U(x), \qquad 0 \leqq x \leqq 1.$$

Proof. Let

$$\varphi(x) = V(x) - V_1(x).$$

Then we may apply Lemma 4.1 to obtain

$$(4.8)\qquad V(x) \leqq V_1(x), \qquad 0 \leqq x \leqq 1.$$

And

$$L_\varepsilon V_1 = g(t, V_1) + [g(t, V) - g(t, V_1)] - \sigma(V - V_1).$$

Since σ is a Lipschitz constant for $g(x, y)$ and (4.8) holds, we obtain (4.6). Finally, if

$$\varphi(x) = V_1(x) - U(x),$$

we have

$$L_\varepsilon[\varphi] - \sigma\varphi \geqq g(t, V) - \sigma V - g(x, U) + \sigma U,$$

and using (4.4a) and the definition of σ we see that we may apply Lemma 4.1 together with (4.8) and obtain (4.7).

THEOREM 4.1. *Suppose $\hat{\alpha}_1 < \hat{\alpha}_2$ are two values such that*

$$(4.9)\qquad H(\hat{\alpha}_1) < H(\hat{\alpha}_2),$$

and $0 < \varepsilon \leqq \varepsilon_0$ implies that

$$(4.10)\qquad \begin{aligned} &Z'(1, \varepsilon, \hat{\alpha}_1) + bZ(1, \varepsilon, \hat{\alpha}_1) < B,\\ &Z'(1, \varepsilon, \hat{\alpha}_2) + bZ(1, \varepsilon, \hat{\alpha}_2) > B. \end{aligned}$$

Let σ be a uniform Lipschitz constant for $g(x, y)$. Let $Z(x, \varepsilon, \hat{\alpha}_1)$ be any solution of (1.1), (1.2) and (1.3′) with $\alpha = \hat{\alpha}_1$. For example, let

$$(4.11)\qquad v_0(x) = M(x, \varepsilon, \hat{\alpha}_1) \quad or \quad v_0(x) = m(x, \varepsilon, \hat{\alpha}_1).$$

Let $V_n(x)$ be defined by the linear boundary value problem

$$(4.12)\qquad \begin{aligned} &L_\varepsilon V_{n+1} - \sigma V_{n+1} = g(x, V_n) - \sigma V_n,\\ &V_{n+1}'(0) - aV_{n+1}(0) = A,\\ &V_{n+1}'(1) + bV_{n+1}(1) = B. \end{aligned}$$

Then the functions $\{V_n(x)\}_{n=1}^{\infty}$ increase to a function $Z(x, \varepsilon, \alpha)$ which satisfies (1.1), (1.2), (1.3) *and*

$$\hat{\alpha}_1 \leqq \alpha \leqq \hat{\alpha}_2. \tag{4.13}$$

Proof. Let

$$U(x) = M(x, \varepsilon, \hat{\alpha}_2).$$

Applying Lemma 4.2 we see that

$$V_n(x) \leqq V_{n+1}(x) \leqq U(x), \qquad 0 \leqq x \leqq 1. \tag{4.14}$$

Remark 1. Glancing back at Theorems 3.2, 3.3 we see that "every other" solution of (1.1), (1.2), (1.3) can be obtained via these iterative methods.

Remark 2. If $g(x, y)$ does not satisfy a uniform Lipschitz condition we may modify $g(x, y)$ for y out of the region of interest, that is, we modify $g(x, y)$ for

$$y \leqq m(x, \varepsilon, \hat{\alpha}_1)$$

and

$$M(x, \varepsilon, \hat{\alpha}_2) \leqq y.$$

Remark 3. Clearly one may find an approximant to $Z(x, \varepsilon, \hat{\alpha}_1)$ by the use of Theorem 2.1. This approximant will be a perfectly good first guess in the iteration described by (4.12).

Remark 4. Using another first iterate (see Theorem 2.1) we can construct a decreasing sequence which would also provide a solution $Z(x, \varepsilon, \alpha_1)$ of (1.1), (1.2), (4.13). Moreover, $Z(x, \varepsilon, \alpha)$ and $Z(x, \varepsilon, \alpha_1)$ would be minimal and maximal solutions of (1.1), (1.2), (1.3) which also satisfy (4.13).

REFERENCES

[1] D. S. COHEN, *Multiple stable solutions of nonlinear boundary value problems arising in chemical reactor theory*, SIAM J. Appl. Math., 20 (1971), pp. 1–13.

[2] ———, *Multiple solutions of singular perturbation problems*, to appear.

[3] E. A. CODDINGTON AND N. LEVINSON, *A boundary value problem for a nonlinear differential equation with a small parameter*, Proc. Amer. Math. Soc., 3 (1952), pp. 73–81.

[4] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics. Vol. II: Partial Differential Equations* (by R. Courant), Interscience, New York, 1962.

[5] F. W. DORR, S. V. PARTER AND L. F. SHAMPINE, *Applications of the maximum principle to singular perturbation problems*, Tech. Rep. 101, Computer Sciences Dept., University of Wisconsin, Madison, 1970.

[6] K. O. FRIEDRICHS, *The identity of weak and strong extensions of differential operators*. Trans. Amer. Math. Soc., 55 (1944), pp. 132–141.

[7] D. GREENSPAN AND S. V. PARTER, *Mildly nonlinear elliptic partial differential equations and their numerical solution II*, Numer. Math., 7 (1965), pp. 129–146.

[8] H. B. KELLER, *Existence theory for multiple solutions of a singular perturbation problem*, to appear.

[9] S. V. PARTER, *Maximal solutions of mildly non-linear elliptic equation*, Numerical Solutions of Nonlinear Differential Equations, Donald Greenspan, ed., John Wiley, New York, 1966, pp. 213–238.

[10] ———, *Nonlinear eigenvalue problems for some fourth order equations I: Maximal solutions*, this Journal, 1 (1970), pp. 437–457.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

A CAUCHY PROBLEM FOR THE NAVIER-STOKES EQUATIONS IN R^n*

GEORGE H. KNIGHTLY†

Abstract. A classical approach to the Cauchy problem for the Navier–Stokes equations in R^n leads directly to a global, smooth solution for small initial data in a class that is defined independently of n. The result supplements recent efforts in related abstract problems and extends earlier work of the author for $n = 3$. Estimates for the solution are obtained, showing that uniform solutions are pointwise asymptotically stable for small disturbances in the class of initial data considered.

1. Introduction. We are concerned with the Cauchy problem for the Navier-Stokes equations:

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = 0, \quad \nabla \cdot u = 0, \tag{1}$$

$$\lim_{t \to 0} u(x, t) = g(x), \tag{2}$$

where $u = u(x, t)$ is an n-dimensional vector field and $p = p(x, t)$ is a scalar field defined for $t > 0$ and x in n-dimensional Euclidean space, R^n. The vector field $g(x)$ is divergence free and constitutes the given initial data for the problem. The viscosity coefficient, which often multiplies the term Δu in (1), has been set equal to unity. In this paper we construct a solution of (1), (2) on $R^n \times (0, \infty)$ when g satisfies

$$|g(x)| \leqq A \min [1, |x|^{-1-s}] \tag{3}$$

for some $s \geqq 0$, $A > 0$, A sufficiently small.

In discussing the question of finding weak solutions of the initial boundary value problem for (1) having more regularity than the Hopf solutions, Serrin [9] points out the failure, when $n \geqq 5$, of methods relying upon standard energy estimates and mentions [9, p. 77] that it is of "greatest interest" to know whether the dimensional restriction is a consequence of the method or an actual property of the equations. Our results support recent evidence [2], [3] indicating, at least for initial value problems without boundary, that the dimensional barrier to solution can be avoided, perhaps being replaced by dimensionally dependent limitations on the data.

When the flow region is a compact n-manifold without boundary, Ebin and Marsden [2] prove, using methods of global analysis and infinite-dimensional geometry, that a unique solution exists for short time (but on an interval independent of viscosity), corresponding to initial data having j square-integrable derivatives, $j > n/2 + 5$. The solution also has j space derivatives in L^2 and is C^1 as a function of (x, t). In [3] Fabes, Jones and Riviére begin with the same representation (9) that we use but seek weak solutions of the Cauchy problem in R^n for

* Received by the editors March 16, 1971, and in revised form September 7, 1971.

† Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01002. This research was supported by the Mathematics Research Center, Madison, Wisconsin, under Contract DA-31-124-ARO-D-462 with the United States Army.

initial data in $L^p(R^n)$. Using the theory of singular integrals they obtain, when $p > n$, a unique weak solution existing for short time. If the data also lie in $L^q(R^n)$, $q < n$, and have small $L^p(R^n)$ and $L^q(R^n)$ norms, then the solution is global in time. The regularity of the solution increases with p.

Our observation is that, with entirely classical techniques, a solution of (1)–(3) can be constructed exhibiting some features not apparent in solutions obtained in other ways. The assumptions on the initial data are different (in some respects milder) than those in [2] and [3] and lead to explicit information on the asymptotic behavior of the solution. The solution is global in time for small A and analytic in all variables. The dimension n acts as a restriction only in determining the "smallness" required of A.

Before stating the main theorem we introduce a relevant class of pairs (v, q), where v is an n-dimensional vector function and q is a scalar function. Specifically, the vector, scalar pair (v, q) lies in the class Λ if there are a constant vector v_∞ and a $T > 0$ such that $v, v_t, \nabla v, \Delta v, q$ and ∇q are all continuous on $R^n \times (0, T)$ and

$$|v(x,t) - v_\infty| = o(1), \quad |\nabla v(x,t)| = o(|x|), \quad |q(x,t)| = o(|x|) \quad \text{as } |x| \to \infty,$$

locally uniformly in t.

THEOREM 1. *Let $g(x)$ be a continuous, divergence-free, n-dimensional vector field defined on R^n and satisfying* (3) *for some $A > 0, s \geqq 0$. If A is sufficiently small, depending on s and n, then there exists a solution of the initial value problem* (1), (2) *on $R^n \times (0, \infty)$, such that*

$$\begin{aligned} &|u(x,t)| \leqq C \min[1, |x|^{-1-s}, t^{-(1+s)/2}] && \text{if } 0 \leqq s < n-1,\\ (4)\quad &|\nabla u(x,t)| \leqq Ct^{-1/2} \min[1, |x|^{-1-s}, t^{-(1+s)/2}] && \text{if } 0 \leqq s < n-1,\\ &|p(x,t)| \leqq Ct^{-1/2} \min[1, |x|^{-1-2s}, t^{-(1+2s)/2}] && \text{if } 0 \leqq s < (n-2)/2,\end{aligned}$$

with constants C depending only upon s and n. This solution is unique in the class Λ and is analytic in all variables.

The proof of Theorem 1 will be sketched in the next section and is patterned after work in [1] and [6], where the case $n = 3$ is treated. Additional details are included in [7].

Theorem 1 may be interpreted as demonstrating the stability of the identically vanishing solution of (1) under small disturbances in the class (3). The following change of variables shows that any *uniform solution* $(v_\infty, 0)$, v_∞ a constant n-dimensional vector, is similarly stable. If (u, p) is a solution coming from Theorem 1, then (v, q),

$$v(x,t) := v_\infty + u(x - tv_\infty, t), \quad q(x,t) := p(x - tv_\infty, t),$$

is a solution of (1) with initial value $v_\infty + g(x)$. Also, relations (4) yield

$$|v(x,t) - v_\infty| \leqq C \min[1, |x - tv_\infty|^{-1-s}, t^{-(1+s)/2}]$$

together with corresponding estimates for ∇v and q. Finn [4] has shown that the uniform solutions are the only time-independent solutions of (1) in all of R^3 having a "physically reasonable" behavior at infinity. This Liouville-type theorem extends to $n > 3$ so that it is natural to investigate stability of the class of uniform solutions in connection with the Cauchy problem in R^n. Finally, we note that in contrast with some approaches to stability, in which it would be shown that $v \to v_\infty$ in some mean sense, the estimates (4) imply that $v \to v_\infty$, $\nabla v \to 0$ and $q \to 0$ *uniformly* on R^n as $t \to \infty$.

2. Construction of the solution. Theorem 1 follows easily from the results of this section; existence is a direct consequence of Theorem 2 and Lemma 3 while uniqueness and analyticity are contained in the corollaries to Theorem 3. The principal ideas are given, except for the proofs of Lemmas 1, 2 and 3. The proofs of these lemmas involve tedious estimation of a fundamental solution (given below) and related integrals, but the demonstrations are largely formal generalizations of those presented for the case $n = 3$ in [1] and [6] and are not repeated here (however, see [7]).

Our study of the system (1) is based upon an integral representation for the solution velocity in terms of a fundamental solution $E = (E_{ij})$, $i, j = 1, \cdots, n$, of the Stokes equations

$$u_t - \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0. \tag{5}$$

E is the n-dimensional analogue of a tensor introduced by Oseen [8] in the case $n = 3$ and its components are defined by the following relations:

$$\phi(r, t) = 2^{-1}(\sqrt{\pi})^{-n} r^{-n+2} \int_0^{r/(2\sqrt{t})} a^{n-3} e^{-a^2}\, da, \tag{6}$$

$$E_{ij}(x - y, t - \tau) = -\Delta\phi(|x - y|, t - \tau)\delta_{ij} + \frac{\partial^2 \phi(|x - y|, t - \tau)}{\partial x_i \partial x_j}. \tag{7}$$

If E_j denotes the jth column of E, then, when $t - \tau > 0$, the pair $(u, p) := (E_j, 0)$ satisfies (5) in the (x, t)-variables and satisfies the adjoint system

$$u_\tau + \Delta u + \nabla p = 0, \quad \nabla \cdot u = 0 \tag{8}$$

in the (y, τ)-variables. In addition, E becomes singular at $(y, \tau) = (x, t)$ in such a way that if (u, p) is a solution of (1), (2) in the class Λ, then integration by parts over $R^n \times (0, t)$ of the identity

$$u \cdot (E_\tau + \Delta E) + E \cdot (u_\tau - \Delta u + \nabla p) = -E \cdot (u \cdot \nabla u)$$

leads to the representation

$$u = H[g] + N[u, u], \tag{9}$$

where

$$H[g](x, t) := -\int_{R^n} \Delta\phi(x - y, t) g(y)\, dy \tag{10}$$

and

$$N[u, u](x, t) := \int_0^t \int_{R^n} u(y, \tau) \cdot |(u(y, \tau) \cdot \nabla) E(x - y, t - \tau)|\, dy\, d\tau. \tag{11}$$

This representation suggests that we find a solution vector u of the initial value problem for (1) by solving (9).

We shall present two existence theorems for (9), the first of which yields a solution global in time. For $s \geqq 0$ define

$$Q_s(x,t) := \min\left[1, (\sqrt{t})^{-s-1}, |x|^{-s-1}\right].$$

THEOREM 2. *Let $g(x)$ be a continuous, n-dimensional vector function satisfying $|g(x)| \leqq A \min[1, |x|^{-1-s}]$ on R^n for some s, $0 \leqq s < n-1$. If A is sufficiently small, depending only on s and n, then there exists a solution $u(x,t)$ of (9) such that $|u(x,t)| \leqq CQ_s(x,t)$, with constant C depending only on A, s and n.*

Theorem 2 follows from the next two lemmas.

LEMMA 1. *If $g(x)$ is an n-dimensional vector function satisfying $|g(x)| \leqq A \min[1, |x|^{-1-s}]$, $0 \leqq s < n-1$, then*

$$|H[g](x,t)| \leqq ACQ_s(x,t),$$

with constant C depending only on s and n.

LEMMA 2. *If $u(x,t)$ and $v(x,t)$ are smooth, n-dimensional vector functions satisfying $|u(x,t)| \leqq UQ_s(x,t)$, $|v(x,t)| \leqq VQ_s(x,t)$ with $0 \leqq s < n/2$, then*

$$|N[u,v](x,t)| \leqq CUVQ_{2s}(x,t)$$

with constant C depending only on s and n.

To construct the u required for the proof of Theorem 2 we insert a parameter λ in front of N in (9) and seek a solution of the form

$$u = \sum_{m=0}^{\infty} \lambda^m u_m \tag{12}$$

valid for $\lambda = 1$. The recursion relations

$$u_0 = H[g], \quad u_m = N[u_0, u_{m-1}] + N[u_1, u_{m-2}] + \cdots + N[u_{m-1}, u_0]$$

result. If we denote by α and β the smallest values of the constants C in Lemmas 1 and 2, respectively, then these lemmas show that

$$\begin{aligned}
|u_0(x,t)| &\leqq \alpha A Q_s(x,t) =: U_0 Q_s(x,t) \leqq U_0 Q_{s/2}(x,t),\\
|u_m(x,t)| &\leqq \beta Q_s(x,t) \sum_{j=0}^{m-1} U_j U_{m-1-j} =: U_m Q_s(x,t)\\
&\leqq U_m Q_{s/2}(x,t).
\end{aligned}$$

It follows that the series (12) is majorized by the series $Q_s(x,t)\sum_{m=0}^{\infty} \lambda^m U_m$. From the definition of the U_m we find that the series

$$U = \sum_{m=0}^{\infty} \lambda^m U_m$$

defines for the quadratic equation $U = U_0 + \lambda\beta U^2$ a solution that is analytic in λ near $\lambda = 0$, provided that the discriminant is positive. Thus the series (12) converges for $\lambda = 1$, uniformly for $(x,t) \in R^n \times (0,\infty)$, if $A < (4\beta\alpha)^{-1}$. Hence (12) defines the solution of (9) specified in Theorem 2.

A similar construction yields a bounded solution of (9) that exists locally in time, if $g(x)$ is only bounded.

THEOREM 3. *Let $g(x)$ be a continuous, n-dimensional vector function satisfying $|g(x)| \leqq A$ on R^n. If T is sufficiently small, depending only on A and n, then there exists a bounded solution, $u(x, t)$ of* (9) *on $R^n \times (0, T)$ such that*

$$\lim_{t \to 0} u(x, t) = g(x). \tag{13}$$

Moreover, u is unique in the class of bounded solutions of (9) *satisfying* (13) *and u is analytic in t.*

The proof of existence follows that given above for Theorem 2, with estimates

$$|H[g](x, t)| \leqq CA, \tag{14}$$

corresponding to Lemma 1 and, if $|u(x, t)| \leqq M_1$ and $|v(x, t)| \leqq M_2$, then

$$|N[u, v](x, t)| \leqq CM_1M_2|t|^{1/2}, \tag{15}$$

corresponding to Lemma 2. The inequalities (14) and (15) are valid for t in a complex neighborhood of $(0, T)$ and the terms of the series (12) are analytic in t. By uniform convergence the solution is analytic in t.

If $t \to 0$ in (9), then (15) shows that the nonlinear term makes no contribution in the limit while $\lim_{t \to 0} H[g](x, t) = g(x)$, according to well-known results for the heat equation. This proves (13).

Finally, to establish uniqueness let u and v be bounded solutions of (9) with the same initial data g. Then

$$u - v = N[u - v, u] + N[v, u - v] \tag{16}$$

with $|u|$, $|v|$ and $|u - v|$ bounded, say by M, on $R^n \times [0, T]$. From (15) it follows that

$$|u(x, t) - v(x, t)| \leqq 2CM^2\sqrt{t_0}$$

if $0 < t \leqq t_0$. Iteration of this estimate in (16) yields $|u(x, t) - v(x, t)| \leqq M(2CM\sqrt{t_0})^m$ if $0 < t \leqq t_0$. The choice $0 < t_0 < (2CM)^{-2}$ reveals that $u(x, t) = v(x, t)$ on $R^n \times (0, t_0)$ since m is arbitrary. Repetition of this argument with successive initial instants $t_0, 2t_0, \cdots$ gives the result on $R^n \times (0, T)$.

The uniqueness and time analyticity statements in Theorem 3 carry over directly to corresponding statements about solutions of (1) in the class Λ since such solutions admit the representation (9). In fact, since the class Λ lies within a wider class of solutions for which Kahane [5] has proved spatial analyticity, we may state the following as corollaries to Theorem 3.

COROLLARY 1. *There is at most one solution (u, p) in the class Λ for the initial value problem* (1), (2).

COROLLARY 2. *Every solution of* (1) *in the class Λ is analytic in all variables.*

If the initial data are divergence free, then the following lemma shows that the solutions of (9) in Theorem 2 are, with suitable scalars p, solutions of (1).

LEMMA 3. *Suppose that g satisfies the hypotheses of Theorem 2 and $\nabla \cdot g = 0$. If u is the resulting solution of* (9) *and p is defined by*

$$p(x, t) := -\int_{R^n} |u(y, t) \cdot \nabla u(y, t)| \cdot \nabla(\omega_n^{-1}|x - y|^{2-n})\, dy,$$

where ω_n is the surface area of the unit sphere in R^n, then (u, p) lies in the class Λ and satisfies (1). *In addition ∇u and p obey the estimates in* (4).

In conclusion we remark that the solutions of (9) in Theorem 3 are also solution vectors of (1), if g is divergence free (see [7]). However, these solutions are not known to lie in any class for which uniqueness holds for (1), (2).

Acknowledgment. The author is grateful to a referee for some constructive suggestions which led to the present form of this paper.

REFERENCES

[1] J. R. CANNON AND GEORGE H. KNIGHTLY, *A note on the Cauchy problem for the Navier-Stokes equations*, SIAM J. Appl. Math., 18 (1970), pp. 641–644.

[2] DAVID G. EBIN AND JERROLD MARSDEN, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math., 92 (1970), pp. 102–163.

[3] E. B. FABES, B. F. JONES AND N. M. RIVIÉRE, *The initial value problem for the Navier-Stokes equations with data in L^p*, to appear.

[4] R. FINN, *Estimates at infinity for stationary solutions of the Navier-Stokes equations*, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine, 3 (53), (1959), pp. 387–418. (See also Proc. Symp. Pure Math., vol. IV, American Mathematical Society, 1961, pp. 143–148.)

[5] C. KAHANE, *On the spatial analyticity of weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal., 33 (1969), pp. 386–405.

[6] G. H. KNIGHTLY, *On a class of global solutions of the Navier-Stokes equations*, Ibid., 21 (1966), pp. 211–245.

[7] ———, *Stability of uniform solutions of the Navier-Stokes equations in n-dimensions*, Tech. Summary Rep. 1085, Mathematics Research Center, United States Army, University of Wisconsin, Madison, 1970.

[8] C. W. OSEEN, *Neuere Methoden und Ergebnisse in der Hydrodynamik*, Akademische Verlagsgesellschaft, Leipzig, 1927.

[9] J. SERRIN, *The initial value problem for the Navier-Stokes equations*, Nonlinear Problems, R. E. Langer, ed., University of Wisconsin Press, Madison, 1963, pp. 69–98.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

SECOND ORDER DIFFERENTIAL EQUATIONS WITH GENERAL BOUNDARY CONDITIONS*

G. J. ETGEN AND S. C. TEFTELLER†

Abstract. This article is concerned with second order linear differential systems involving a parameter together with integral boundary conditions. The objective is to establish the existence of eigenvalues for the boundary problems and to determine the oscillatory behavior of the associated solutions. The methods employed in the paper are derived primarily from the work of W. M. Whyburn, and the results presented here represent an extension of some of his work on similar boundary problems.

1. Introduction. In [4], W. M. Whyburn studied the second order differential system:

$$\frac{dy}{dx} = k(x, \lambda)z, \qquad \frac{dz}{dx} = g(x, \lambda)y, \tag{1}$$

where $k(x, \lambda)$ and $g(x, \lambda)$ are real-valued functions on $X: a \leqq x \leqq b, L: \lambda^* - \delta < \lambda < \lambda^* + \delta$, $-\infty < a < b < \infty$, $0 < \delta \leqq \infty$, together with boundary conditions of the form

$$\alpha(\lambda)y(a, \lambda) - \beta(\lambda)z(a, \lambda) = 0, \tag{2a}$$

$$\int_a^b A(x, \lambda)y(x, \lambda)\,dx = 0, \tag{2b}$$

or

$$\int_a^b B(x, \lambda)z(x, \lambda)\,dx = 0. \tag{2c}$$

By comparing the boundary problems (1), (2a), (2b) and (1), (2a), (2c) with simpler two-point boundary problems, Whyburn proved that eigenvalues for each of the problems exist and he established the oscillatory behavior of the associated solutions.

The purpose of this paper is to extend some of Whyburn's work by considering the differential system (1) under somewhat less restrictive hypotheses on the functions $k(x, \lambda)$ and $g(x, \lambda)$ than those imposed by Whyburn, and by studying (1) together with boundary conditions which are more general than (2a), (2b) and (2a), (2c). In particular, we shall consider (1) with each of the boundary conditions:

$$\alpha(\lambda)y(a, \lambda) - \beta(\lambda)z(a, \lambda) = 0, \tag{3a}$$

$$\gamma_1(\lambda)y(a, \lambda) + \delta_1(\lambda)z(a, \lambda) = \gamma_2(\lambda)y(b, \lambda) + \delta_2(\lambda)z(b, \lambda) + H(b, \lambda), \tag{3b}$$

$$\alpha(\lambda)y(a, \lambda) - \beta(\lambda)z(a, \lambda) = 0, \tag{4a}$$

$$\gamma_1(\lambda)y(a, \lambda) + \delta_1(\lambda)z(a, \lambda) = \gamma_2 y(b, \lambda) + \delta_2 z(b, \lambda) + J(b, \lambda), \tag{4b}$$

* Received by the editors April 19, 1971, and in revised form September 27, 1971.
† Department of Mathematics, University of Houston, Houston, Texas 77004.

where $H(x, \lambda) = \int_a^x h(t, \lambda)z(t, \lambda)\, dt$ and $J(x, \lambda) = \int_a^x j(t, \lambda)y(t, \lambda)\, dt$. The methods employed in the paper are analogous to those used in [2] and are different from those developed by Whyburn.

The following hypotheses on the coefficients involved in (1), (3a), (3b) and (4a), (4b) will be assumed throughout:

(H_1) For each $x \in X$, each of $k(x, \lambda)$, $g(x, \lambda)$, $h(x, \lambda)$ and $j(x, \lambda)$ is continuous on L.

(H_2) For each λ on L, each of $k(x, \lambda)$, $g(x, \lambda)$, $h(x, \lambda)$ and $j(x, \lambda)$ is measurable on X.

(H_3) There exists a Lebesgue integrable function $M(x)$ on X such that $|k(x, \lambda)| \leqq M(x)$, $|g(x, \lambda)| \leqq M(x)$, $|h(x, \lambda)| \leqq M(x)$ and $|j(x, \lambda)| \leqq M(x)$ on XL.

(H_4) $k(x, \lambda) > 0$ on XL.

(H_5) Each of the functions $\alpha(\lambda), \beta(\lambda), \gamma_i(\lambda), \delta_i(\lambda), i = 1, 2$, is continuous on L.

(H_6) $\alpha^2(\lambda) + \beta^2(\lambda) > 0$ on L. In particular, without loss of generality, we assume

$$\alpha^2(\lambda) + \beta^2(\lambda) \equiv 1 \quad \text{on} \quad L.$$

(H_7) $\beta(\lambda) \geqq 0$ on L.

2. Existence of eigenvalues. We establish the existence of values of λ on L for which there corresponds a nontrivial solution of (1) satisfying (3a), (3b) or satisfying (4a), (4b). Such values of λ are called *eigenvalues* of the respective boundary problems. By a nontrivial solution of (1) we mean a solution pair $\{y(x, \lambda), z(x, \lambda)\}$ of (1) such that $y^2(x, \lambda) + z^2(x, \lambda) > 0$ on XL.

Hypotheses H_1–H_3 allow the application of fundamental existence and uniqueness theorems [1, Chap. 2] for differential systems to obtain the existence of a unique solution pair $\{y(x, \lambda), z(x, \lambda)\}$ of (1) on XL such that

$$y(a, \lambda) \equiv \beta(\lambda), \quad z(a, \lambda) \equiv \alpha(\lambda) \tag{5}$$

on L.

Applying the polar coordinate transformation to the solution pair $\{y(x, \lambda), z(x, \lambda)\}$ of (1), (5) (for example, see [5]), we obtain

$$\begin{aligned} y(x, \lambda) &= r(x, \lambda) \sin v(x, \lambda), \\ z(x, \lambda) &= r(x, \lambda) \cos v(x, \lambda), \end{aligned} \tag{6}$$

where $r(x, \lambda)$ and $v(x, \lambda)$ are the solution of

$$\begin{aligned} \frac{dv}{dx} &= k(x, \lambda) \cos^2 v - g(x, \lambda) \sin^2 v, \\ \frac{dr}{dx} &= r \cdot [k(x, \lambda) + g(x, \lambda)] \sin v \cdot \cos v, \end{aligned} \tag{7}$$

satisfying

$$\begin{aligned} &r(a, \lambda) \equiv 1, \\ &\sin v(a, \lambda) \equiv \beta(\lambda), \quad \cos v(a, \lambda) \equiv \alpha(\lambda), \quad 0 \leqq v(a, \lambda) < 2\pi. \end{aligned} \tag{8}$$

In fact, since $\beta(\lambda) \geqq 0$ on L, $v(a, \lambda)$ may be assumed to satisfy $0 \leqq v(a, \lambda) \leqq \pi$ on L.

For each λ on L, $y^2(x, \lambda) + z^2(x, \lambda) = r^2(x, \lambda)$ on X and since $r(x, \lambda)$ is a solution of a first order linear differential equation and is positive at $x = a$, we conclude $y^2(x, \lambda) + z^2(x, \lambda) > 0$ on XL and the solution pair $\{y(x, \lambda), z(x, \lambda)\}$ is nontrivial.

THEOREM 1. *Let* $\{y(x, \lambda), z(x, \lambda)\}$ *be the solution of* (1), (5) *and let* $v(x, \lambda)$ *and* $r(x, \lambda)$ *be defined by* (7) *and* (8). *Then* $v(b, \lambda) \geqq 0$ *on* L. *In addition to* H_1–H_7, *let the following conditions hold*:

(i) $h(x, \lambda)/k(x, \lambda)$ *is integrable, nonnegative and nondecreasing on* X *for each* λ *on* L;

(ii) $0 < \gamma_1^2(\lambda) + \delta_1^2(\lambda) \leqq 1$ *and* $\gamma_2(\lambda) \geqq 1$ *on* L;

(iii) $r(b, \lambda) \geqq 1$ *on* L *and* $r(b, \lambda) \geqq r(x, \lambda)$ *on* X *for each* λ *on* L.

If m *is the least nonnegative integer such that* $\inf_{\lambda \in L} v(b, \lambda) < (2m + 1)\pi/2$, n *is an integer such that* $\sup_{\lambda \in L} v(b, \lambda) > (2n + 1)\pi/2$, *and if* $n \geqq m + 1$, *then there exist at least* p, $p = n - m$, *nonempty sets of eigenvalues* $T_0, T_1, \cdots, T_{p-1}$ *for the boundary problem* (1), (3a), (3b). *Moreover, the number of distinct eigenvalues for* (1), (3a), (3b) *is at least* $p/2$ *if* p *is even and at least* $(p + 1)/2$ *if* p *is odd.*

Proof. The continuity conditions on the coefficients of the boundary problem imply $v(x, \lambda)$ is continuous on XL. Fix any λ on L. Since $v(a, \lambda) \geqq 0$ and since $v'(x, \lambda) > 0$ whenever $y(x, \lambda) = 0$, we conclude $v(x, \lambda) \geqq 0$ on X. In particular, $v(b, \lambda) \geqq 0$ and it follows that $v(b, \lambda) \geqq 0$ on L.

Let m and n be the integers with the properties described in the hypothesis. Using the continuity of $v(b, \lambda)$, there is a value of λ, say λ_0, such that $v(b, \lambda_0) = (2m + 1)\pi/2$ and a value of λ, say λ_p, such that $v(b, \lambda_p) = (2n + 1)\pi/2$. Of course $\lambda_0 \neq \lambda_p$, so we assume, without loss of generality, $\lambda_0 < \lambda_p$.

Using the polar representation of $\{y(x, \lambda), z(x, \lambda)\}$, boundary condition (3b) becomes

$$[\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{1/2} \sin [v(a, \lambda) + \theta(\lambda)] = r(b, \lambda) f(b, \lambda) + H(b, \lambda), \tag{9}$$

where

$$\sin \theta(\lambda) = \frac{\delta_1(\lambda)}{[\gamma_1^2(\gamma) + \delta_1^2(\lambda)]^{1/2}}, \quad \cos \theta(\lambda) = \frac{\gamma_1(\lambda)}{[\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{1/2}},$$
$$f(b, \lambda) = \gamma_2(\lambda) \sin v(b, \lambda) + \delta_2(\lambda) \cos v(b, \lambda). \tag{10}$$

Define $Q(\lambda)$ by

$$Q(\lambda) = r(b, \lambda) f(b, \lambda) + H(b, \lambda). \tag{11}$$

Fix λ on L and consider

$$H(b, \lambda) = \int_a^b h(t, \lambda) z(t, \lambda)\, dt = \int_a^b \frac{h(t, \lambda)}{k(t, \lambda)} y'(t, \lambda)\, dt.$$

Condition (i) allows the application of a mean value theorem for integrals [3, Theorem 244, p. 164] to obtain

$$\frac{h(b, \lambda)}{k(b, \lambda)} \min_{x \in X} \int_x^b y'(t, \lambda)\, dt \leqq H(b, \lambda) \leqq \frac{h(b, \lambda)}{k(b, \lambda)} \max_{x \in X} \int_x^b y'(t, \lambda)\, dt. \tag{12}$$

Let $\bar{x}$ and x^* be the values of x on X such that

$$\min_{x \in X} \int_x^b y'(t, \lambda)\, dt = \int_{\bar{x}}^b y'(t, \lambda)\, dt,$$

and

$$\max_{x \in X} \int_x^b y'(t, \lambda)\, dt = \int_{x^*}^b y'(t, \lambda)\, dt.$$

We now have

$$\begin{aligned} &r(b, \lambda)\left\{ f(b, \lambda) + \frac{h(b, \lambda)}{k(b, \lambda)}\left[\sin v(b, \lambda) - \frac{r(\bar{x}, \lambda)}{r(b, \lambda)} \sin v(\bar{x}, \lambda) \right] \right\} \\ &\leqq Q(\lambda) \leqq r(b, \lambda)\left\{ f(b, \lambda) + \frac{h(b, \lambda)}{k(b, \lambda)}\left[\sin v(b, \lambda) - \frac{r(x^*, \lambda)}{r(b, \lambda)} \sin v(x^*, \lambda) \right] \right\}. \end{aligned} \tag{13}$$

Since $n = m + p$, $p \geqq 1$, and since $v(b, \lambda)$ is continuous in λ, there are $p - 1$ values of $\lambda, \lambda_1, \cdots, \lambda_{p-1}$ on (λ_0, λ_p) such that $v(b, \lambda_j) = [2(m + j) + 1]\pi/2$, $j = 1, 2, \cdots, p - 1$. Moreover, we may assume $\lambda_1 < \lambda_2 < \cdots < \lambda_{p-1}$. Now, choose any integer j, $0 \leqq j \leqq p - 1$, and, without loss of generality, assume $\sin v(b, \lambda_j) = +1$. Then $\sin v(b, \lambda_{j+1}) = -1$. From (13), we have

$$Q(\lambda_j) \geqq r(b, \lambda_j)\left\{ \gamma_2(\lambda_j) + \frac{h(b, \lambda_j)}{k(b, \lambda_j)}\left[1 - \frac{r(\bar{x}, \lambda_j)}{r(b, \lambda_j)} \sin v(\bar{x}, \lambda_j) \right] \right\} \geqq 1,$$

$$\begin{aligned} Q(\lambda_{j+1}) &\leqq r(b, \lambda_{j+1})\left\{ -\gamma_2(\lambda_{j+1}) + \frac{h(b, \lambda_{j+1})}{k(b, \lambda_{j+1})}\left[-1 - \frac{r(x^*, \lambda_{j+1})}{r(b, \lambda_{j+1})} \sin v(x^*, \lambda_{j+1}) \right] \right\} \\ &\leqq -1, \end{aligned}$$

using conditions (ii) and (iii). Thus, as λ increases from λ_j to λ_{j+1}, $Q(\lambda)$ changes continuously in value from not less than $+1$ to not more than -1 (or vice versa) for $j = 0, 1, \cdots, p - 1$.

Now, using condition (ii) and the continuity of $\sin [v(a, \lambda) + \theta(\lambda)]$ in λ, we have

$$[\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{1/2} |\sin [v(a, \lambda) + \theta(\lambda)]| \leqq 1.$$

Thus there will exist at least one value of λ on $[\lambda_j, \lambda_{j+1}]$ with the property that (3b) is satisfied. Let $T_j = \{\lambda \in [\lambda_j, \lambda_{j+1}] |$ (3b) is satisfied$\}$, $j = 0, 1, \cdots, p - 1$.

Our work above establishes that the continuous curves $S(\lambda) = [\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{1/2} \sin [v(a, \lambda) + \theta(\lambda)]$ and $Q(\lambda)$ must intersect at least once on each of the intervals $[\lambda_j, \lambda_{j+1}]$, $j = 0, 1, \cdots, p - 1$. Of course, it could happen that they intersect only at alternate endpoints, that is, at $\lambda_1, \lambda_3, \cdots$, with $\lambda_{2j+1}, j = 0, 1, \cdots$, serving as the eigenvalue for both $[\lambda_{2j}, \lambda_{2j+1}]$ and $[\lambda_{2j+1}, \lambda_{2j+2}]$. We find therefore that there will be at least $p/2$ or $(p + 1)/2$ distinct eigenvalues for (1), (3a), (3b) depending on whether p is even or odd. This completes the proof of the theorem.

Concerning the eigenvalues for the boundary problem (1), (3a), (3b), we note that there may exist additional eigenvalues outside the interval $[\lambda_0, \lambda_p]$. We note, also, that each of the nonempty sets of eigenvalues T_j, $j = 0, 1, \cdots, p - 1$, can be finite, countable or uncountable. Finally, T_j and T_{j+1} may have an eigenvalue in common, namely, λ_{j+1}.

We remark that condition (iii) of the hypothesis concerns the amplitudes of the solution pair $\{y(x, \lambda), z(x, \lambda)\}$. This condition can be verified by solving (7) to obtain

$$r(x, \lambda) = \exp \int_a^x [k(t, \lambda) + g(t, \lambda)] \sin v(t, \lambda) \cos v(t, \lambda)\, dt,$$

and noting, for example, that if

$$\int_a^b (k + g) \sin v \cos v \geqq \int_a^x (k + g) \sin v \cos v, \quad x \in X,$$

and

$$\int_a^b (k + g) \sin v \cos v \geqq 0 \quad \text{on } L,$$

then (iii) is satisfied. The case of system (1) with $g(x, \lambda) \equiv -k(x, \lambda)$ provides a simple example of a system in which condition (iii) is satisfied.

COROLLARY 1. *Under the hypotheses of the theorem, if the integer n can be chosen arbitrarily large, then there exist infinitely many nonempty sets of eigenvalues $T_0, T_1, \cdots$ for the boundary problem* (1), (3a), (3b). *Moreover, in this case, there are infinitely many distinct eigenvalues for the boundary problem.*

COROLLARY 2. *Under the hypotheses of the theorem, there exist p nonempty sets of eigenvalues $J_0, J_1, \cdots, J_{p-1}$ for the boundary problem* (1), (3a), (3b) *such that, if $\rho_j \in J_j$, $j = 0, 1, \cdots, p - 1$, then $v(b, \rho_j) \geqq [2(m + j) + 1]\pi/2$. Moreover, if $\rho_j \in J_j$, then the corresponding solution $\{y(x, \rho_j), z(x, \rho_j)\}$ has the property that $y(x, \rho_j)$ has at least $m + j - 1$ zeros on X, assuming j has the property $m + j - 1 \geqq 0$.*

Proof. Let $\{y(x, \lambda), z(x, \lambda)\}$ be the solution of (1), (5) and define $v(x, \lambda)$ by (7), (8). Since $v(b, \lambda)$ is continuous in λ and since $v(b, \lambda)$ ranges in value from less than $(2m + 1)\pi/2$ to more than $(2n + 1)\pi/2$ on L, we select λ_0 and λ_p as in the proof of the theorem. If $\lambda_0 < \lambda_p$, then, using the continuity of $v(b, \lambda)$, select $\lambda_1, \lambda_2, \cdots, \lambda_{p-1}$ such that $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{p-1} < \lambda_p$ and $v(b, \lambda) \geqq v(b, \lambda_j)$ for $\lambda \geqq \lambda_j$, $j = 0, 1, \cdots, p - 1$. If $\lambda_0 > \lambda_p$, then select $\lambda_1, \lambda_2, \cdots, \lambda_{p-1}$ such that $\lambda_p < \lambda_{p-1} < \cdots < \lambda_1 < \lambda_0$ and $v(b, \lambda) \geqq v(b, \lambda_j)$ for $\lambda \leqq \lambda_j$. Let J_j be the set of all λ on the closed interval with endpoints λ_j and λ_{j+1} such that (3b) is satisfied. As in the proof of the theorem, each J_j is nonempty.

Now fix λ on L. From the polar representation of $y(x, \lambda)$, $y(x, \lambda)$ has at least q zeros on X if and only if $v(x, \lambda) \equiv 0 \pmod{\pi}$ at least q times. This will occur if $v(b, \lambda) - v(a, \lambda) \geqq q\pi$.

Now, choose any integer j, $0 \leqq j \leqq p - 1$, and let $\rho_j \in J_j$. Then

$$v(b, \rho_j) - v(a, \rho_j) \geqq \frac{[2(m + j) + 1]\pi}{2} - \pi > (m + j - 1)\pi.$$

Assuming j has the property $m + j - 1 \geqq 0$, $y(x, \rho_j)$ has at least $m + j - 1$ zeros on X.

In Theorem 1, we required $0 < \gamma_1^2 + \delta_1^2 \leqq 1$ and $\gamma_2^2 \geqq 1$ on L. We may use the proof of Theorem 1, however, to establish the existence of eigenvalues for the boundary problem consisting of (1), (3a) and

$$H(b, \lambda) = \int_a^b h(t, \lambda) z(t, \lambda)\, dt = 0, \tag{3b'}$$

obtaining a result which is analogous to Whyburn's work in [4].

THEOREM 2. *Let* $\{y(x, \lambda), z(x, \lambda)\}$ *be the solution of* (1), (5). *Then* $v(b, \lambda) \geqq 0$ *on* L, $v(x, \lambda)$ *defined by* (7) *and* (8). *Let the following conditions hold:*

(i) *For each* λ *on* L, $h(x, \lambda)/k(x, \lambda)$ *is integrable, nonnegative and nondecreasing on* X.

(ii) *For each* λ *on* L, $r(b, \lambda) \geqq r(x, \lambda)$ *on* X.

If m, $m \geqq 0$, *is the least nonnegative integer such that* $\inf v(b, \lambda) < (2m + 1)\pi/2$, n *is an integer such that* $\sup v(b, \lambda) > (2n + 1)\pi/2$, *and if* $n \geqq m + 1$, *then there exist at least* p, $p = n - m$, *nonempty sets of eigenvalues* $T_0, T_1, \cdots, T_{p-1}$ *for the boundary problem* (1), (3a), (3b').

Proof. This proof is a simplification of the proof of Theorem 1. Using the statements and notations of that theorem, inequality (13) becomes

$$\begin{aligned} r(b, \lambda)\frac{h(b, \lambda)}{k(b, \lambda)}\left[\sin v(b, \lambda) - \frac{r(\bar{x}, \lambda)}{r(b, \lambda)} \sin v(\bar{x}, \lambda)\right] &\leqq H(b, \lambda) \\ \leqq r(b, \lambda)\frac{h(b, \lambda)}{k(b, \lambda)}\left[\sin v(b, \lambda) - \frac{r(x^*, \lambda)}{r(b, \lambda)} \sin v(x^*, \lambda)\right]&. \end{aligned} \tag{14}$$

Defining $\lambda_0, \lambda_1, \cdots, \lambda_{p-1}, \lambda_p$ as in the proof of Theorem 1, we have, from (14), that $H(b, \lambda_j)$ and $H(b, \lambda_{j+1})$ have opposite sign. We conclude, therefore, that (3b') is satisfied for at least one value of λ on the closed interval with endpoints λ_j and $\lambda_{j+1}, j = 0, 1, \cdots, p - 1$. Let T_j be the nonempty set of eigenvalues on the closed interval with endpoints λ_j and $\lambda_{j+1}, j = 0, 1, \cdots, p - 1$.

It is clear that corollaries analogous to those of Theorem 1 can be stated for Theorem 2.

We now adapt the proof of Theorem 1 to prove the following theorem establishing the existence of eigenvalues for the boundary problem (1), (4a), (4b).

THEOREM 3. *Let* $\{y(x, \lambda), z(x, \lambda)\}$ *be the solution pair of* (1), (5). *Then* $v(b, \lambda) \geqq 0$ *on* L, $v(x, \lambda)$ *defined by* (7) *and* (8). *Let the following conditions hold:*

(i) $g(x, \lambda)$ *is not identically zero on any subinterval of* X *for each* λ *on* L *and is not identically zero on any subinterval of* L *for each* x *on* X.

(ii) $j(x, \lambda)$ *is a function such that* $j(x, \lambda)/g(x, \lambda)$ *is defined, integrable, nonnegative and nondecreasing on* X *for each* λ *on* L.

(iii) $0 < \gamma_1^2(\lambda) + \delta_1^2(\lambda) \leqq 1$ *and* $\delta_2(\lambda) \geqq 1$ *on* L.

(iv) $r(b, \lambda) \geqq 1$ *on* L *and* $r(b, \lambda) \geqq r(x, \lambda)$ *on* X *for each* λ *on* L.

If m *is the least nonnegative integer such that* $\inf v(b, \lambda) < m\pi$, n *is an integer such that* $\sup v(b, \lambda) > n\pi$, *and if* $n \geqq m + 1$, *then there exist at least* p, $p = n - m$, *nonempty sets of eigenvalues* $T_0, T_1, \cdots, T_{p-1}$ *for the boundary problem* (1), (4a), (4b). *Moreover, the number of distinct eigenvalues for* (1), (4a), (4b) *is at least* $p/2$ *if* p *is even, or at least* $(p + 1)/2$ *if* p *is odd.*

Proof. Using the polar representation of the solution pair $\{y(x, \lambda), z(x, \lambda)\}$, boundary condition (4b) becomes

$$[\gamma_1^2(\lambda) + \delta_1^2(\lambda)]^{1/2} \sin [v(a, \lambda) + \theta(\lambda)] = r(b, \lambda) f(b, \lambda) + J(b, \lambda), \tag{15}$$

where $\theta(\lambda)$ and $f(b, \lambda)$ are defined by (10). Define $R(\lambda)$ by

$$R(\lambda) = r(b, \lambda) f(b, \lambda) + J(b, \lambda). \tag{16}$$

Fixing λ on L and using condition (ii) together with the mean value theorem employed in the proof of Theorem 1, we have

$$\frac{j(b, \lambda)}{g(b, \lambda)} \min_{x \in X} \int_x^b z'(t, \lambda)\, dt \leqq J(b, \lambda) \leqq \frac{j(b, \lambda)}{g(b, \lambda)} \max_{x \in X} \int_x^b z'(t, \lambda)\, dt. \tag{17}$$

Defining $\bar{x}$ and x^* by

$$\min_{x \in X} \int_x^b z'(t, \lambda)\, dt = \int_{\bar{x}}^b z'(t, \lambda)\, dt, \quad \max_{x \in X} \int_x^b z'(t, \lambda)\, dt = \int_{x^*}^b z'(t, \lambda)\, dt,$$

we have

$$r(b, \lambda)\left\{ f(b, \lambda) + \frac{j(b, \lambda)}{g(b, \lambda)}\left[\cos v(b, \lambda) - \frac{r(\bar{x}, \lambda)}{r(b, \lambda)} \cos v(\bar{x}, \lambda)\right]\right\} \leqq R(\lambda)$$
$$\leqq r(b, \lambda)\left\{ f(b, \lambda) + \frac{j(b, \lambda)}{g(b, \lambda)}\left[\cos v(b, \lambda) - \frac{r(x^*, \lambda)}{r(b, \lambda)} \cos v(x^*, \lambda)\right]\right\}.$$

Now paraphrasing the proof of Theorem 1, choose $\lambda_0, \lambda_1, \cdots, \lambda_p$ such that $v(b, \lambda_j) = (m + j)\pi$, $j = 0, \cdots, p$. The continuity of $v(b, \lambda)$ in λ together with conditions (iii) and (iv) implies $R(\lambda)$ varies continuously from a value less than or equal to -1 to a value greater than or equal to $+1$ (or vice versa) on the closed interval with endpoints λ_j and λ_{j+1}, $j = 0, \cdots, p - 1$. The proof is now completed exactly as in Theorem 1.

The following corollaries are the analogues of those given for Theorem 1.

COROLLARY 1. *Under the hypotheses of the theorem, if the integer n can be chosen arbitrarily large, then there exist infinitely many nonempty sets of eigenvalues $T_0, T_1, \cdots$, for the boundary problem* (1), (4a), (4b). *Moreover, in this case, there are infinitely many distinct eigenvalues for the boundary problem.*

COROLLARY 2. *Under the hypotheses of the theorem, there exist p nonempty sets of eigenvalues, $J_0, J_1, \cdots, J_{p-1}$, for the boundary problem* (1), (4a), (4b) *such that if $\rho_j \in J_j$, $0 \leqq j \leqq p - 1$, then $v(b, \rho_j) \geqq (m + j)\pi$. Moreover, if $\rho_j \in J_j$, then the corresponding solution $\{y(x, \rho_j), z(x, \rho_j)\}$ has the property that $y(x, \rho_j)$ has at least $m + j$ zeros on X, assuming $m + j \geqq 0$.*

We conclude with an extension of Theorem 2 which is suggested by Whyburn's work [4]. Consider the differential system (1) together with the boundary conditions

$$\begin{aligned} \alpha(\lambda) y(a, \lambda) - \beta(\lambda) z(a, \lambda) &= 0, \\ M(b, \lambda) &= 0, \end{aligned} \tag{18}$$

where $M(x, \lambda) = \int_a^x p(t, \lambda)[\sigma(\lambda)y(t, \lambda) - \tau(\lambda)z(t, \lambda)]\,dt$ and $\alpha(\lambda)\tau(\lambda) - \beta(\lambda)\sigma(\lambda) \equiv 1$ on L. Define the functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ by

$$\begin{aligned} \varphi(x, \lambda) &= \alpha(\lambda)y(x, \lambda) - \beta(\lambda)z(x, \lambda), \\ \psi(x, \lambda) &= \sigma(\lambda)y(x, \lambda) - \tau(\lambda)z(x, \lambda). \end{aligned} \tag{19}$$

Then

$$\begin{aligned} \varphi' &= [\alpha\sigma k - \beta\tau g]\varphi + [\beta^2 g - \alpha^2 k]\psi, \\ \psi' &= [\sigma^2 k - \tau^2 g]\varphi + [\beta\tau g - \alpha\sigma k]\psi. \end{aligned} \tag{20}$$

Define Φ, Ψ, K, G and ω by

$$\begin{aligned} &\Phi = \varphi\, e^{-\omega}, \quad \Psi = \psi\, e^{\omega}, \quad K = [\rho^2 g - \alpha^2 k]\, e^{-2\omega}, \\ &G = [\sigma^2 k - \tau^2 g]\, e^{2\omega}, \quad \omega = \int_a^x [\alpha\sigma k - \beta\tau g]\, dt. \end{aligned} \tag{21}$$

In terms of these functions, the boundary problem (1), (21) is transformed into

$$\begin{aligned} \Phi' &= K(x, \lambda)\Psi, \\ \Psi' &= G(x, \lambda)\Phi, \end{aligned} \tag{22}$$

$$\begin{aligned} \Phi(a, \lambda) &= 0, \\ W(b, \lambda) &= 0, \end{aligned} \tag{23}$$

where $W(x, \lambda) = \int_a^x p(t, \lambda)\, e^{-\omega(t,\lambda)}\Psi(t, \lambda)\, dt$. Imposing the same conditions on K, G and $p(x, \lambda)\, e^{-\omega(x,\lambda)}$ that were assigned to k, g and h, respectively, in Theorem 2, we obtain the existence of eigenvalues for the boundary problem (22), (23). Of course, the existence of eigenvalues for (22), (23) can be translated into the existence of eigenvalues for (1), (18).

REFERENCES

[1] E. A. CODDINGTON AND N. LEVISON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[2] G. J. ETGEN AND S. C. TEFTELLER, *A two-point boundary problem for nonlinear second order differential systems*, this Journal, 2 (1971), pp. 64–71.

[3] H. KESTELMAN, *Modern Theories of Integration*, Dover, New York, 1960.

[4] W. M. WHYBURN, *Second-order differential systems with integral and k-point boundary conditions*, Trans. Amer. Math. Soc., 30 (1928), pp. 630–640.

[5] ———, *A non-linear boundary value problem for second order differential systems*, Pacific J. Math., 5 (1955), pp. 147–160.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

SINGULAR PERTURBATIONS OF A GENERAL BOUNDARY VALUE PROBLEM*

K. W. CHANG†

Abstract. This paper treats the boundary problem

$$y' = A(t)y + B(t)z,$$
$$\varepsilon z' = C(t)y + D(t)z,$$
$$M(\varepsilon)\begin{pmatrix} y(0,\varepsilon) \\ z(0,\varepsilon) \end{pmatrix} + N(\varepsilon)\begin{pmatrix} y(1,\varepsilon) \\ z(1,\varepsilon) \end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}.$$

The main difference of our approach and that of earlier writers is that we are able to reduce the system to a purely diagonalized form under even less stringent assumptions.

1. Introduction. Consider the boundary value problem consisting of $m + n$ equations

$$y' = A(t)y + B(t)z, \qquad \varepsilon z' = C(t)y + D(t)z, \tag{1}$$

and $m + n$ boundary conditions

$$M(\varepsilon)\begin{pmatrix} y(0,\varepsilon) \\ z(0,\varepsilon) \end{pmatrix} + N(\varepsilon)\begin{pmatrix} y(1,\varepsilon) \\ z(1,\varepsilon) \end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}, \tag{2}$$

on the interval $0 \leqq t \leqq 1$. Here y, c_1 and z, c_2 are respectively real m-dimensional and n-dimensional vectors and A, B, C, D, M, N are square matrices of appropriate orders. We assume that A, B, C, D are continuous functions for $0 \leqq t \leqq 1$ and $M(\varepsilon) = M(0) + 0(\varepsilon)$, $N(\varepsilon) = N(0) + 0(\varepsilon)$, $c_i(\varepsilon) = c_i(0) + 0(\varepsilon)$, $i = 1, 2$, where $0(\varepsilon)$ is a standard order symbol referring to $\varepsilon \to 0+$.

Harris [4], [5] and, more recently, O'Malley [6] have analyzed similar boundary value problems involving powers of ε. Their approach is to reduce (1) to a simpler form:

$$v' = (A - BD^{-1}C + 0(\varepsilon))v + 0(\varepsilon)w,$$
$$\varepsilon w' = 0(\varepsilon)v + (Q^{-1}DQ + 0(\varepsilon))w,$$

by means of the transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} = U(t,\varepsilon)\begin{pmatrix} v \\ w \end{pmatrix}$$

with

$$U(t,\varepsilon) = \begin{pmatrix} I_m & \varepsilon BD^{-1} \\ -QD^{-1}C & Q \end{pmatrix}$$

* Received by the editors September 8, 1971, and in revised form November 17, 1971.

† Department of Mathematics, University of Calgary, Calgary, Alberta, Canada. This research was supported by the National Research Council, Canada, under Grant 5593, and was completed at S.R.I., Edmonton.

and Q such that

$$Q^{-1}DQ = \operatorname{diag}[D_-, D_+],$$

where the eigenvalues of the matrices D_- and D_+ have, respectively, negative and positive real parts for $0 \leqq t \leqq 1$. To carry out this transformation, Harris and O'Malley assume that $U(t, \varepsilon)$ and hence BD^{-1}, $D^{-1}C$ and Q are continuously differentiable. Such a Q definitely exists if D is assumed continuously differentiable and its eigenvalues have nonzero real parts for $0 \leqq t \leqq 1$ (cf. [2]). However, as shown by the counterexample in [2], such a Q may not exist if D is continuous but not continuously differentiable.

The main purpose of this paper is to weaken the assumptions of Harris and O'Malley to:

(I) A, B, C, D are continuous and all eigenvalues of D have nonzero real part for $0 \leqq t \leqq 1$.

We shall show in the next section that under assumption (I) we can reduce (1) to a purely diagonalized form

$$v' = (A - BT)v,$$
$$\varepsilon w' = (D + \varepsilon TB)w,$$

by using the transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} I_m & -\varepsilon S \\ -T & I_n + \varepsilon TS \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

in place of the transformation indicated above, where T, S are bounded solutions of

$$T' = \varepsilon^{-1}DT - TA + TBT - \varepsilon^{-1}C,$$
$$S' = [A - BT]S - \varepsilon^{-1}S[D + \varepsilon TB] - \varepsilon^{-1}B,$$

respectively.

The main result is given in the end as a theorem.

2. Reduction into block diagonalization. From our assumption on $D(t)$ it follows that $D(t)$ is invertible and has the constant number p, $0 \leqq p \leqq n$, of eigenvalues with negative real part for $0 \leqq t \leqq 1$. Moreover, since the interval $[0, 1]$ is compact, there exists $\mu > 0$ such that the real part of every eigenvalue of $D(t)$ has absolute value $\geqq 2\mu$. Therefore, by Lemma 1 in [2], the linear equation

$$\varepsilon z' = D(t)z \tag{3}$$

has a fundamental matrix $Z(t) = Z(t, \varepsilon)$ satisfying the inequalities

$$\begin{aligned} |Z(t)PZ^{-1}(s)| &\leqq L \exp(-\mu(t - s)/\varepsilon) \quad \text{for} \quad 1 \geqq t \geqq s \geqq 0, \\ |Z(t)(I_n - P)Z^{-1}(s)| &\leqq L \exp(-\mu(s - t)/\varepsilon) \quad \text{for} \quad 1 \geqq s \geqq t \geqq 0, \end{aligned} \tag{4}$$

where L is a positive constant independent of ε and P is the projection

$$P = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix},$$

where I_p is the unit $p \times p$ matrix.

Since $A(t)$ is continuous and therefore bounded on [0, 1], there exists $\sigma > 0$ such that the norm $\|A(t)\| \leqq \sigma$ and the equation

$$y' = A(t)y \tag{5}$$

has a fundamental matrix $Y(t)$ such that

$$|Y(t)Y^{-1}(s)| \leqq \exp(\sigma|t - s|) \quad \text{for} \quad 0 \leqq t, s \leqq 1. \tag{6}$$

Then we have the following result.

LEMMA. *There exists* $\varepsilon_0 > 0$ *such that the equations*

$$T' = \varepsilon^{-1}D(t)T - TA(t) + TB(t)T - \varepsilon^{-1}C(t), \tag{7}$$

$$S' = [A(t) - B(t)T(t,\varepsilon)]S - \varepsilon^{-1}S[D(t) + \varepsilon T(t,\varepsilon)B(t)] - \varepsilon^{-1}B(t), \tag{8}$$

have respectively solutions $T = T(t, \varepsilon)$, $S = S(t, \varepsilon)$ *which are uniformly bounded for* $0 \leqq t \leqq 1$ *and* $0 < \varepsilon \leqq \varepsilon_0$.

Moreover, for $0 < t < 1$, $T(t, 0) = \lim_{\varepsilon \to 0} T(t, \varepsilon) = D^{-1}(t)C(t)$ *and* $S(t, 0) = \lim_{\varepsilon \to 0} S(t, \varepsilon) = -B(t)D^{-1}(t)$.

Furthermore, the change of variables

$$w = z + T(t,\varepsilon)y, \quad v = y + \varepsilon S(t,\varepsilon)w \tag{9}$$

transforms (1) *into the block diagonal form*:

$$\begin{aligned} v' &= [A(t) - B(t)T(t,\varepsilon)]v, \\ \varepsilon w' &= [D(t) + \varepsilon T(t,\varepsilon)B(t)]w. \end{aligned} \tag{10}$$

Proof. The existence of a bounded solution $T(t, \varepsilon)$ of (7) follows from the theorem in [1]. Clearly, $\lim_{\varepsilon \to 0} T(t, \varepsilon) = D^{-1}(t)C$ for $0 < t < 1$.

To obtain a bounded solution of (8), let $V(t, \varepsilon)$ be a fundamental matrix of the first equation of (10). Since [0, 1] is compact, there exists $\tilde{\sigma} > 0$ such that $\|A(t) - B(t)T(t, \varepsilon)\| \leqq \tilde{\sigma}$ which implies

$$|V(t,\varepsilon)V^{-1}(s,\varepsilon)| \leqq \exp(\tilde{\sigma}|t - s|) \quad \text{for} \quad 0 \leqq t, s \leqq 1.$$

Also, by Theorem 2 in [3], the second equation of (10) has, for all sufficiently small $\varepsilon > 0$, a fundamental matrix $W(t, \varepsilon)$ such that

$$\begin{aligned} |W(t,\varepsilon)PW^{-1}(s,\varepsilon)| &\leqq \tilde{L}\exp(-\mu(t - s)/2\varepsilon) \quad \text{for} \quad 1 \geqq t \geqq s \geqq 0, \\ |W(t,\varepsilon)(I_n - P)W^{-1}(s,\varepsilon)| &\leqq \tilde{L}\exp(-\mu(s - t)/2\varepsilon) \quad \text{for} \quad 1 \geqq s \geqq t \geqq 0, \end{aligned} \tag{4'}$$

where $\tilde{L}$ is a positive constant independent of ε.

It can easily be verified by differentiation that

$$\begin{aligned} S(t,\varepsilon) = &\int_0^t V(t,\varepsilon)V^{-1}(s,\varepsilon)[-\varepsilon^{-1}B(s)]W(s,\varepsilon)(I_n - P)W^{-1}(t,\varepsilon)\,ds \\ &- \int_t^1 V(t,\varepsilon)V^{-1}(s,\varepsilon)[-\varepsilon^{-1}B(s)]W(s,\varepsilon)PW^{-1}(t,\varepsilon)\,ds \end{aligned}$$

is a solution of (8), and for $0 < \varepsilon < \mu/2\tilde{\sigma}$,

$$\begin{aligned}\|S(t,\varepsilon)\| &\leqq \tilde{L}\varepsilon^{-1}\|B\|\left\{\int_0^t \exp\left[(\tilde{\sigma} - \mu/2\varepsilon)(t-s)\right] ds \right.\\ &\qquad \left. + \int_t^1 \exp\left[(\tilde{\sigma} - \mu/2\varepsilon)(s-t)\right] ds\right\}\\ &\leqq 2\tilde{L}\|B\|(\mu - 2\varepsilon\tilde{\sigma})^{-1}.\end{aligned}$$

Thus $S(t, \varepsilon)$ is bounded, and moreover, $\lim_{\varepsilon\to 0} S(t,\varepsilon) = -B(t)D^{-1}(t)$ for $0 < t < 1$. Consequently, the change of variables (9) transforms the system (1) into (10).

3. Theorem and proof. Applying (9) now to the boundary conditions (2), we obtain

$$\tilde{M}(\varepsilon)\begin{pmatrix} v(0,\varepsilon) \\ w(0,\varepsilon)\end{pmatrix} + \tilde{N}(\varepsilon)\begin{pmatrix} v(1,\varepsilon) \\ w(1,\varepsilon)\end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon)\end{pmatrix}, \tag{11}$$

where

$$\tilde{M}(\varepsilon) = M(\varepsilon)H(0,\varepsilon), \quad \tilde{N}(\varepsilon) = N(\varepsilon)H(1,\varepsilon)$$

and

$$H(t,\varepsilon) = \begin{pmatrix} I_m & -\varepsilon S(t,\varepsilon) \\ -T(t,\varepsilon) & I_n + \varepsilon T(t,\varepsilon)S(t,\varepsilon)\end{pmatrix}.$$

Clearly, $H(t, \varepsilon)$ is nonsingular for all small ε for which $S(t, \varepsilon)$ and $T(t, \varepsilon)$ exist.

We have now transformed the original problem (1), (2) into a more tractable problem (10), (11), which we treat in the same way as O'Malley, except for a modification due to $D(t)$ not having block diagonal form. One can readily verify by differentiation that the functions

$$\begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon)\end{pmatrix} = \begin{pmatrix} V(t,\varepsilon) & 0 \\ 0 & W(t,\varepsilon)PW^{-1}(0,\varepsilon) + W(t,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon)\end{pmatrix}\begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon)\end{pmatrix} \tag{12}$$

are a solution of (10), where α_1, α_2 are arbitrary constant vectors. It only remains to choose α_1, α_2 to satisfy the boundary conditions (11). Substitution into (11) yields

$$\Delta(\varepsilon)\begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon)\end{pmatrix} = \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon)\end{pmatrix},$$

where

$$\begin{aligned}\Delta(\varepsilon) &= \tilde{M}(\varepsilon)\operatorname{diag}\left[V(0,\varepsilon), W(0,\varepsilon)PW^{-1}(0,\varepsilon) + W(0,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon)\right]\\ &\quad + \tilde{N}(\varepsilon)\operatorname{diag}\left[V(1,\varepsilon), W(1,\varepsilon)PW^{-1}(0,\varepsilon) + W(1,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon)\right].\end{aligned} \tag{13}$$

If the inverse $\Delta^{-1}(\varepsilon)$ exists, then

$$\begin{pmatrix} \alpha_1(\varepsilon) \\ \alpha_2(\varepsilon)\end{pmatrix} = \Delta^{-1}(\varepsilon)\begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon)\end{pmatrix}$$

and

$$(14)\quad \begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} V(t,\varepsilon) & 0 \\ 0 & W(t,\varepsilon)PW^{-1}(0,\varepsilon) + W(t,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon) \end{pmatrix} \cdot \Delta^{-1}(\varepsilon) \begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}$$

is a solution of the boundary value problem (10), (11).

Let us analyze $\Delta(\varepsilon)$. Since $W(1,\varepsilon)PW^{-1}(0,\varepsilon) = 0(\varepsilon)$ and

$$W(0,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon) = 0(\varepsilon)$$

as $\varepsilon \to 0$, we can express (13) as

$$\Delta(\varepsilon) = \tilde{M}(0)\,\mathrm{diag}\,[V(0), W(0)PW^{-1}(0)] + \tilde{N}(0)\,\mathrm{diag}\,[V(1), W(1)(I_n - P)W^{-1}(1)] + 0(\varepsilon),$$

where

$$V(0) = \lim_{\varepsilon\to 0} V(0,\varepsilon), \quad W(0) = \lim_{\varepsilon\to 0} W(0,\varepsilon), \quad \text{etc.}$$

This is equivalent to

$$\Delta(\varepsilon) = (\tilde{M}_1(0)V(0) + \tilde{N}_1(0)V(1) : \tilde{M}_2(0)W(0)PW^{-1}(0) + \tilde{N}_2(0)W(1)(I_n - P)W^{-1}(1)) + 0(\varepsilon)$$

if we partition $\tilde{M}, \tilde{N}$ as

$$\tilde{M}(\varepsilon) = (\tilde{M}_1(\varepsilon) : \tilde{M}_2(\varepsilon)), \quad \tilde{N}(\varepsilon) = (\tilde{N}_1(\varepsilon) : \tilde{N}_2(\varepsilon)),$$

such that $\tilde{M}_1, \tilde{N}_1$ and V have the same number m of columns, and $\tilde{M}_2, \tilde{N}_2$ and W have the same number n of columns. Therefore, for all sufficiently small ε, the inverse $\Delta^{-1}(\varepsilon)$ exists if we make the following assumption:

(II) The matrix

$$\begin{aligned}\Delta(0) &= \tilde{M}(0)\,\mathrm{diag}\,[V(0), W(0)PW^{-1}(0)] + \tilde{N}(0)\,\mathrm{diag}\,[V(1), W(1)(I_n - P)W^{-1}(1)] \\ &= (\tilde{M}_1(0)V(0) + \tilde{N}_1(0)V(1) : \tilde{M}_2(0)W(0)PW^{-1}(0) + \tilde{N}_2(0)W(1)(I_n - P)W^{-1}(1))\end{aligned}$$

is nonsingular.

We note that $\Delta(0)$ may be checked immediately since it depends only on the leading coefficients of the problem (1), (2). However, if it were singular, then a higher order analysis of $\Delta(\varepsilon)$ would be necessary to see if it could be nonsingular.

We next analyze the form of the solution within [0, 1] as $\varepsilon \to 0$. In view of (4′) it follows from (12) that for $0 < t < 1$,

$$x(t) \equiv \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \equiv \lim_{\varepsilon\to 0} \begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} \lim_{\varepsilon\to 0} V(t,\varepsilon) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1(0) \\ \alpha_2(0) \end{pmatrix} = \begin{pmatrix} \lim_{\varepsilon\to 0} V(t,\varepsilon)\,\alpha_1(0) \\ 0 \end{pmatrix},$$

that is, $x(t)$ satisfies the degenerate system of (10):

$$x_1' = [A(t) - B(t)T(t,0)]x_1 = [A(t) - B(t)D^{-1}(t)C(t)]x_1,$$
$$0 = D(t)x_2.$$

Also, $x(t)$ satisfies the first m boundary conditions of $\Delta^{-1}(0)\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}$. In fact, on partitioning $\tilde{M}(0)$, $\tilde{N}(0)$, $\Delta(0)$, $\Delta^{-1}(0)$ after the first m rows and columns as

$$\tilde{M}(0) = \begin{pmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{pmatrix}, \qquad \tilde{N}(0) = \begin{pmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{pmatrix},$$

$$\Delta(0) = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}, \qquad \Delta^{-1}(0) = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix},$$

we find that the first m rows of $\Delta^{-1}(0)\tilde{M}(0)x(0) + \Delta^{-1}(0)\tilde{N}(0)x(1)$ are

$$[d_{11}(\tilde{M}_{11}V(0) + \tilde{N}_{11}V(1)) + d_{12}(\tilde{M}_{21}V(0) + \tilde{N}_{21}V(1))]\alpha_1(0)$$
$$= (d_{11}\delta_{11} + d_{12}\delta_{12})\alpha_1(0) = \alpha_1(0),$$

that is, they are the first m rows of $\Delta^{-1}(0)\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}$.

To sum up, we have proved the following theorem.

THEOREM. *Let assumptions* (I), (II) *hold. Then for all sufficiently small* ε *the boundary value problem* (10), (11) *has the solution*

$$\begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} V(t,\varepsilon) & 0 \\ 0 & W(t,\varepsilon)PW^{-1}(0,\varepsilon) + W(t,\varepsilon)(I_n - P)W^{-1}(1,\varepsilon) \end{pmatrix} \Delta^{-1}(\varepsilon)\begin{pmatrix} c_1(\varepsilon) \\ c_2(\varepsilon) \end{pmatrix}$$

for $0 \leqq t \leqq 1$, *where* $\Delta(\varepsilon)$ *is given by* (13).

Moreover, as $\varepsilon \to 0$ *this solution* $(v(t,\varepsilon), w(t,\varepsilon)) \to (x_1(t), x_2(t))$ *for* $0 < t < 1$, *where* $(x_1(t), x_2(t))$ *is the solution of the degenerate system*

$$x_1' = [A(t) - B(t)D^{-1}(t)C(t)]x_1,$$
$$0 = D(t)x_2,$$

and the first m equations of

$$\Delta^{-1}(0)\tilde{M}(0)\begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} + \Delta^{-1}(0)\tilde{N}(0)\begin{pmatrix} x_1(1) \\ x_2(1) \end{pmatrix} = \Delta^{-1}(0)\begin{pmatrix} c_1(0) \\ c_2(0) \end{pmatrix}.$$

Returning to the original variables, for all sufficiently small ε *the boundary value problem* (1), (2) *has the solution*

$$\begin{pmatrix} y(t,\varepsilon) \\ z(t,\varepsilon) \end{pmatrix} = \begin{pmatrix} I_m & -\varepsilon S(t,\varepsilon) \\ -T(t,\varepsilon) & I_n + \varepsilon T(t,\varepsilon)S(t,\varepsilon) \end{pmatrix} \begin{pmatrix} v(t,\varepsilon) \\ w(t,\varepsilon) \end{pmatrix}$$

for $0 \leqq t \leqq 1$. *Moreover, for* $0 < t < 1$, *this solution tends, as* $\varepsilon \to 0$, *to the solution* $(\bar{y}(t), \bar{z}(t))$ *of the degenerate boundary value problem consisting of*

$$\bar{y}' = A(t)\bar{y} + B(t)\bar{z},$$
$$0 = C(t)\bar{y} + D(t)\bar{z}$$

and the first m equations of

$$\Delta^{-1}(0)M(0)\begin{pmatrix}\bar{y}(0)\\ \bar{z}(0)\end{pmatrix} + \Delta^{-1}(0)N(0)\begin{pmatrix}\bar{y}(1)\\ \bar{z}(1)\end{pmatrix} = \Delta^{-1}(0)\begin{pmatrix}c_1(0)\\ c_2(0)\end{pmatrix}.$$

REFERENCES

[1] K. W. CHANG, *Remarks on a certain hypothesis in singular perturbations*, Proc. Amer. Math. Soc., 23 (1969), pp. 41–45.

[2] K. W. CHANG AND W. A. COPPEL, *Singular perturbations of initial value problems over a finite interval*, Arch. Rational Mech. Anal., 32 (1969), pp. 268–280.

[3] W. A. COPPEL, *Dichotomies and reducibility*, J. Differential Equations, 3 (1967), pp. 500–521.

[4] W. A. HARRIS, JR., *Singular perturbations of two-point boundary problems for systems of ordinary differential equations*, Arch. Rational Mech. Anal., 5 (1960), pp. 212–225.

[5] ———, *Singular perturbations of two-point boundary problems*, J. Math. Mech., 11 (1962), pp. 371–382.

[6] R. E. O'MALLEY, JR., *Boundary value problems for linear systems of ordinary differential equations involving many small parameters*, Ibid., 18 (1969), pp. 835–855.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

EXISTENCE AND REPRESENTATION THEOREMS FOR A SEMILINEAR SOBOLEV EQUATION IN BANACH SPACE*

R. E. SHOWALTER†

Abstract. An existence theory is developed for a semilinear evolution equation in Banach space which is modeled on boundary value problems for partial differential equations of Sobolev type. The operators are assumed to be measurable and to satisfy coercive estimates which are not necessarily uniform in their time dependence, and to satisfy Lipschitz conditions on the nonlinear term. Applications are briefly indicated.

1. Introduction. We shall consider the abstract Cauchy problem for the nonlinear evolution equation

$$\mathscr{M}(t)u'(t) + \mathscr{L}(t)u(t) = f(t, u(t))$$

in a separable and reflexive Banach space. The linear operators $\mathscr{M}(t)$ are assumed to be weakly measurable in t and to satisfy nonuniform coercive estimates over the Banach space which permit them to degenerate for certain values of t. The family of linear operators $\mathscr{L}(t)$ are assumed to be weakly measurable in t. The nonlinear term $f(t, u)$ is measurable in t and Lipschitz in u.

Three types of solution are considered: weak, mild, and strong. A mild solution is (essentially) a weak solution which permits a certain integral representation, and we shall prove that these two notions differ by a measurability assumption. A strong solution is a weak solution for which each term in the equation belongs to a specified Hilbert space for almost every t.

The plan of the paper is as follows. Section 2 contains some technical results and notation we shall use. These include measurability of vector- and operator-valued functions, Gronwall's inequality, and an elementary fixed-point theorem for Banach space-valued functions.

The weak solution is defined in § 3, where we obtain results on uniqueness, local existence and global existence under various hypotheses. These results are used in § 4 to construct the linear propagator (which resolves the linear equation with $f \equiv 0$) and thereby to introduce the notion of a mild solution. We prove that mild solutions (local and global) exist with the same hypotheses as used for existence of weak solutions.

Strong solutions are introduced in § 5. We give sufficient conditions for a mild solution to be strong; these conditions are essentially that the operators $\mathscr{M}(t)$ dominate the operators $\mathscr{L}(t)$. Finally we obtain independently a sufficient condition for the existence (and uniqueness) of a strong solution; this condition requires that the function f be dominated by the operators $\mathscr{M}(t)$.

2. Preliminaries. For notation and standard material in functional analysis except as noted below, we shall refer to [11]. The space of continuous linear operators from the normed linear space X to the normed linear space Y will be

* Received by the editors June 24, 1971, and in revised form November 18, 1971.

† Department of Mathematics, University of Texas, Austin, Texas 78712. This research was supported in part by a National Science Foundation grant.

denoted by $L(X, Y)$, and $L(X)$ means $L(X, X)$. The space $L(X, Y)$ with the uniform, strong and weak operator topologies is indicated by $L_u(X, Y)$, $L_s(X, Y)$ and $L_w(X, Y)$, respectively. Absolutely continuous (strongly, weakly) is abbreviated by AC (respectively, SAC, WAC). An operator-valued function $h:[0, 1] \to L(X, Y)$ is called SAC from $[0, 1]$ to $L_s(X, Y)$ if for each $x \in X$ the map $t \to h(t)x$ is SAC from $[0, 1]$ to Y, and h is SAC from $[0, 1]$ to $L_u(X, Y)$ if it is SAC from $[0, 1]$ to the normed linear space $L_u(X, Y)$ (see [11, pp. 40–41, 52–53]).

All linear spaces will be over the field C of complex numbers. Each of our results will hold if the spaces are over the real field R and if conjugate-linearity is replaced by linearity. The modifications will be obvious.

The antidual of the normed linear space X is the Banach space X' of conjugate-linear continuous maps from X to C. If $x \in X$, the map $\phi \to \overline{\phi(x)}: X' \to C$ is continuous and conjugate-linear and hence determines an element $Jx \in X''$. This defines a linear isometry $J: X \to X''$ by the identity $\langle Jx, \phi\rangle_{X''-X'} = \overline{\langle \phi, x\rangle_{X'-X}}$. We say X is reflexive if J is onto, and we identify each $Jx \in X''$ with $x \in X$ (see [11, pp. 32–33]).

We shall need to discuss the adjoint of a map $T \in L(X, X')$. If $x \in X$, the map $y \to \overline{\langle Ty, x\rangle_{X'-X}}: X \to C$ is continuous and linear, so this determines a $T'x \in X'$ for which $\langle T'x, y\rangle_{X'-X} = \overline{\langle Ty, x\rangle_{X'-X}}$ for all $y \in X$. This defines the map $T' \in L(X, X')$. The adjoint of the map $T \in L(X, X')$ is the map $T^* \in L(X'', X')$ defined by $\langle T^*y, x\rangle_{X'-X} = \langle y, Tx\rangle_{X''-X'}$ for $y \in X''$, $x \in X$. Comparing this with the above, we have for $x, y \in X, \langle T^* \circ Jy, x\rangle_{X'-X} = \langle Jy, Tx\rangle_{X''-X'} = \overline{\langle Tx, y\rangle_{X'-X}}$ $= \langle T'y, x\rangle_{X'-X}$. This shows that $T^* \circ J = T'$, so when we identify X and $J(X)$ we see that T^* is an extension of T'. When X is reflexive, we have $T^* = T'$ under the indicated identification, and this will simplify many of the duality arguments to follow (see [11, pp. 42-43]).

For strongly measurable functions $t \to x(t)$ from the real interval I to the Banach space X we shall use exclusively the Bochner integral with respect to Lebesgue measure on I. If $1 \leqq p < \infty$, $L^p(I, X)$ is the Banach space of strongly measurable functions $x(\cdot): I \to X$ for which

$$\|x\|_{L^p(X)} = \left(\int_I |x(t)|_X^p \, dt\right)^{1/p} < \infty,$$

and $\|x\|_{L^p(X)}$ is the norm. Similarly, $L^\infty(I, X)$ is the Banach space of strongly measurable functions $x(\cdot): I \to X$ for which the norm $\|x\|_{L^\infty(X)} = \text{ess sup}\,\{|x(t)|_X : t \in I\}$ is finite (see [11, pp. 71–89]).

When the Banach space X is separable, the notions of weak measurability and strong measurability of X-valued functions are equivalent. The following similar result for operator-valued functions will be useful.

PROPOSITION 2.1. *Let X and Y be separable Banach spaces and $h:[0, 1] \to L_s(X, Y)$ a bounded function. Then h is measurable* (*in the strong operator topology*) *if and only if there is a sequence of countably-valued measurable functions $h_n:[0, 1] \to h([0, 1]) \subseteq L(X, Y)$ such that $h_n(t) \to h(t)$ in $L_s(X, Y)$, uniformly in $t \in [0, 1]$.*

LEMMA 2.2. *Let X and Y be separable Banach spaces and $\{T_\alpha : \alpha \in A\} \subseteq L(X, Y)$. There is a countable subset $\{T_{\alpha_n} : n \geqq 1\}$ which is strongly dense in $\{T_\alpha : \alpha \in A\}$.*

Proof. By considering subsets of the form $\{T_\alpha : n - 1 \leqq \|T_\alpha\|_{L(X,Y)} < n\}$, we may assume the T_α are uniformly bounded. Consider the space $l^1(Y) \equiv \{(y_n) : y_n \in Y$ and $\sum_{n=1}^{\infty} |y_n|_Y < \infty\}$. Since Y is separable there is a sequence $\{\eta_n\}$ dense in Y; those sequences in $l^1(Y)$ of the form $(\eta_{n_1}, \eta_{n_2}, \cdots, \eta_{n_k}, 0, 0, \cdots)$ are dense in $l^1(Y)$, so $l^1(Y)$ is separable.

Let the sequence $\{\xi_n\}$ be dense in X and define a map $\phi : L(X, Y) \to l^1(Y)$ by $\phi(T) = (T(\xi_k)/|\xi_k|_X 2^k : k \geqq 1)$ for $T \in L(X, Y)$. Since $\{\phi(T_\alpha) : \alpha \in A\}$ is a subset of the separable $l^1(Y)$, it is separable, and hence has a dense subset of the form $\{\phi(T_{\alpha_n}) : n \geqq 1\}$. Thus for any T_β, $\beta \in A$, there is a sequence $(\phi(T_m) : T_m \in \{T_{\alpha_n}\})$ such that $\phi(T_m) \to \phi(T_\beta)$ in $l^1(Y)$. Then $T_m(\xi_k) \to T_\beta(\xi_k)$ in Y for every $k \geqq 1$. But $\{\xi_k\}$ dense in X and $\{T_\alpha\}$ bounded imply that $T_m \to T_\beta$ in $L_s(X, Y)$.

Proof of Proposition 2.1. Let $\{T_n : n \geqq 1\}$ be a strongly dense subset of the range $h([0, 1])$. Since $h([0, 1])$ is bounded, the topology induced on it by $L_s(X, Y)$ is metrizable, and the metric is given by $\rho(T, U) = \sum_{j=1}^{\infty} |(T - U)x_j|_Y / (1 + |(T - U)x_j|_Y)2^j$, where $\{x_j : j \geqq 1\}$ is dense in the unit sphere of X. If h is measurable in $L_s(X, Y)$, then each of the maps $t \to |(h(t) - T_n)x_j|_Y$ is measurable [11, p. 72] and so then is $t \to \rho(h(t), T_n)$. For any $\varepsilon > 0$, each of the sets $E_n = \{t \in [0, 1] : \rho(h(t), T_n) < \varepsilon\}$ is measurable with $\bigcup\{E_n : n \geqq 1\} = [0, 1]$. The function defined on $[0, 1]$ by $h_\varepsilon(t) = T_n$ for $t \in E_n \sim \bigcup\{E_j : 1 \leqq j < n\}$ is measurable, countably-valued in $h([0, 1])$ and $\rho(h(t), h_\varepsilon(t)) < \varepsilon$ on $[0, 1]$. The converse is clear.

Finally we cite an elementary inequality and corresponding fixed-point theorem [4], [9].

LEMMA 2.3. *Let $Z(\cdot) \in L^\infty([0, 1], R)$ satisfy for some $\alpha \geqq 0$ the inequality*

$$0 \leqq Z(t) \leqq \alpha + \int_0^t K(\tau) Z(\tau)\, d\tau \tag{2.1}$$

for $t \in [0, 1]$, where $K(\cdot) \in L^1([0, 1], R)$, $K(t) \geqq 0$. Then

$$Z(t) \leqq \alpha \exp\left\{\int_0^t K(\tau)\, d\tau\right\} \tag{2.2}$$

for $t \in [0, 1]$.

LEMMA 2.4. *Let X be a Banach space and F a map of the closed and bounded subset M of $L^\infty([0, 1], X)$ into itself satisfying*

$$|(Fu)(t) - (Fv)(t)|_X \leqq \int_0^t K(\tau) |u(\tau) - v(\tau)|_X\, d\tau$$

for $t \in [0, 1]$, where $K(\cdot) \in L^1([0, 1], R)$ and each $K(t) \geqq 0$. Then there exists exactly one solution in M of the equation $F(u) = u$.

3. The weak solution. Let V be a reflexive and separable Banach space; the norm is given by $|v|_V$ and the $V' - V$ antiduality by $\langle \phi, v \rangle$. Let $a > 0$ and assume that for each $t \in I_a = [0, a]$ we are given a continuous sesquilinear form $m(t; \cdot, \cdot)$ on V. This defines a family of operators $\mathscr{M}(t) \in L(V, V')$ by the identity

$$m(t; x, y) = \langle \mathscr{M}(t)x, y \rangle, \qquad x, y \in V. \tag{3.1}$$

Let $b > 0$, $x_0 \in V$ and $B_b(x_0) = \{x \in V : |x - x_0|_V \leqq b\}$. Assume that we are given a function $f : I_a \times B_b(x_0) \to V'$.

DEFINITION. A function $x: I_a \to V$ is a *weak solution* of the Cauchy problem

$$\mathcal{M}(t)x'(t) = f(t, x(t)), \quad x(0) = x_0 \tag{3.2}$$

if it is SAC with range in $B_b(x_0)$, weakly differentiable a.e. on I_a and (3.2) is satisfied for a.e. $t \in I_a$.

Remark. It follows [11, p. 88] that $x' \in L^1(I_a, V)$ is a strong derivative a.e. with $x(t) - x(s) = \int_s^t x'(\tau)\,d\tau$. It suffices to require that x be WAC and a.e. have a weak derivative $x' \in L^1(I_a, V)$.

The results of this section on weak solutions of (3.2) are obtained from combinations of the following assumptions listed here for reference.

(I) There is a measurable function $k: I_a \to (0, \infty)$ such that $|m(t; x, x)| \geqq k(t)|x|_V^2$ for $x \in V$, a.e. on I_a.

(II) There is a measurable function $Q: I_a \to [1, \infty)$ such that $|f(t, x) - f(t, y)|_{V'} \leqq Q(t)|x - y|_V$ for $x, y \in B_b(x_0)$ a.e. on I_a, and $Q/k \in L^1(I_a, R)$.

(III) For each pair, $x, y \in V$, the function $t \to m(t; x, y): I_a \to C$ is measurable.

(IV) For each $x \in B_b(x_0)$, the function $t \to f(t, x): I_a \to V'$ is (weakly) measurable. For a.e. $t \in I_a$, the function $x \to f(t, x)$ is continuous from $B_b(x_0)$ with the norm topology to V' with the weak (= weak*) topology.

Suppose (I) holds; then for a.e. $t \in I_a$ the operator $\mathcal{M}(t): V \to V'$ is an isomorphism with $\|\mathcal{M}^{-1}(t)\|_{L(V',V)} \leqq k(t)^{-1}$. To see this, note from (3.1) and (I) that $k(t)|x|_V^2 \leqq |\mathcal{M}(t)|_{V'}|x|_V$, and hence $k(t)|x|_V \leqq |\mathcal{M}(t)x|_{V'}$. This shows that $\mathcal{M}(t)$ is injective with closed range in V'. Hence the range of $\mathcal{M}(t)$ is the annihilator in V' of the null space of the adjoint $\mathcal{M}(t)'$ [4, pp. 180–181], [11, p. 44]. But $\mathcal{M}(t)'$ satisfies the same conditions as $\mathcal{M}(t)$, so it has a trivial null space. Thus $\mathcal{M}(t)$ is onto V' and the result follows from the inequalities above.

Let x_1 and x_2 be weak solutions of (3.2) on I_a and assume (I) holds. Then we obtain the estimate

$$|x_1'(t) - x_2'(t)|_V \leqq k(t)^{-1}|f(t, x_1(t)) - f(t, x_2(t))|_{V'}.$$

Since $x_1 - x_2: I_a \to V$ is SAC with summable derivative, we have

$$x_1(t) - x_2(t) = x_1(0) - x_2(0) + \int_0^t (x_1'(s) - x_2'(s))\,ds \tag{3.3}$$

on I_a. If we also assume (II), then we obtain the estimate

$$|x_1(t) - x_2(t)|_V \leqq |x_1(0) - x_2(0)|_V + \int_0^t k(s)^{-1}Q(s)|x_1(s) - x_2(s)|_V\,ds. \tag{3.4}$$

This yields the following.

THEOREM 1. *Assume* (I) *and* (II). *Then there is at most one weak solution of* (3.2) *on* I_a. *If* $\{x_n(\cdot)\}$ *is a sequence of weak solutions of the equation* (3.2) *with initial conditions* $x_n(0)$, $n \geqq 0$, *then* $x_n(0) \to x_0(0)$ *in* V *implies that* $x_n(t) \to x_0(t)$ *in* V, *uniformly on* I_a.

Proof. These results follow from the preceding inequality (3.4) and Lemma 2.3 with $Z(t) = |x_1(t) - x_2(t)|_V$. That this function is bounded follows from (3.3), since $x_1', x_2' \in L^1(I_a, V)$.

We consider next the existence of solutions.

LEMMA 3.1. *Assume* (I) *and* (III). *Then the operator-valued map* $t \to \mathcal{M}^{-1}(t): I_a \to L_s(V', V)$ *is measurable.*

Proof. Since $k(t) > 0$ on I_a and k is measurable, the sets defined by $J_n = \{t \in I_a : k(t) \geqq 1/n\}$, $n \geqq 1$, are measurable and $\cup \{J_n : n \geqq 1\} = I_a$.

The function $\mathscr{M} : I_a \to L_w(V, V')$ is measurable by (III) and $V'' = V$; hence it is measurable $I_a \to L_s(V, V')$ since V' is separable [11, pp. 34, 74–75]. Let $m \geqq 1$; the restriction of $\mathscr{M}$ to J_m is strongly measurable, so by Proposition 2.1 there is a sequence of countably-valued measurable functions $\mathscr{M}_k : J_m \to \mathscr{M}(J_m) \subseteq L_s(V, V')$ such that, for $t \in J_m$, $\mathscr{M}_k(t) \to \mathscr{M}(t)$ in $L_s(V, V')$ as $k \to \infty$. Since each $\mathscr{M}_t(t) \in \mathscr{M}(J_m)$, we have $\|\mathscr{M}_k^{-1}(t)\|_{L(V',V)} \leqq k(t)^{-1} \leqq m$, so for $\phi \in V'$, $|\mathscr{M}_k^{-1}(t)\phi - \mathscr{M}^{-1}(t)\phi|_V = |\mathscr{M}_k^{-1}(t)[(\mathscr{M}(t)x - \mathscr{M}_k(t)x]|_V \leqq m|\mathscr{M}(t)x - \mathscr{M}_k(t)x|_V \to 0$ as $k \to \infty$. Hence $\mathscr{M}^{-1} : J_m \to L_s(V', V)$ is measurable for every $m \geqq 1$, and this yields the desired result.

LEMMA 3.2. *Assume* (I), (III) *and* (IV). *If* $x : I_a \to B_b(x_0)$ *is measurable*, $I_a \to V$, *then the function* $t \to \mathscr{M}^{-1}(t) \circ f(t, x(t)) : I_a \to V$ *is measurable.*

Proof. For every $\phi \in V'$ we have

$$\langle \phi, \mathscr{M}^{-1}(t) f(t, x(t)) \rangle = \overline{\langle f(t, x(t)), \mathscr{M}^{-1}(t)'\phi \rangle},$$

where $\mathscr{M}^{-1}(t)' : V' \to V$ is the adjoint of $\mathscr{M}^{-1}(t)$, so it suffices to show that $f(t, x(t))$ and $\mathscr{M}^{-1}(t)'\phi$ are measurable in V' and V, respectively.

By Lemma 3.1, $\mathscr{M}^{-1}(t)$ is measurable, so the identity $\langle \psi, \mathscr{M}^{-1}(t)'\phi \rangle = \overline{\langle \phi, \mathscr{M}^{-1}(t)\psi \rangle}$ for ψ in V' implies that $\mathscr{M}^{-1}(t)'\phi$ is weakly (hence strongly) measurable.

Since V' is separable and V is reflexive, the measurability of $f(t, x(t))$ will follow from that of $t \to \langle f(t, x(t)), v \rangle$ for every $v \in V$. Suppose first that $x : I_a \to V$ is a countably-valued function assuming the value x_j on G_j, where $\{G_j : j \geqq 1\}$ is a measurable partition of I_a. Let $\phi_j(t)$ be the characteristic function of G_j. Then we have $f(t, x(t)) = \sum \{f(t, x_j)\phi_j(t) : j \geqq 1\}$ on I_a; each term is measurable by (IV), so $f(t, x(t))$ is measurable when $x(t)$ is countably-valued. But any measurable function is a strong limit of countably-valued functions, so the result follows from the continuity requirement in (IV).

THEOREM 2. *Assume* (I), (II), (III) *and* (IV). *Let* $x_0 \in V$ *be such that* $|f(t, x_0)|_{V'} \leqq Q(t)b_0$ *on* I_c, *where* $c \in I_a$ *is chosen so that* $\int_0^c Q(t)k(t)^{-1}\,dt \leqq b(b_0 + b)^{-1}$. *Then there exists a* (*unique*) *weak solution of* (3.2) *on* I_c.

Proof. Define M to be those continuous functions $x \in L^\infty(I_c, V)$ for which $x(t) \in B_b(x_0)$ on I_c. For any $x \in M$ the function $t \to \mathscr{M}^{-1}(t) f(t, x(t))$ is measurable $I_c \to V$ by Lemma 3.2, and we have the estimate $|\mathscr{M}^{-1}(t) f(t, x(t))|_V \leqq k(t)^{-1}Q(t) \cdot (b_0 + b)$ on I_c. Hence the function is integrable, and we can define on I_c the function

$$[Fx](t) = x_0 + \int_0^t \mathscr{M}^{-1}(s) f(s, x(s))\,ds. \tag{3.5}$$

It follows that $|[Fx](t) - x_0|_V \leqq b$, so F maps M into itself. Finally we have from (II) that $|[Fx](t) - [Fy](t)|_V \leqq \int_0^t Q(s)k(s)^{-1}|x(s) - y(s)|_V\,ds$, so Lemma 2.4 asserts that there is a unique $x \in M$ for which

$$x(t) = x_0 + \int_0^t \mathscr{M}^{-1}(s) f(s, x(s))\,ds \tag{3.6}$$

on I_c. But this is equivalent to being a weak solution of (3.2) (see [11, p. 88]), so the result follows.

THEOREM 3. *Assume* (I), (II), (III), (IV) *and* $B_b(x_0) = V$. *Let* $x_0 \in V$ *be such that* $|f(t, x_0)|_{V'} \leqq Q(t)b_0$ *on* I_a. *Then there exists a unique weak solution of* (3.2) *on* I_a.

Proof. Let $u(t) = b_0 \exp \int_0^t k(s)^{-1} Q(s) ds$ and define M to be the continuous functions in $L^\infty(I_a, V)$ for which $|x(t) - x_0|_V \leqq u(t) - b_0$ for all $t \in I_a$. For any $x \in M$ we have

$$|\mathscr{M}^{-1}(s) f(s, x(s))|_V \leqq k(s)^{-1} Q(s)(|x(s) - x_0|_V + b_0), \tag{3.7}$$

so the boundedness of x implies that we can define $Fx \in L^\infty(I_a, V)$ by (3.5). Also we have from (3.7) the estimate

$$|[Fx](t) - x_0|_V \leqq \int_0^t k(s)^{-1} Q(s) u(s)\, ds = u(t) - b_0$$

so $Fx \in M$. Lemma 2.4 applies again to give the result.

Remark. The estimate (3.7) is a growth condition on the second term in $f(t, x)$ and results from the Lipschitz condition in (II) and the above estimate on $f(t, x_0)$. This combination of hypotheses has advantages in applications. (See, for example, the discussion following (6.9).) In particular, it applies directly to linear equations.

4. The mild solution. In addition to the forms $\{m(t; \cdot, \cdot) : t \in I_a\}$, the function f, and the space V as in § 3, suppose we are given a second family $\{l(t; \cdot, \cdot) : t \in I_a\}$ of continuous sesquilinear forms on V. As before each of these determines an operator $\mathscr{L}(t) \in L(V, V')$ by the identity

$$l(t; x, y) = \langle \mathscr{L}(t)x, y \rangle, \qquad x, y \in V.$$

We shall consider weak solutions of the equation

$$\mathscr{M}(t)x'(t) + \mathscr{L}(t)x(t) = f(t, x(t)) \tag{4.1}$$

and its linear homogeneous counterpart

$$\mathscr{M}(t)x'(t) + \mathscr{L}(t)x(t) = 0 \tag{4.2}$$

under assumptions like the following.

(V) For each pair $x, y \in V$, the function $t \to l(t; x, y) : I_a \to C$ is measurable, and there is a measurable function $K : I_a \to R$ such that $|l(t; x, y)| \leqq K(t)|x|_V|y|_V$, $x, y \in V$ a.e. on I_a, and $K/k \in L^1(I_a, R)$.

Our purpose in considering (4.1) is to separate the nonlinear term and characterize those weak solutions which have an integral representation sharper than (3.6). With the assumption (V), the equation (4.1) is certainly no more general than (3.2), since the assumptions (II) and (IV) hold for $f(t, x) - \mathscr{L}(t)x$ whenever they hold for $f(t, x)$. Hence the results of § 3 apply to (4.1) when we assume (V).

Consider the linear equation (4.2). If we assume (I), (III) and (V), then Theorem 3 asserts that for each $x_0 \in V$ and $s \in I_a$ there is a unique weak solution $x(t)$ of (4.2) which satisfies $x(s) = x_0$. This solution is characterized by the integral equation

$$x(t) = x_0 - \int_s^t \mathscr{M}^{-1}(\xi)\mathscr{L}(\xi)x(\xi)\, d\xi, \qquad t \in I_a. \tag{4.3}$$

From Lemma 2.3 it follows that

$$|x(t)|_V \leqq |x_0|_V \exp\left|\int_s^t (K(\xi)/k(\xi))\,d\xi\right|. \tag{4.4}$$

For each $t \in I_a$, we see from Theorem 1 that the dependence of $x(t)$ on x_0 is linear and from (4.4) that it is continuous from V to V. Hence, for each $t, s \in I_a$ there is a unique $G(t, s) \in L(V)$ defined by $G(t, s)x_0 = x(t)$, where $x(t)$ is given by (4.3). We summarize this construction as the following result.

PROPOSITION 4.1. *Assume* (I), (III) *and* (V). *Then there is a function* $G: I_a \times I_a \to L(V)$ *for which*:

(i) *for each* $x_0 \in V$ *the function* $x(t) = G(t, s)x_0$ *is the unique solution of* (4.3);

(ii) *G is a linear propagator* [5]: $G(t, s) = G(t, \xi)G(\xi, s), G(t, t) = I$ *for* $t, s, \xi \in I_a$;

(iii) $\|G(t, s)\|_{L(V)} \leqq \exp|\int_s^t (K(\xi)/k(\xi))\,d\xi|$;

(iv) *for each* $s \in I_a$, $G(\cdot, s): I_a \to L_s(V)$ *is SAC*;

(v) *for each* $t \in I_a$, $G(t, \cdot): I_a \to L_s(V)$ *is continuous.*

COROLLARY. *In addition to the above, assume that both of the functions* $\mathscr{M}$ *and* $\mathscr{L}: I_a \to L_u(V, V')$ *are a.e. separably-valued. Then for each* $s \in I_a$ *the function* $G(\cdot, s): I_a \to L_u(V)$ *is the unique continuous solution of*

$$G(t, s) = I - \int_s^t \mathscr{M}^{-1}(\xi)\mathscr{L}(\xi)G(\xi, s)\,d\xi,$$

(vi) $G(\cdot, s): I_a \to L_u(V)$ *is SAC, and*

(vii) *for each* $t \in I_a$, $G(t, \cdot): I_a \to L_u(V)$ *is SAC.*

Proof. $\mathscr{M}$ and $\mathscr{L}$ are weakly measurable and a.e. separably-valued so they are uniformly measurable [11, p. 75]. Thus the map $t \to \mathscr{M}^{-1}(t)\mathscr{L}(t): I_a \to L_u(V)$ is summable, and Lemma 2.4, with $X = L_u(V)$ and M the set of continuous $x \in X$ for which $\|x(t) - I\| \leqq \exp(|\int_s^t k(\xi)^{-1}K(\xi)\,d\xi|) - 1$, shows there is an operator-valued function which satisfies

$$M(t, s) = I - \int_s^t \mathscr{M}^{-1}(\xi)\mathscr{L}(\xi)M(\xi, s)\,d\xi.$$

But for $x_0 \in V$, the function $t \to M(t, s)x_0$ is the unique solution of (4.3), so $M = G$. That each $G(\cdot, s)$ is SAC in $L_u(V)$ follows from the integral representation above, and this fact, the identity

$$G(t, s_1) - G(t, s_2) = G(t, s_2)\{G(s_2, 0) - G(s_1, 0)\}G(0, s_1)$$

and the uniform boundedness of G imply the last result.

Assume (I), (III) and (V), and let $x(t)$ be a weak solution of (4.1). Since $s \to G(s, 0)$ is a.e. strongly differentiable and $G(0, s) = G^{-1}(s, 0)$ is strongly continuous, it follows [4, pp. 136-137] that $G(0, s)$ is differentiable and $(d/ds)G(0, s) = -G(0, s)[(d/ds)G(s, 0)]G(0, s) = +G(0, s)\mathscr{M}^{-1}(s)\mathscr{L}(s)$ a.e. on I_a, where d/ds denotes the strong derivative. Since $x(t)$ is differentiable a.e. we have

$$(d/ds)[G(0, s)x(s)] = G(0, s)[x'(s) + \mathscr{M}^{-1}(s)\mathscr{L}(s)x(s)] \tag{4.5}$$

and hence from (4.1) follows

$$(d/ds)[G(0, s)x(s)] = G(0, s)\mathscr{M}^{-1}(s)f(s, x(s)) \tag{4.6}$$

a.e. on I_a. Since $x(t)$ is a weak solution it follows that the right side of (4.5) is in $L^1(I_a, V)$ so we may integrate (4.6). If $\mathscr{M}$ and $\mathscr{L}$ are a.e. separably-valued in $L_u(V)$, then SAC of $G(0, s)$ in $L_u(V)$ and that of $x: I_a \to V$ imply that $G(0, s)x(s)$ is SAC in V and we integrate (4.6) to obtain (after operating with $G(t, 0)$) [11, p. 88]

$$x(t) = G(t, 0)x_0 + \int_0^t G(t, s)\mathscr{M}^{-1}(s)f(s, x(s))ds. \tag{4.7}$$

This is the desired integral representation.

DEFINITION. Assume (I), (III) and (V). A *mild solution* of (4.1) is a continuous function $x: I_a \to V$ which satisfies (4.7). (In particular, the integrand belongs to $L^1(I_a, V)$ for each $t \in I_a$.)

In the special case of equation (3.1), which is obtained from setting $\mathscr{L} = 0$ and hence $G(t, s) = I$, it follows by comparing (4.7) with (3.6) that mild solutions are equivalent to weak solutions. Our next result states the relation between weak and mild solutions in the general case.

THEOREM 4. *Assume* (I), (III) *and* (V). *Then a mild solution of* (4.1) *is a weak solution of* (4.1); *a weak solution is a mild solution if* $\mathscr{M}$ *and* $\mathscr{L}$ *are a.e. separably-valued.*

Proof. The second statement was proved in the discussion preceding the definition of mild solution. If $x: I_a \to V$ is a mild solution, then from

$$x(t) = G(t, 0)\left\{x_0 + \int_0^t G(0, s)\mathscr{M}^{-1}(s)f(s, x(s))\, ds\right\}$$

it follows that x is strongly differentiable a.e., satisfies (4.1) a.e. and $x' \in L^1(I_a, V)$. Thus we need only to verify that x is WAC (see Remark following definition of weak solution).

Let $v \in V$ and $\phi \in V'$. Applying ϕ to the identity

$$G(t, 0)v = v - \int_0^t \mathscr{M}^{-1}(\xi)\mathscr{L}(\xi)G(\xi, 0)v\, d\xi$$

and then taking the indicated adjoints give us the weak integral identity

$$\langle G^*(t, 0)\phi, v\rangle = \langle \phi, v\rangle - \int_0^t \langle G^*(\xi, 0)\mathscr{L}^*(\xi)\mathscr{M}^{-1}(\xi)^*\phi, v\rangle\, d\xi.$$

From this we obtain the strong integral

$$G^*(t, 0)\phi = \phi - \int_0^t G^*(\xi, 0)\mathscr{L}^*(\xi)\mathscr{M}^{-1}(\xi)^*\phi\, d\xi$$

in V' from estimates like (iii) of Proposition 4.1 and the measurability of adjoints. From this we see that $t \to G^*(t, 0)\phi: I_a \to V'$ is SAC. But we already know $t \to x_0 + \int_0^t G(0, s)\mathscr{M}^{-1}(s)f(s, x(s))\, ds: I_a \to V$ is SAC, so it follows from $\langle \phi, x(t)\rangle = \langle G^*(t, 0)\phi, x_0 + \int_0^t G(0, s)\mathscr{M}^{-1}(s)f(s, x(s))\, ds\rangle$ that $x(t)$ is WAC on I_a.

COROLLARY. *Assume* (I), (II), (III) *and* (V). *Then there is at most one mild solution of* (4.1).

Proof. Every mild solution is a weak solution and there is at most one weak solution.

THEOREM 5. *Assume* (I), (II), (III), (IV) *and* (V). *Let* $x_0 \in V$ *be such that* $|f(t, x_0)|_{V'} \leqq Q(t)b_0$ *on* I_a. *Then there is a* c, $0 < c \leqq a$, *such that there exists a* (*unique*) *mild solution of* (4.1) *on* I_c. *If additionally* $B_b(x_0) = V$, *then there is a mild solution on* I_a.

Proof. Let $x(\cdot)$ be strongly continuous from I_a to V and $x(t) \in B_b(x_0)$ for all $t \in I_a$. For any $\phi \in V'$, the map $s \to \langle \phi, G(t, s)\mathscr{M}^{-1}(s)f(s, x(s))\rangle = \langle G^*(t, s)\phi, \mathscr{M}^{-1}(s)f(s, x(s))\rangle$ is measurable by Lemma 3.2 and Proposition 4.1(v). Also we have the estimates of Proposition 4.1(iii) and (II), which show that the map $s \to G(t, s)\mathscr{M}^{-1}(s)f(s, x(s))$ is in $L^1(I_a, V)$ for any $t \in I_a$. Also, for $t, \tau \in I_a$ we have

$$\begin{aligned} &\int_0^t G(t, s)\mathscr{M}^{-1}(s)f(s, x(s))\, ds - \int_0^\tau G(\tau, s)\mathscr{M}^{-1}(s)f(s, x(s))\, ds \\ &\quad = [G(t, 0) - G(\tau, 0)] \int_0^\tau G(0, s)\mathscr{M}^{-1}(s)f(s, x(s))\, ds \\ &\quad\quad + \int_\tau^t G(t, s)\mathscr{M}^{-1}(s)f(s, x(s))\, ds \end{aligned} \tag{4.8}$$

and this difference converges to zero in V as $t \to \tau$. Thus, for any x as above we define a continuous function by

$$[Fx](t) = G(t, 0)x_0 + \int_0^t G(t, s)\mathscr{M}^{-1}(s)f(s, x(s))\, ds.$$

Finally, the estimate

$$|[Fx](t) - x_0|_V \leqq |G(t, 0)x_0 - x_0|_V + \exp\left\{\int_0^a (K/k)\right\}(b_0 + b)\int_0^t (Q(s)/k(s))\, ds$$

shows that for c sufficiently small, F maps the set M of those continuous functions $x: I_c \to V$ with every $x(t) \in B_b(x_0)$ into itself. The estimate of Lemma 2.4 follows from (II), so F has a unique fixed point. When $B_b(X_0) = V$, we may proceed as in Theorem 3.

5. The strong solution. Let V be the reflexive and separable Banach space of § 3. Let H be a Hilbert space with norm and inner product given by $|h|_H$ and $(h_1, h_2)_H$, respectively. Assume V is a dense subset of H (so H is separable) and the injection $V \hookrightarrow H$ is continuous. Thus we have $|v|_H \leqq c|v|_V$ on V for some $c > 0$. If we identify H and its antidual H' by the theorem of F. Riesz [4, pp. 43-44], we then have $V \hookrightarrow H \hookrightarrow V'$, the second injection following by duality from the first, and also $(x, y)_H = \langle x, y\rangle$ on $H \times V$ under the indicated identification.

Let $\{\mathscr{M}(t): t \in I_a\}$ and $\{\mathscr{L}(t): t \in I_a\}$ be the families of operators in $L(V, V')$ constructed in §§ 3 and 4. Define $M(t)$ and $L(t)$ to be the respective restrictions of $\mathscr{M}(t)$ and $\mathscr{L}(t)$ to H. These restrictions are unbounded operators on H with respective domains given by $D(M(t)) = \{x \in V: \mathscr{M}(t)x \in H\}$ and $D(L(t)) = \{x \in V: \mathscr{L}(t)x \in H\}$. Note that an element $x \in V$ is in $D(M(t))$ if and only if the conjugate-linear map $y \to m(t; x, y): V \to C$ is continuous with respect to the topology induced by H on V (which is weaker than that of V). (See [4, pp. 62–67] for an elementary discussion and references.)

DEFINITION. A *strong solution* of (4.1) is a weak solution for which each term of the equation is in H a.e. on I_a. This is equivalent to writing

$$M(t)x'(t) + L(t)x(t) = f(t, x(t)) \quad \text{a.e.} \tag{5.1}$$

We note that (5.1) is an equation in H whereas (4.1) is an equation in V'.

Our first result is a sufficient condition for the linear propagator to generate strong solutions.

PROPOSITION 5.1. *Assume* (I), (III), (V) *and in addition*:

(VI) *for a.e.* $t \in I_a$, *we have* $D(M(s)) \subseteq D(L(t))$ *for* $s \in I_t$, *and there is a* $K_1 \in L^1(I_a, R)$ *such that* $\|L(t)M^{-1}(s)\|_{L(H)} \leqq K_1(s)$ *when* $0 \leqq s \leqq t \leqq a$.

Then for $x_0 \in D(M(0))$ *the function* $x(t) = G(t, 0)x_0$ *is the strong solution of the homogeneous equation*

$$M(t)x'(t) + L(t)x(t) = 0. \tag{5.2}$$

Proof. Consider the linear space X of all elements $x \in L^\infty(I_a, V)$ for which

$$x(t) \in D(L(t)) \quad \text{a.e. on } I_a$$

and

$$\|L(\cdot)x(\cdot)\|_{L^\infty(I_a,H)} = \operatorname{ess\,sup}\{|L(t)x(t)|_H : t \in I_a\} < \infty.$$

(Note that for each $v \in V$, the map $t \to (L(t)x(t), v)_H = \langle \mathscr{L}'(t)v, x(t)\rangle$ is measurable by (V) and the measurability of $x(\cdot)$; V is dense in H, so this means $t \to L(t)x(t): I_a \to H$ is measurable.) Since each $L(t)$ is closed, it follows that X with the norm $\|x\|_X = \max\{\|x\|_{L^\infty(I_a,V)}, \|L(\cdot)x(\cdot)\|_{L^\infty(I_a,H)}\}$ is a Banach space.

Let $x \in X$ and $t \in [0, a]$. For $\xi \in I_t = [0, t]$ we have $M^{-1}(\xi)L(\xi)x(\xi) \in D(M(\xi)) \subseteq D(L(t))$ by (VI), and from (I) the estimate $|M^{-1}(\xi)L(\xi)x(\xi)|_H \leqq c|M^{-1}(\xi)L(\xi)x(\xi)|_V \leqq ck(\xi)^{-1}|L(\xi)x(\xi)|_H \leqq ck(\xi)^{-1}\|x\|_X$. In (V) we may assume $K \geqq 1$, and hence $k(\xi)^{-1}$ is in $L^1(I_a, R)$, without loss of generality. Since Lemma 3.2 implies $\xi \to M^{-1}(\xi)L(\xi)x(\xi)$ is measurable $I_a \to V$ and since $V \hookrightarrow H$ is continuous, the map is measurable $I_a \to H$, hence in $L^1(I_a, H)$. Also from (VI) follows the estimate $|L(t)M^{-1}(\xi)L(\xi)x(\xi)|_H \leqq K_1(\xi)|L(\xi)x(\xi)|_H \leqq K_1(\xi)\|x\|_X$. For each $v \in V$ the map $\xi \to (L(t)M^{-1}(\xi)L(\xi)x(\xi), v)_H = \langle \mathscr{L}'(t)v, \mathscr{M}^{-1}(\xi)\mathscr{L}(\xi)x(\xi)\rangle$ is measurable and V is dense in H, so the map $\xi \to L(t)M^{-1}(\xi)L(\xi)x(\xi)$ is in $L^1(I_t, H)$. Since $L(t)$ is closed we have $\int_0^t M^{-1}(\xi)L(\xi)x(\xi)\,d\xi$ belongs to $D(L(t))$ and

$$L(t)\int_0^t M^{-1}(\xi)L(\xi)x(\xi)\,d\xi = \int_0^t L(t)M^{-1}(\xi)L(\xi)x(\xi)\,d\xi \tag{5.3}$$

(see [11, p. 83] for a proof).

Consider the function defined a.e. on I_a by (5.3). From the estimates

$$\left|L(t)\int_0^t M^{-1}(\xi)L(\xi)x(\xi)\,d\xi\right|_H \leqq \|K_1\|_{L^1(I_a,R)}\|x\|_X$$

and

$$\left|\int_0^t M^{-1}(\xi)L(\xi)x(\xi)\,d\xi\right|_V \leqq \|(K/k)\|_{L^1(I_a,R)}\|x\|_X,$$

it follows that this function is in X. Finally, since $x_0 \in D(M(0))$ and $|L(t)x_0|_H \leqq K_1(0)|M(0)x_0|_H$ the function F defined by

$$[Fx](t) = x_0 - \int_0^t M^{-1}(\xi)L(\xi)x(\xi)\,d\xi, \qquad t \in I_a, \tag{5.4}$$

maps X into itself and satisfies

$$\|Fx_1 - Fx_2\|_X \leqq \max\{\|(K/k)\|_{L^1(I_a,R)}, \|K_1\|_{L^1(I_a,R)}\}\|x_1 - x_2\|_X.$$

By the usual arguments, it follows that there is a unique $x \in X$ for which $Fx = x$, and this is the strong solution of (5.2). By the uniqueness of weak solutions it follows that $x(t) = G(t, 0)x_0$.

COROLLARY. *Assume* (I), (III), (V) *and* (VI). *Then for* $x_0 \in D(M(s))$, $s \in [0, a]$, *the function* $x(t) = G(t, s)x_0$ *is the strong solution of* (5.2) *on* $[s, a]$.

THEOREM 6. *Assume* (I), (III), (V) *and* (VI). *Let* $x_0 \in D(M(0))$ *and* $f: I_a \times B_b(x_0) \to H$ *be given with* $|f(t, x)|_H \leqq Q_1(t)g(|x|_V)$, *where* $g:[0, \infty] \to [0, \infty]$ *maps bounded sets into bounded sets and the measurable function* $Q_1(\cdot)$ *is such that* $Q_1K_1 \in L^1(I_a, R)$. *Then any mild solution of* (4.1) *is a strong solution.*

Proof. Let $u(\cdot)$ be a mild solution. For $s \in [0, a]$, define $y(s) = M^{-1}(s)f(s, u(s))$. The function $t \to x(t, s) = G(t, s)y(s):[s, a] \to D(L(t))$ is the strong solution of (5.2), and hence has the representation $x(t, s) = y(s) - \int_s^t M^{-1}(\xi)L(\xi)x(\xi, s)\,d\xi$. This follows from the previous corollary since $y(s) \in D(M(s))$. From (5.3) we obtain the estimates

$$\begin{aligned}|L(t)x(t, s)|_H &\leqq |L(t)y(s)|_H + \int_s^t K_1(\xi)|L(\xi)x(\xi, s)|_H \\ &\leqq K_1(s)Q_1(s)g(|u(s)|_V) + \int_s^t K_1(\xi)|L(\xi)x(\xi, s)|_H\,d\xi\end{aligned}$$

and Lemma 2.3 thus gives

$$|L(t)x(t, s)|_H \leqq K_1(s)Q_1(s)g(|u(s)|_V)\exp\left\{\int_s^t K_1(\xi)\,d\xi\right\}$$

for $0 \leqq s \leqq t \leqq a$. By an argument like that preceding (5.3), one can use Proposition 4.1(v) to show that $s \to L(t)x(t, s):[0, t] \to H$ is measurable and the above estimate shows that

$$\int_0^t |L(t)G(t, s)M^{-1}(s)f(s, u(s))|_H\,ds < \infty.$$

(We have used the fact that $\{g(|u(s)|_V): s \in [0, a]\}$ is bounded since $u: I_a \to V$ is bounded.) Finally, the map $s \to G(t, s)M^{-1}(s)f(s, u(s)) = G(t, s)\mathcal{M}^{-1}(s)f(s, u(s))$ is in $L^1(I_t, V) \subseteq L^1(I_t, H)$ by the definition of mild solution, so we have (see [11, p. 83])

$$\int_0^t G(t, s)M^{-1}(s)f(s, u(s))\,ds \in D(L(t)).$$

The result is now immediate from (4.7) and Proposition 5.1.

Remark. The requirement that x_0 belong to $D(M(0))$ is unnecessarily restrictive when $D(L(t))$ is independent of t. It is only necessary to have the map $t \to L(t)x_0$ in $L^1(I_a, H)$ (see the argument preceding (5.4)).

Our final result is a sufficient condition for the existence of a strong solution of (3.2). In applications the function f will contain spatial derivatives of order as high as those of the leading operators, whereas Theorem 6 requires that f contain spatial derivatives of order at most half of the order of those of the leading operator.

THEOREM 7. *Assume* (I) *and* (III), *with* V *and* H *as given above. Suppose there is a separable and reflexive Banach space* D, *dense and continuously imbedded in* H *for which* $D(M(t)) \subseteq D$ *and* $|M(t)x|_H \geqq k(t)\|x\|_D$ *for* $x \in D(M(t))$ *and a.e.* $t \in I_a$. *For each* $x \in D$, *the function* $f(\cdot, x)$ *is measurable from* I_a *to* H, *and for each* $t \in I_a$ *we have* $|f(t, x) - f(t, y)|_H \leqq Q(t)|x - y|_D$. *Assume that* $Q(t)/k(t)$ *is in* $L^1(I_a, R)$. *Then for each* x_0 *in* D *such that* $|f(t, x_0)|_H \leqq b_0 Q(t)$, *there is a unique SAC function* $x : I_a \to D$ *for which the strong derivative* $x'(t)$ *exists a.e.*, $x' \in L^1(I_a, D)$, $x'(t) \in D(M(t))$ *a.e.*, $x(0) = x_0$ *and*

$$M(t)x'(t) = f(t, x(t)) \quad \text{a.e. } t \in I_a.$$

The proof of Theorem 7 can be patterned after the techniques above. There are certain measurability results that must be obtained, but these can also be handled as above. If D is continuously imbedded in V, this function is a strong solution.

6. Applications. We shall present a rather general realization of the abstract evolution equations (4.1) and (5.1) as a mixed initial and boundary value problem for a partial differential equation of third order. The same technique yields similar results for higher order equations [4], [15]. Problems of this type arise in the flow of fluid through fissured rocks. [3], thermodynamics [6], shear in second order fluids [8], [12], consolidation of clay [23], and others [10]. Certain examples of a linear and time-independent version of our model have been studied [1], [7], [16], [17], [18], [24]. Time-dependent and nonlinear variations have also been studied. In particular, [26] contains results for a linear equation like (4.1) in which the operators are strongly-differentiable, and [13] applies to the linearized form of equation (5.1) when the operators are realizations of regular elliptic boundary value problems and the time-dependence is continuous in the *uniform* operator topology. The nonlinear equation (3.2) is considered in [10] with the added assumption that the linear operators are independent of time. The Lipschitz assumptions in [10] imply ours, and "solution" in [10] means "weak solution" in our notation, so the existence results of [10] are contained in ours.

Our abstract results imply that each of the boundary value problems in the preceding applications is well-posed in an appropriate function space. The intent in the following is to display the types of nonlinear problems to which the abstract results apply, so we do not consider properties of solutions. Such properties as regularity [13], [19], [20] and asymptotic behavior [7], [14], [17], [21] have been discussed for linear equations. We refer to [2], [4], [15], [19] for references to unsupported results on regular elliptic boundary value problems and additional models like those below. Finally we remark that we assume no continuity in the time-dependence of the operators in our models below. In fact we simply require that the coefficients be measurable in the space and time variables and not "too degenerate" in time. The third example does not require ellipticity of the leading operator.

Let Ω be an open set in R^n with boundary $\partial\Omega$ an $(n-1)$-dimensional manifold with Ω on one side of $\partial\Omega$. Γ_0 is a measurable subset of $\partial\Omega$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. H^m will denote the space of (equivalence classes of) functions $\phi \in H \equiv L^2(\Omega)$ such that $D^\alpha\phi \in L^2(\Omega)$ when $|\alpha| \leqq m$, where D^α is a partial derivative of order $|\alpha|$. Then H^m is a Hilbert space with inner product

$$(\phi, \psi)_m = \sum\left(\int_\Omega D^\alpha\phi\overline{D^\alpha\psi}\,dx : |\alpha| \leqq m\right).$$

Let V be the closed subspace of $H^1(\Omega)$ consisting of those $\phi \in H^1(\Omega)$ for which (the *trace* of) ϕ vanishes on Γ_0. Then V is a reflexive and separable Banach space with the norm $|\phi|_V = \sqrt{(\phi, \phi)_1}$. We shall assume that $\partial\Omega$ is sufficiently smooth for the divergence theorem to apply: there is a unit outward normal $n(s) = (n_1(s), \cdots, n_n(s))$ at each $s \in \partial\Omega$ for which

$$\int_\Omega D_j\phi\,dx = \int_{\partial\Omega} n_j(s)\phi(s)\,ds, \qquad j = 1, 2, \cdots, n,$$

for all smooth functions ϕ, where $D_j = \partial/\partial x_j$.

Let $I = [0, 1]$ and functions $m_0 \in L^\infty(\Omega \times I)$, $\alpha \in L^\infty(\Gamma_1 \times I)$ be given, for which $\operatorname{Re}\alpha(s, t) \geqq 0$ and $\operatorname{Re} m_0(x, t) \geqq k$ for some number $k \in (0, 1]$. Let $\eta : I \to I$ be measurable and assume $\int_0^1 (\eta(t))^{-1}\,dt < \infty$. Then for ϕ, ψ in V we define

$$m(t; \phi, \psi) = \eta(t)\int_\Omega \sum_{j=1}^n D_j\phi\overline{D_j\psi}\,dx + \int_\Omega m_0(x, t)\phi\bar{\psi}\,dx + \int_{\Gamma_1} \alpha(s, t)\phi\bar{\psi}\,ds.$$

The assumptions (I) and (III) are satisfied with $k(t) = k \cdot \eta(t)$. The restriction of $\mathscr{M}(t)\phi \in V'$ to $C_0^\infty(\Omega)$ is the distribution given by

$$\mathscr{M}(t)\phi = -\eta(t)\sum_{j=1}^n D_jD_j\phi + m_0(\cdot, t)\phi. \tag{6.1}$$

By the regularity theory of elliptic operators, the domain of the restriction to $H = L^2(\Omega)$ is given by

$$D(M(t)) = \{\phi \in V \cap H^2 : m(t; \phi, \psi) = (M(t)\phi, \psi)_{L^2(\Omega)}, \psi \in V\}.$$

The condition that $\phi \in V$ means ϕ vanishes on Γ_0, while from the second condition we see that

$$\eta(t)\int_\Omega \left(\sum_{j=1}^n D_j\phi \cdot \overline{D_j\psi} + \sum_{j=1}^n D_jD_j\phi \cdot \bar{\psi}\right)dx + \int_{\Gamma_1} \alpha\phi\bar{\psi}\,ds = 0$$

for all ψ in V. But elements of V are (essentially) arbitrary on Γ_1, so the divergence theorem asserts that this is a variational boundary condition

$$\eta(t)D_n\phi(s) + \alpha(s, t)\phi(s) = 0, \qquad s \in \Gamma_1. \tag{6.2}$$

Here $D_n = \sum_{j=1}^n n_j(s)D_j$ denotes the normal (directional) derivative on $\partial\Omega$. Thus we have

$$(6.3) \qquad D(M(t)) = \{\phi \in H^2 : \phi = 0 \text{ on } \Gamma_0, \eta(t)D_n\phi + \alpha\phi = 0 \text{ on } \Gamma_1\},$$

where the equations on $\partial\Omega$ are interpreted as above.

Assume we are given functions l_{ij}, l_j, $l_0 \in L^\infty(\Omega \times I)$, $i, j = 1, 2, \cdots, n$, and $\beta \in L^\infty(\Gamma_1 \times I)$. For $\phi, \psi \in V$ define

$$l(t; \phi, \psi) = \int_\Omega \left\{ \sum_{i,j=1}^n l_{ij}(x,t)D_j\phi \cdot \overline{D_i\psi} + \sum_{j=1}^n l_j(x,t)D_j\phi \cdot \bar{\psi} + l_0(x,t)\phi \cdot \bar{\psi} \right\} dx + \int_{\Gamma_1} \beta(s,t)\phi(s)\overline{\psi(s)}\, ds.$$

As above we have

$$(6.4) \qquad \mathscr{L}(t)\phi = -\sum_{i,j=1}^n D_i(l_{ij}(\cdot, t)D_j\phi) + \sum_{j=1}^n l_j(\cdot, t)D_j\phi + l_0(\cdot, t)\phi$$

and the domain of the restriction to H is given by

$$D(L(t)) = \left\{ \phi \in V : \sum_{i,j=1}^n D_i(l_{ij}(\cdot, t)D_j\phi) \in L^2(\Omega),\ l(t; \phi, \psi) = (L(t)\phi, \psi)_{L^2(\Omega)} \text{ for all } \psi \in V \right\}.$$

The second condition is the variational boundary condition

$$\sum_{i,j=1}^n n_i(s)l_{ij}(s,t)D_j\phi(s) + \beta(s,t)\phi(s) = 0, \qquad s \in \Gamma_1.$$

The assumption (V) is satisfied with $K(t) = K$ depending on the $L^\infty(\Omega \times I)$ norms of the coefficients in (6.4) and the norm of β in $L^\infty(\Gamma_1 \times I)$.

A sufficient condition that the condition (VI) hold is that

$$(6.5) \qquad \eta(t) \equiv 1, \quad l_{ij}(x,t) = \delta_{ij}, \quad \alpha = \beta \text{ is independent of } t.$$

In this case we see that $D(M(t)) = D(L(t))$ is independent of t though the operators may vary with t through the lower order terms. A second sufficient condition for (VI) is that

$$(6.6) \qquad l_{ij}(x,t) = 0 \quad \text{and} \quad \beta = 0.$$

Then $D(L(t)) = V$ for $t \in I$. The estimate in (VI) is easily obtained and K_1 depends on k, η and the coefficients in $\mathscr{L}(t)$. Other variations are possible; we may require (6.5) to hold for $t \in [0, 1/2]$ and (6.6) for $t \in [1/2, 1]$, but we cannot interchange the order of these requirements.

Example 1. We consider the semilinear equation

$$(6.7a) \qquad \mathscr{M}(t)D_tu(x,t) + \mathscr{L}(t)u(x,t) = F(x,t,u(x,t),D_ju(x,t)), \qquad (x,t) \in \Omega \times I,$$

with the linear conditions

$$(6.7b) \qquad \begin{aligned} &u(s,t) = 0, \quad s \in \Gamma_0; \quad \eta(t)D_nD_tu(s,t) + \alpha(s,t)D_tu(s,t) + \beta(s,t)u(s,t) \\ &\quad + \sum_{i,j=1}^n n_i(s)l_{ij}(s,t)D_ju(s,t) = 0, \quad s \in \Gamma_1; \quad t \in I, \end{aligned}$$

and the initial condition

$$u(x, 0) = u_0(x), \qquad x \in \Omega. \tag{6.7c}$$

The measurable function $F:\Omega \times I \times C^{n+1} \to C$ is assumed to satisfy the Lipschitz condition

$$|F(x, t, \xi) - F(x, t, \eta)| \leqq Q(t) \sum_{i=0}^{n} |\xi_i - \eta_i|, \qquad \xi, \eta \in C^{n+1}, \tag{6.8}$$

where $Q(t) \geqq 1$ is measurable and $Q(\cdot)/\eta(\cdot) \in L^1(I, R)$. From the Cauchy–Schwarz inequality we then obtain the estimate

$$\|F(\cdot, t, \phi, D_j\phi) - F(\cdot, t, \psi, D_j\psi)\|_{L^2(\Omega)} \leqq Q(t)\sqrt{n+1}|\phi - \psi|_V$$

for ϕ, ψ in V. Similarly, if F satisfies

$$|F(x, t, \xi)| \leqq Q(t)\left(q(x) + \sum_{j=0}^{n} |\xi_j|\right), \qquad x \in \Omega, \quad \xi \in C^{n+1}, \tag{6.9}$$

where $q(x) \geqq 0$ and $q \in L^2(\Omega)$, then we obtain the estimate

$$\|F(\cdot, t, \phi, D_j\phi)\|_{L^2(\Omega)} \leqq Q(t)g(|\phi|_V),$$

where $g(x) = [(n + 2)(\|q\|^2_{L^2(\Omega)} + x^2)]^{1/2}$, $x \geqq 0$.

From Theorem 5 we obtain the following. Let the spaces V and H and sesquilinear forms $m(t; \cdot, \cdot)$ and $l(t; \cdot, \cdot)$ be given as above. Let the measurable function F be given and satisfy (6.8). Thus $f(t, \phi)(x) = F(x, t, \phi(x), D_j\phi(x))$ defines a function $f: I \times V \to H$; since H is continuously embedded in V', f satisfies (II) and (IV). Assume that a u_0 is given in V for which $\|f(t, u_0)\|_{L^2(\Omega)} \leqq Q(t)b_0$ on I for some $b_0 > 0$. (This estimate is automatically true if (6.9) holds.) Then there exists exactly one mild solution $u(t)$ of (4.1) on I. This mild solution satisfies the partial differential equation (6.7a) a.e. on I in the sense of distributions, the initial condition (6.7c) is satisfied a.e. on Ω, and the first boundary condition in (6.7b) is satisfied in the sense of traces on Γ_0. Finally, we have the identity

$$m(t; u'(t), \phi) + l(t; u(t), \phi) = (\mathscr{M}(t)u'(t) + \mathscr{L}(t)u(t), \phi)_H$$

for all ϕ in V, and this is a variational boundary condition on Γ_1 which by the divergence theorem implies the second condition in (6.7b). If we furthermore assume (6.5) and (6.9) and that u_0 is given in $D(M(0))$, then Theorem 6 asserts that $u(t)$ is a strong solution, so the boundary conditions (6.7b) are strengthened to require that $u(t) \in D(L(t)) = D(M(t))$ for every $t \in I$. This also implies a regularity result, i.e., that $u(t) \in H^2(\Omega)$ for $t \in I$ (see (6.3)).

Example 2. The techniques above are applicable to solutions of the quasilinear equation

$$\mathscr{M}(t)D_t u(x, t) + \sum_{i=1}^{n} D_i(F_i(x, t, u(x, t), D_j u(x, t))) = F_0(x, t, u(x, t), D_j u(x, t)) \tag{6.10a}$$

with the nonlinear boundary conditions

$$\begin{aligned} &u(s, t) = 0, \quad s \in \Gamma_0; \quad \eta(t)D_n D_t u(s, t) + \alpha(s, t)D_t u(s, t) \\ &\qquad - \sum_{i=1}^{n} n_i(s)F_i(s, t, u(s, t), D_j u(s, t)) = G(s, t, u(s, t)), \quad s \in \Gamma_1, \end{aligned} \tag{6.10b}$$

and an initial condition (6.7c). Here we assume $F_i:\Omega \times I \times C^{n+1} \to C$, $i = 0, 1, 2, \cdots, n$, are given which are measurable and satisfy (6.8). $G:\Gamma_1 \times I \times C \to C$ is measurable and satisfies

$$|G(s,t,\xi) - G(s,t,\eta)| \leqq Q(t)|\xi - \eta|, \qquad \xi, \eta \in C.$$

Then we define $f:I \times V \to V'$ by

$$\langle f(t,\phi),\psi\rangle = \int_\Omega \sum_{i=0}^{n} F_i(x,t,\phi(x),D_j\phi(x))\overline{D_i\psi(x)}\,dx + \int_{\Gamma_1} G(s,t,\phi(s))\overline{\psi(s)}\,ds,$$

where $D_0 = 1$. Weak (= mild) solutions are obtained from Theorem 3 (Theorem 5) for u_0 appropriately chosen. Strong solutions are obtained from Theorem 7 if $u_0 \in D \equiv V \cap H^2$, (6.5) holds, and $G = 0$.

Example 3. Let $I = \{0, 1\}$ and $\Omega = I \times I$. V is the closure of $C_0^\infty(\Omega)$ in the norm $|\cdot|_V$, where $|\phi|_V^2 = \int_\Omega (|D_1^2\phi|^2 + |D_2\phi|^2)\,dx$. For any ϕ in C_0^∞ we have

$$\int_0^1 \{|\phi(x_1,x_2)|^2 + x_1 D_1|\phi(x_1,x_2)|^2\}\,dx_1 = x_1|\phi(x_1,x_2)|^2|_0^1 = 0,$$

$$\begin{aligned}\int_0^1 \{|\phi(x_1,x_2)|^2\,dx_1 &\leqq 2\int_0^1 x_1|\phi(x_1,x_2)|\,|D_1\phi(x_1,x_2)|\,dx_1\\ &\leqq \frac{1}{2}\int_0^1 |\phi(x_1,x_2)|^2\,dx_1 + 2\int_0^1 |D_1\phi(x_1,x_2)|^2\,dx_1,\end{aligned}$$

and hence,

$$\int_0^1 |\phi(x_1,x_2)|^2\,dx_1 \leqq 4\int_0^1 |D_1\phi(x_1,x_2)|^2\,dx_1.$$

Integrating this with respect to x_2 on I we obtain

$$\|\phi\|_{L^2(\Omega)} \leqq 2\|D_1\phi\|_{L^2(\Omega)} \tag{6.11}$$

for all ϕ in $C_0^\infty(\Omega)$. Thus $V = \{\phi \in L^2(\Omega): D_1\phi, D_1^2\phi, D_2\phi \in L^2(\Omega)$ and $\phi = 0$ on $\partial\Omega$, $D_1\phi = 0$ when $x_1 = 0$ or $1\}$. Define $m(t;\phi,\psi) = \int_\Omega (D_1^2\phi D_1^2\psi + D_2\phi D_2\psi)\,dx$ on V and

$$l(t;\phi,\psi) = \sum_{j=0}^{2}\sum_{k=0}^{1} l_{jk}(x,t)D_1^j\phi D_2^k\psi\,dx,$$

where $l_{jk} \in L^\infty(\Omega \times I)$. Then (I), (III) and (V) are satisfied; we use (6.11) to verify the boundedness of $l(t;\phi,\psi)$. Thus we have

$$\mathscr{M}(t)\phi = D_1^4\phi - D_2^2\phi \quad\text{and}\quad \mathscr{L}(t)\phi = \sum_{j=0}^{2}\sum_{k=0}^{1}(-1)^k D^k(l_{jk}(\cdot,t)D^j\phi).$$

Nonlinear terms and coefficients in $\mathscr{L}(t)$ could be added as above. Theorem 2 asserts the existence of a weak solution $u(t)$ of (4.2) which satisfies the partial differential equation (4.2) in the sense of distributions on Ω, the boundary conditions built into the space V at each $t \in I$, and the initial condition $u(x,0) = u_0(x)$ a.e. on Ω, where u_0 is given in V. (We note that $\mathscr{M}(t)$ is *not elliptic*.)

REFERENCES

[1] D. E. AMOS, *On half-space solutions of a modified heat equation*, Quart. Appl. Math., 27 (1969), pp. 359–369.

[2] S. AGMON, *Remarks on self-adjoint and semi-bounded elliptic boundary value problems*, Proc. Internat. Sympos. on Linear Spaces, 1–13, Hebrew University, Jerusalem, July 5–12, 1960.

[3] G. BARENBLAT, I. ZHELTOV AND I. KOCHIVA, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech., 24 (1960), pp. 1286–1303.

[4] R. CARROLL, *Abstract Methods in Partial Differential Equations*, Harper and Row, New York, 1969.

[5] ———, *On the propagator equation*, Illinois J. Math., 11 (1967), pp. 506–527.

[6] P. J. CHEN AND M. E. GURTIN, *On a theory of heat conduction involving two temperatures*, Z. Angew. Math. Phys., 19 (1968), pp. 614–627.

[7] B. D. COLEMAN, R. J. DUFFIN AND V. J. MIZEL, *Instability, uniqueness and nonexistence theorems for the equation* $u_t = u_{xx} - u_{xtx}$ *on a strip*, Arch. Rational Mech. Anal., 19 (1965), pp. 100–116.

[8] B. D. COLEMAN AND W. NOLL, *An approximation theorem for functionals, with applications to continuum mechanics*, Ibid., 6 (1960), pp. 355–370.

[9] C. FOIAS, G. GUSSI AND V. POENARU, *Sur les solutions generalisées de certaines équations linéares et quasi-linéares dans l'espace de Banach*, Rev. Roumaine Math. Pures Appl., 3 (1958), pp. 283–304.

[10] H. GAJEWSKI AND K. ZACHARIAS, *Zur starken Konvergenz des Galerkinverfahrens bei einer Klasse pseudoparabolischer partialler Differentialgleichungen*, Math. Nachr., 47 (1970), pp. 365–376.

[11] E. HILLE AND R. PHILLIPS, *Functional Analysis and Semi-Groups*, Colloquium Publications, vol. 31, American Mathematical Society, New York, 1957.

[12] R. HUILGOL, *A second order fluid of the differential type*, Internat. J. Non-linear Mech., 3 (1968), pp. 471–482.

[13] J. LAGNESE, *General boundary value problems for differential equations of Sobolov type*, this Journal, 3 (1972), pp. 105–119.

[14] ———, *Exponential stability of solutions of differential equations of Sobolev type*, this Journal, to appear.

[15] J. L. LIONS, *Equations Differentielles Operationnelles et Problemes aux Limites*, Grundlehren B. 111, Springer, Berlin, 1961.

[16] L. PROKOPENKO, *Cauchy problem for Sobolev's type of equation*, Dokl. Akad. Nauk SSSR, 122 (1968), pp. 990–993.

[17] R. E. SHOWALTER, *Well-posed problems for a partial differential equation of order* $2m + 1$, this Journal, 1 (1970), pp. 214–231.

[18] ———, *The Sobolev equation, I*, J. Applicable Anal., to appear.

[19] ———, *The Sobolev equation, II*, Ibid., to appear.

[20] R. E. SHOWALTER AND T. W. TING, *Pseudo-parabolic partial differential equations*, this Journal, 1 (1970), pp. 1–26.

[21] ———, *Asymptotic behavior of solutions of pseudo-parabolic differential equations*, Ann. Mat. Pura Appl. (IV), 90 (1971), pp. 241–258.

[22] S. SOBOLEV, *Some new problems in mathematical physics*, Izv. Akad. Nauk SSSR Ser. Mat., 18 (1954), pp. 3–50.

[23] D. TAYLOR, *Research on Consolidation of Clays*, Massachusetts Institute of Technology Press, Cambridge, Mass., 1952.

[24] T. W. TING, *Certain non-steady flows of second-order fluids*, Arch. Rational Mech. Anal., 14 (1963), pp. 1–26.

[25] ———, *Parabolic and pseudo-parabolic partial differential equations*, J. Math. Soc. Japan, 21 (1969), pp. 440–453.

[26] M. I. VISIK, *The Cauchy problem for equations with operator coefficients; mixed boundary value problem for systems of differential equations and approximation methods for their solution*, Mat. Sb., 39 (81) (1956), pp. 51–148, English transl.; Amer. Math. Soc. Transl. (2), 24 (1963), pp. 173–278.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

AN INEQUALITY RELATED TO POISSON'S EQUATION*

ALAN R. ELCRAT†

Abstract. The inequality

$$\|u\|_{W_2^2(\mathscr{R})} \leqq \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2}\right)^{1/2} \|\Delta u\|_{L_2(\mathscr{R})},$$

for $u \in W_{2,0}^2(\mathscr{R})$, is presented. The ($n$-dimensional) region $\mathscr{R}$ has a piecewise smooth boundary with nonnegative mean curvature, and λ is the fundamental frequency of $\mathscr{R}$. If $\mathscr{R}$ is a convex polyhedron, the inequality is sharp, so that the fundamental frequency of $\mathscr{R}$ is characterized.

Suppose that $\mathscr{R}$ is an n-dimensional region with piecewise smooth boundary. The Hilbert space of functions which are square summable over $\mathscr{R}$ together with their first and second order distribution derivatives is denoted $W_2^2(\mathscr{R})$, and the closure therein of smooth functions vanishing on $\partial\mathscr{R}$ by $W_{2,0}^2(\mathscr{R})$.

THEOREM. *If $\partial\mathscr{R}$ has nonnegative mean curvature almost everywhere,*

$$\text{(A)} \qquad \begin{aligned} [\|u\|_{W_2^2(\mathscr{R})}]^2 &= \int_{\mathscr{R}} u^2\,dx + \int_{\mathscr{R}} |\nabla u|^2\,dx + \int_{\mathscr{R}} \sum |D^2 u|^2\,dx \\ &\leqq \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2}\right) \int_{\mathscr{R}} (\Delta u)^2\,dx \end{aligned}$$

for u in $W_{2,0}^2(\mathscr{R})$, where λ is the lowest eigenvalue of $\Delta u + \lambda u = 0$ in $\mathscr{R}$. If $\mathscr{R}$ is a convex polyhedron, then (A) *is sharp.*

Proof. The inequality (A) was established in [3] for smoothly bounded $\mathscr{R}$. The proof given there applies without change here. If $\mathscr{R}$ is a convex polyhedron, the results of [4] imply that the eigenfunction u_0 corresponding to λ is an element of $W_{2,0}^2(\mathscr{R})$. By a well-known identity [2, p. 171],

$$\begin{aligned} &\int_{\mathscr{R}} u_0^2\,dx + \int_{\mathscr{R}} |\nabla u_0|^2\,dx + \int_{\mathscr{R}} \sum |D^2 u_0|^2\,dx \\ &\quad = \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2}\right) \int_{\mathscr{R}} (\Delta u_0)^2\,dx - \int_{\partial\mathscr{R}} H\left(\frac{\partial u_0}{\partial n}\right)^2 ds. \end{aligned}$$

Since H vanishes almost everywhere for a polyhedron, the result follows.

It is perhaps worth remarking here that if $\partial\mathscr{R}$ is locally convex, then the results of [5] imply that Δ is an invertible operator from $W_{2,0}^2(\mathscr{R})$ into $L_2(\mathscr{R})$ and (A) provides an upper bound for $\|\Delta^{-1}\|$.

We now observe that since $\psi(\lambda) = 1 + \lambda^{-1} + \lambda^{-2}$ establishes a one-to-one correspondence between $(0, \infty)$ and $(1, \infty)$, the quantity

$$\inf_{u \in W_{2,0}^2(\mathscr{R})} \frac{\int_{\mathscr{R}} u^2\,dx + \int_{\mathscr{R}} |\nabla u|^2\,dx + \int_{\mathscr{R}} \sum |D^2 u|^2\,dx}{\int_{\mathscr{R}} (\Delta u)^2\,dx}$$

* Received by the editors June 4, 1971, and in revised form October 4, 1971.

† Department of Mathematics, Wichita State University, Wichita, Kansas 67208.

determines the first eigenvalue of a convex polyhedron. This appears to yield a new characterization of this quantity for convex polyhedra. It is natural to conjecture that this characterization also holds for $\mathscr{R}$ convex, or for more general regions $\mathscr{R}$, but the author is thus far unable to accomplish such an extension.

Acknowledgment. The author is indebted to the referee for several helpful suggestions.

REFERENCES

[1] S. L. SOBOLEV, *Applications of Functional Analysis in Mathematical Physics*, American Mathematical Society, Providence, R.I., 1963.

[2] C. MIRANDA, *Partial Differential Equations of Elliptic Type*, Springer-Verlag, New York, 1970.

[3] A. R. ELCRAT, *A constructive existence theorem for a nonlinear elliptic equation*, this Journal, 2 (1971), pp. 368–374.

[4] J. KADLEC, *The regularity of the solution of the Poisson problem in a domain whose boundary is similar to that of a convex domain*, Czechoslovak Math. J., 89 (1964), pp. 386–393.

[5] V. A. KONDRATEV, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Moscow Math. Soc., 1968, pp. 227–314.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

MODIFIED ABEL EXPANSION AND A SUBCLASS OF COMPLETELY CONVEX FUNCTIONS*

S. P. PETHE AND A. SHARMA†

Abstract. We consider a modified Abel series which corresponds to an interpolation problem considered in 1936 by I. J. Schoenberg. This leads to a subclass of completely convex functions which in turn lead to the necessary and sufficient condition for a function to have an absolutely convergent generalized Abel series expansion.

1. Introduction. In the notation of Schoenberg [7], the Lidstone series corresponds to the interpolation problem given by the incidence matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & 0 & \cdots \end{pmatrix}. \tag{1.1}$$

The nonzero entries indicate that only even order derivatives are prescribed at two points 0 and 1. In 1942 Widder [9] introduced the class of completely convex functions and showed its close relationship with Lidstone series. A function $f \in C^{\infty}[0, 1]$ is said to be completely convex if

$$(-1)^k f^{(2k)}(x) \geqq 0, \qquad k = 0, 1, \cdots, \quad 0 \leqq x \leqq 1.$$

Widder proved that such a function is necessarily entire. He also obtained necessary and sufficient conditions for a function to have an absolutely convergent Lidstone series expansion.

Earlier Schoenberg [6], Whittaker [8] and Poritsky [4] had considered a series expansion corresponding to the interpolation problem given by the incidence matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \end{pmatrix}. \tag{1.2}$$

Schoenberg [6] solved the unicity question connected with the interpolation problems (1.1) and (1.2).

Our object here is to study a modified Abel series expansion corresponding to the interpolation problem (1.2) and to find necessary and sufficient conditions for this expansion to be absolutely convergent. It is interesting that the results are easily formulated in terms of a class of functions (c–c* functions) which forms a subclass of completely convex functions.

In § 2 we introduce the modified Abel expansion and its fundamental polynomials. Sections 3 and 4 deal with their relation with Bernoulli and Euler polynomials and their estimates. The class of c–c* functions is defined in § 5. In § 6,

* Received by the editors March 30, 1971, and in final revised form November 5, 1971.

† Department of Mathematics, The University of Alberta, Edmonton 7, Alberta, Canada. This work was supported by the United States Army under Contract DA-31-124-ARO-D-462.

we introduce the class of minimal c–c* functions which gives the main result of the paper in Theorem 6.5. In § 7, we give a representation theorem for c–c* functions following a similar result of Boas [1] for completely convex functions.

This theorem and its proof are due to the kindness of the referee for which we are very grateful.

2. Modified Abel expansion. Let $f \in C^{\infty}[0, 1]$. Consider the formal expansion

$$f(x) = \sum_{n=0}^{\infty} f^{(2n)}(1)\mu_n(x) + \sum_{n=0}^{\infty} f^{(2n+1)}(0)\nu_n(x), \tag{2.1}$$

where $\mu_n(x)$ and $\nu_n(x)$ are the fundamental polynomials of the interpolation problem (1.2). The series on the right in (2.1) will be called the modified Abel series. The Lidstone series corresponds to the interpolation problem (1.1) and is given by

$$f(x) = \sum_{n=0}^{\infty} f^{(2n)}(1)\Lambda_n(x) + \sum_{n=0}^{\infty} f^{(2n)}(0)\Lambda_n(1 - x), \tag{2.2}$$

where $\Lambda_n(x)$ is a polynomial of degree $2n + 1$ given by the generating function (see [9])

$$\frac{\sinh xt}{\sinh t} = \sum_{k=0}^{\infty} t^{2k}\Lambda_k(x). \tag{2.3}$$

Similarly it is known [6] that

$$\frac{\cosh xt}{\cosh t} = \sum_{k=0}^{\infty} t^{2k}\mu_k(x) \tag{2.4}$$

and

$$\nu_k(x) = \mu'_{k+1}(x - 1), \qquad k = 0, 1, 2, \cdots. \tag{2.5}$$

Adding (2.3) and (2.4) and observing that

$$\frac{\sinh xt}{\sinh t} + \frac{\cosh xt}{\cosh t} = \frac{2 \sinh (1 + x)t}{\sinh 2t},$$

we easily see that

$$\Lambda_k(x) + \mu_k(x) = 2^{2k+1}\Lambda_k\left(\frac{1 + x}{2}\right), \qquad k = 0, 1, 2, \cdots. \tag{2.6}$$

Since $\Lambda_0(x) = x$, $\Lambda_1(x) = \frac{1}{6}(x^3 - x)$, from (2.6) we have

$$\mu_0(x) = 1, \quad \mu_1(x) = \tfrac{1}{2}(x^2 - 1) \quad \text{and} \quad \mu_2(x) = (x^2 - 1)(x^2 - 5)/24.$$

Also from (2.4) we have

$$\begin{aligned} &\mu''_k(x) = \mu_{k-1}(x), \\ &\mu_k(1) = 0, \quad \mu'_k(0) = 0, \qquad k = 1, 2, \cdots. \end{aligned} \tag{2.7}$$

Further, we have

$$\mu_k(-x) = \mu_k(x), \qquad k = 0, 1, 2, \cdots. \tag{2.8}$$

The differential system (2.7) suggests that the Green's function for this system is given by

$$K(x,t) = \begin{cases} x - 1, & 0 \leqq t < x \leqq 1, \\ t - 1, & 0 \leqq x \leqq t \leqq 1. \end{cases} \tag{2.9}$$

Then

$$\mu_n(x) = \int_0^1 \mu_{n-1}(t) K(x,t)\, dt.$$

Setting

$$\begin{aligned} K_1(x,t) &= K(x,t), \\ K_n(x,t) &= \int_0^1 K(x,y) K_{n-1}(y,t)\, dy, \qquad n = 2, 3, \cdots, \end{aligned} \tag{2.10}$$

we see that, since $\mu_0(t) = 1$ and $v_0(t) = t - 1$,

$$\begin{aligned} \mu_n(x) &= \int_0^1 K_n(x,t)\, dt, \\ v_n(x) &= \int_0^1 K_n(x,t)(t-1)\, dt. \end{aligned} \tag{2.11}$$

These remarks lead us to formulate the following lemma.

LEMMA 2.1. *If $f(x) \in C^{(2n)}[0,1]$, then*

$$\begin{aligned} f(x) = f(1) + f'(0)(x-1) &+ \sum_{i=1}^{n-1} f^{(2i)}(1) \int_0^1 K_i(x,t)\, dt \\ &+ \sum_{i=1}^{n-1} f^{(2i+1)}(0) \int_0^1 K_i(x,t)(t-1)\, dt + R_n(f;x), \end{aligned} \tag{2.12}$$

where

$$R_n(f;x) = \int_0^1 f^{(2n)}(t) K_n(x,t)\, dt.$$

Since $K_1(x,t) \leqq 0$ for $0 \leqq x,\ t \leqq 1$ we see that $(-1)^n K_n(x,t) \geqq 0$. Hence, from (2.11) we get

$$(-1)^n \mu_n(x) \geqq 0 \quad \text{and} \quad (-1)^{n+1} v_n(x) \geqq 0, \qquad n = 0, 1, \cdots. \tag{2.13}$$

3. Relations with Bernoulli and Euler polynomials. Since

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!},$$

where $B_m(x)$ denote Bernoulli polynomials, it follows easily that

$$\begin{aligned}\Lambda_m(x) &= \frac{2^{2m}}{(2m+1)!}\left[B_{2m+1}\left(\frac{1+x}{2}\right) - B_{2m+1}\left(\frac{1-x}{2}\right)\right]\\ &= \frac{2^{2m+1}}{(2m+1)!}B_{2m+1}\left(\frac{1+x}{2}\right).\end{aligned}\tag{3.1}$$

Therefore it follows from (2.6) and the known relation

$$B_n(1-x) = (-1)^n B_n(x)$$

that

$$\begin{aligned}\mu_k'(x) &= 2^{2k}\Lambda_k'\left(\frac{1+x}{2}\right) - \Lambda_k'(x)\\ &= \frac{2^{2k}}{(2k)!}\left[2^{2k}B_{2k}\left(\frac{3+x}{4}\right) - B_{2k}\left(\frac{1+x}{2}\right)\right]\\ &= \frac{2^{2k}}{(2k)!}\left[2^{2k}B_{2k}\left(\frac{1}{2}\cdot\frac{3+x}{2}\right) - B_{2k}\left(\frac{3+x}{2}\right) + (2k)\left(\frac{1+x}{2}\right)^{2k-1}\right],\end{aligned}$$

where we have used the known relation

$$B_n(x) = B_n(x+1) - nx^{n-1}.$$

Since

$$E_n(x) + E_n(x+1) = 2x^n,$$

where $E_n(x)$ denote the Euler polynomials, it follows from simple calculations that

$$\mu_k'(x) = \frac{2^{2k-1}}{(2k-1)!}E_{2k-1}\left(\frac{1+x}{2}\right),\tag{3.2}$$

whence

$$v_k(x) = \frac{2^{2k+1}}{(2k+1)!}E_{2k+1}\left(\frac{x}{2}\right).\tag{3.3}$$

Since $(-1)^{k+1}E_{2k+1}(x/2) > 0$ for $0 < x < 1$, this again confirms the second part of (2.13).

It is known [3] that

$$B_{2m+1}(x) = 2(-1)^{m+1}(2m+1)!\sum_{k=1}^{\infty}\frac{\sin(2k\pi x)}{(2k\pi)^{2m+1}}, \qquad 0 \leqq x < 1.$$

This gives

$$\begin{aligned}\Lambda_m(x) &= \frac{2^{2m+1}}{(2m+1)!}B_{2m+1}\left(\frac{1+x}{2}\right)\\ &= \frac{2(-1)^{m+1}}{\pi^{2m+1}}\sum_{k=1}^{\infty}\frac{(-1)^k\sin k\pi x}{k^{2m+1}}, \qquad 0 \leqq x < 1.\end{aligned}\tag{3.4}$$

Using (2.6), we have after some simplification,

$$\mu_n(x) = \frac{(-1)^n 2^{2n+2}}{\pi^{2n+1}} \sum_{k=0}^{\infty} (-1)^k \frac{\cos(k+\frac{1}{2})\pi x}{(2k+1)^{2n+1}}, \qquad 0 \leqq x < 1, \tag{3.5}$$

whence from (2.5) we have

$$v_n(x) = \frac{(-1)^{n+1} 2^{2n+3}}{\pi^{2n+2}} \sum_{k=0}^{\infty} \frac{\cos(k+\frac{1}{2})\pi x}{(2k+1)^{2n+2}}, \qquad 0 \leqq x < 1. \tag{3.6}$$

4. Estimates for $\mu_n(x)$ and $v_n(x)$. In the following, $C_0, C_1, \cdots$ will always denote positive constants independent of n even though this is not specifically stated.

LEMMA 4.1. *There exist constants C_0, C_1 such that*

$$\left|(-1)^n \mu_n(x) - \frac{2^{2n+2}}{\pi^{2n+1}} \cos\frac{\pi x}{2}\right| < \frac{C_0}{\pi^{2n+1}}, \qquad 0 \leqq x \leqq 1, \quad n = 0, 1, \cdots, \tag{4.1}$$

$$\left|(-1)^{n+1} v_n(x) - \frac{2^{2n+3}}{\pi^{2n+2}} \cos\frac{\pi x}{2}\right| \leqq \frac{C_1}{\pi^{2n}}, \qquad 0 \leqq x \leqq 1, \quad n = 0, 1, \cdots. \tag{4.2}$$

Proof. From (3.5), we have, for $0 \leqq x < 1$,

$$\left|(-1)^n \mu_n(x) - \frac{2^{2n+2}}{\pi^{2n+1}} \cos\frac{\pi x}{2}\right| \leqq \frac{2}{\pi^{2n+1}} \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{2n+1}$$

which proves (4.1), since the sum of the right side is a Dirichlet series which tends to 1 as $n \to \infty$. Equation (4.2) follows similarly from (3.6).

The results are trivially true for $x = 1$.

LEMMA 4.2. *For $0 \leqq x \leqq 1$, we have*

$$0 \leqq (-1)^n \mu_n(x) \leqq C_2(\pi/2)^{-2n}, \qquad n = 0, 1, 2, \cdots, \tag{4.3}$$

and

$$0 \leqq (-1)^{n+1} v_n(x) \leqq C_3(\pi/2)^{-2n}, \qquad n = 0, 1, 2, \cdots. \tag{4.4}$$

Proof. The proof follows at once from (4.1) and (4.2).

LEMMA 4.3. *For $0 < x_0 < 1$, there exist constants C_2, C_3 independent of n such that*

$$(-1)^n \mu_n(x_0) \geqq C_2(\pi/2)^{-2n}, \qquad n = 0, 1, 2, \cdots, \tag{4.5}$$

and

$$(-1)^{n+1} v_n(x_0) \geqq C_3(\pi/2)^{-2n}, \qquad n = 0, 1, 2, \cdots. \tag{4.6}$$

Proof. From (3.5) we have

$$\lim_{n\to\infty} \frac{(-1)^n \mu_n(x_0)}{\cos(\pi x_0/2)} \left(\frac{\pi}{2}\right)^{2n} = \frac{4}{\pi}.$$

This yields (4.5). Equation (4.6) follows similarly from (3.6).

DEFINITION 4.4. *A real entire function $f(x)$ belongs to the class P^** if there exists a number $p < \pi/2$ such that

$$f^{(n)}(0) = O(p^n) \quad \text{as } n \to \infty.$$

The following lemma is due to Widder [9, p. 392] and is proved by using Taylor's formula.

LEMMA 4.5. *If $f(x) \in P^*$, then there exists a positive number $p < \pi/2$ such that*

$$f^{(n)}(x) = O(p^n), \tag{4.7}$$

uniformly for $0 \leqq x \leqq 1$.

THEOREM 4.6. *If $f(x) \in P^*$, then*

$$\begin{aligned} f(x) = f(1)\mu_0(x) &+ f'(0)\nu_0(x) + f''(1)\mu_1(x) \\ &+ f'''(0)\nu_1(x) + \cdots, \end{aligned} \tag{4.8}$$

the series converging uniformly in $0 \leqq x \leqq 1$.

Proof. If $S_{2n}(x)$ denotes the sum of the first $2n$ terms in the series in (4.8), then $f(x) - S_{2n}(x)$ has the property that all its derivatives of order $2k$ vanish at $x = 1$ and those of order $2k + 1$ vanish at $x = 0$ for $k = 0, 1, 2, \cdots, n - 1$. By the definition of $K_n(x, t)$ in (2.9) and (2.10) it is clear that if

$$F^{(2n)}(x) = \phi(x),$$

with

$$F^{(2k)}(1) = F^{(2k+1)}(0) = 0, \qquad k = 0, 1, \cdots, n - 1,$$

then

$$F(x) = \int_0^1 K_n(x, t)\phi(t)\, dt.$$

Hence,

$$f(x) - S_{2n}(x) = \int_0^1 K_n(x, t) f^{(2n)}(t)\, dt.$$

Using (2.11) and (4.3), we have

$$\begin{aligned} |f(x) - S_{2n}(x)| &\leqq \int_0^1 (-1)^n K_n(x, t)|f^{(2n)}(t)|\, dt \\ &\leqq C_4 \cdot p^{2n} \int_0^1 (-1)^n K_n(x, t)\, dt \\ &= C_4 p^{2n} |\mu_n(x)| \\ &\leqq C_5 p^{2n} (\pi/2)^{-2n}, \qquad n = 0, 1, \cdots \end{aligned} \tag{4.9}$$

Also,

$$S_{2n+1}(x) = S_{2n}(x) + f^{(2n)}(1)\mu_n(x),$$

so that we easily have

$$|f(x) - S_{2n+1}(x)| \leqq |f(x) - S_{2n}(x)| + |f^{(2n)}(1)|(-1)^n\mu_n(x) \leqq C_6 p^{2n}(\pi/2)^{-2n}. \tag{4.10}$$

The result then follows from (4.9) and (4.10).

THEOREM 4.7. *If the series*

$$b_0\mu_0(x) - a_0\nu_0(x) - b_1\mu_1(x) + a_1\nu_1(x) + \cdots \tag{4.11}$$

converges for a single value of x *in* $0 < x < 1$, *it converges uniformly in* $[0, 1]$ *to a function* $f(x)$. *Also the series*

$$b_0 + a_0(\pi/2)^{-1} + b_1(\pi/2)^{-2} + a_1(\pi/2)^{-3} \cdots \tag{4.12}$$

converges and

$$(-1)^k f^{(2k)}(x) = b_k\mu_0(x) - a_k\nu_0(x) - b_{k+1}\mu_1(x) + a_{k+1}\nu_1(x) + \cdots \tag{4.13}$$

for $0 \leqq x \leqq 1$ *and* $k = 0, 1, 2, \cdots$.

Proof. If (4.11) converges for $x = x_0$, then $\lim_{n\to\infty} b_n\mu_n(x_0) = 0$ and $\lim_{n\to\infty} a_n \cdot \nu_n(x_0) = 0$. By Lemma 4.3, $b_n = O((\pi/2)^{2n})$ and $a_n = O((\pi/2)^{2n})$. Using (4.1) and (4.2), we see then that

$$\begin{aligned}&\left\{\sum_{n=0}^{\infty}(-1)^n b_n \quad \mu_n(x_0) - \frac{(-1)^n 2^{2n+2}\cos(\pi x_0/2)}{\pi^{2n+1}}\right\}\\ &\quad + (-1)^n a_n\left\{\nu_n(x_0) - \frac{(-1)^{n+1}2^{2n+3}}{\pi^{2n+2}}\cos\frac{\pi x_0}{2}\right\}\end{aligned} \tag{4.14}$$

converges absolutely. Subtracting (4.11), which is convergent for $x = x_0$, from (4.14), we see that the series

$$-\frac{4}{\pi}\cos\frac{\pi x_0}{2}\left\{\sum_{n=0}^{\infty} b_n\left(\frac{\pi}{2}\right)^{-2n} + a_n\left(\frac{\pi}{2}\right)^{-2n-1}\right\} \tag{4.15}$$

must converge. This proves (4.12).

Replacing x_0 by x in (4.14) and (4.15) it is clear that both series converge uniformly in $0 \leqq x \leqq 1$. The same is then true for their difference, the series (4.11).

In order to prove (4.13), we observe that the series

$$b_{n+k}\left[\mu_n(x) - \frac{(-1)^n 2^{2n+2}}{\pi^{2n+1}}\cos\frac{\pi x}{2}\right] + \left[(-1)^{n+1}a_{n+k}\nu_n(x) - \frac{(-1)^{n+1}2^{2n+3}}{\pi^{2n+2}}\cos\frac{\pi x}{2}\right]$$

and

$$\frac{4}{\pi}\cos\frac{\pi x}{2}\left\{\sum_{n=0}^{\infty}(-1)^n b_{n+k}\left(\frac{\pi}{2}\right)^{-2n} + (-1)^{n+1}a_{n+k}\left(\frac{\pi}{2}\right)^{-2n-1}\right\}$$

both converge uniformly in $0 \leqq x \leqq 1$ because of (4.15), (4.1) and (4.2). Hence their sum also converges uniformly in $0 \leqq x \leqq 1$. This completes the proof of Theorem 4.7.

5. c–c* Functions. We shall now consider a class of functions which is a subclass of completely convex functions. We call these functions completely convex* or c–c* functions.

DEFINITION 5.1. A real function $f(x)$ is said to be *c–c* on* $a \leqq x \leqq b$ if:

(i) $f \in C^\infty[a, b]$,

(ii) $(-1)^k f^{(2k)}(x) \geqq 0, \quad a \leqq x \leqq b, \quad k = 0, 1, 2, \cdots,$

(iii) $(-1)^{k+1} f^{(2k+1)}(a) \geqq 0, \quad k = 0, 1, 2, \cdots.$

Thus $\cos(\pi x/2)$ is c–c* on $[0, 1]$, but $\sin(\pi x/2)$ is not.

The following results are easy to prove, and give some of the basic properties of c–c* functions.

LEMMA 5.2. *If f is c–c* in $0 \leqq x \leqq 1$, then*

$$f^{(2n)}(1) = O((\pi/2)^{2n}),$$
$$f^{(2n+1)}(0) = O((\pi/2)^{2n+1}).$$

Proof. The proof follows from (2.13) and the identity (2.12) which give the inequalities

$$0 \leqq f^{(2n)}(1)\mu_n(x) \leqq f(x), \qquad 0 \leqq x \leqq 1, \quad n = 0, 1, 2, \cdots,$$
$$0 \leqq f^{(2n+1)}(0)\nu_n(x) \leqq f(x), \qquad 0 \leqq x \leqq 1, \quad n = 0, 1, 2, \cdots.$$

The result follows by putting $x = \frac{1}{2}$ and applying Lemma 4.3.

LEMMA 5.3. *If $f(x)$ is c–c* in $0 \leqq x \leqq 1$, then there exist constants C_6, C_7 such that*

$$\begin{aligned} &0 \leqq (-1)^n f^{(2n)}(x) \leqq C_6(\pi/2x)^{2n}, \\ &0 \leqq (-1)^{n+1} f^{(2n+1)}(x) \leqq C_7(\pi/(2(1-x)))^{2n+1}, \qquad n \to \infty. \end{aligned} \tag{5.1}$$

Proof. If $f(x)$ is c–c* in $a \leqq x \leqq b$, then $F(x) = f(a + x(b-a))$ is c–c* in $0 \leqq x \leqq 1$. By Lemma 5.2, we have for $0 \leqq a < b \leqq 1$,

$$\begin{aligned} F^{(2n+1)}(0) &= f^{(2n+1)}(a)(b-a)^{2n+1} = O((\pi/2)^{2n+1}), \\ F^{(2n)}(1) &= f^{(2n)}(b)(b-a)^{2n} = O((\pi/2)^{2n}). \end{aligned} \tag{5.2}$$

Setting $a = x > 0, b = 1$ in the first of equations (5.2) and $b = x < 1, a = 0$ in the second, we obtain (5.1), which proves the lemma.

THEOREM 5.4. *If $f(x)$ is c–c* in $a \leqq x \leqq b$ with $b - a > 1$, then $f(x) \in P^*$ and (4.8) holds.*

Proof. Reasoning as in the proof of Lemma 5.3, we have for a suitable constant C_8,

$$|f^{(2n-1)}(x)| \leqq C_8(\pi/(2(b-x)))^{2n-1}, \qquad a \leqq x \leqq b.$$

Choose c so near to a that $b - c > 1$. Then

$$|f^{(2n-1)}(x)| \leqq C_8 p^{2n-1} \quad \text{and} \quad |f^{(2n+1)}(x)| \leqq C_8 p^{2n+1}$$

with $p = \pi/(2(b-c))$. Now the well-known Hadamard's inequality [9, p. 177, Lemma 20a] after some simple computation yields

$$f^{(2n)}(x) = O(p^{2n}).$$

That is, $f^{(n)}(x) = O(p^n)$ as $n \to \infty$ uniformly in $a \leqq x \leqq c$. This shows that $f(x)$ is entire and that $f(x+a) \in P^*$. This completes the proof.

6. Minimal c–c* functions. The sufficient condition of Theorem 5.4 for the representation of a function as a generalized Abel expansion is not necessary. The example $f(x) = \cosh x$ shows this, since it is not c–c* in [0, 1], yet it has the representation

$$\cosh x = \cosh 1 \sum_{k=0}^{\infty} \mu_k(x).$$

In order to obtain conditions which are both necessary and sufficient, we introduce a class of minimal c–c* functions.

DEFINITION 6.1. A function $f(x)$ is called *minimal c–c** in [0, 1] if it is c–c* in [0, 1] and if $f(x) - \varepsilon \cos(\pi x/2)$ is not c–c* for any positive ε.

Thus the functions 0 and $\cos x$ are minimal c–c* in [0, 1].

THEOREM 6.2. *If the series*

$$b_0\mu_0(x) - a_0\nu_0(x) - b_1\mu_1(x) + a_1\nu_1(x) + \cdots, \tag{6.1}$$
$$a_n \geqq 0, \quad b_n \geqq 0, \quad n = 0, 1, 2, \cdots,$$

converges to $f(x)$, *then* $f(x)$ *is minimal c–c* in* [0, 1].

Proof. Differentiating (6.1) $2k$ times, we have

$$(-1)^k f^{(2k)}(x) = \sum_{n=0}^{\infty} (-1)^n b_{n+k}\mu_n(x) + (-1)^{n+1} a_{n+k}\nu_n(x) \geqq 0, \qquad 0 < x < 1.$$

Also

$$(-1)^{k+1} f^{(2k+1)}(0) = -\sum_{n=0}^{\infty} (-1)^n b_{n+k}\mu_n'(0) + (-1)^{n+1} a_{n+k}\nu_n'(0)$$
$$= a_k$$

because of (2.7) and (3.3). Thus $f(x)$ is c–c*.

By Lemma 4.2, we have

$$(-1)^k f^{(2k)}(x) \leqq C \sum_{n=0}^{\infty} b_{n+k}\left(\frac{\pi}{2}\right)^{-2n} - a_{n+k}\left(\frac{\pi}{2}\right)^{-2n-1}$$
$$= C(\pi/2)^{2k} R_k,$$

where

$$R_k = \sum_{n=k}^{\infty} b_n(\pi/2)^{-2n} - a_n(\pi/2)^{-2n-1}.$$

Due to (4.12), $R_k \to 0$ as $k \to \infty$. Therefore, for a given positive ε and a number x_0, $0 < x_0 < 1$, we can find an integer k so large that $CR_k - \varepsilon\cos(\pi x_0/2) < 0$. That is,

$$(-1)^k \left[f(x) - \varepsilon \cos \frac{\pi x}{2} \right]_{x=x_0}^{(2k)} < 0.$$

Hence $f(x) - \varepsilon \cos(\pi x/2)$ does not belong to the class c–c* in $0 \leqq x \leqq 1$ which proves the result.

LEMMA 6.3. *Suppose that*

(i) $f(x) \geqq 0$, $0 \leqq x \leqq 1$,

(ii) $-f''(x) \geqq 0$, $0 \leqq x \leqq 1$,

(iii) $f'(0) \leqq 0$, *and*

(iv) $f(x_0) > \varepsilon\pi/2$ *for some number* x_0 *in* $0 \leqq x \leqq 1$.

Then

$$f(x) > \varepsilon \cos \frac{\pi x}{2}, \qquad 0 \leqq x \leqq 1.$$

Proof. If $x_0 = 1$, then from (ii) and (iii) it follows that $f(x)$ is a concave function and has a negative slope at $x = 0$. Therefore, it has its maximum at $x = 0$ and so $f(x) > \varepsilon\pi/2$ and a fortiori $f(x) > \varepsilon \cos(\pi x/2)$.

If $x_0 = 0$, the inequality follows by considering the function $f(1 - x)$. Thus we have to prove the result for $0 < x_0 < 1$.

Since $f'(x) = f'(0) + \int_0^x f'(t)\,dt$ it follows because of (ii) that $f'(x) \leqq f'(0) \leqq 0$ for $x \geqq 0$. Also since

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(x)\,dt$$

we have, for $0 \leqq x \leqq x_0$, $f(x) \geqq f(x_0)$.

From the concavity of $f(x)$ (that is, condition (ii)) the curve $y = f(x)$ is above the chord joining $(1, 0)$ and $(x_0, f(x_0))$. Then

$$f(x) \geqq f(x_0) \cdot \frac{1 - x}{1 - x_0} \geqq f(x_0)(1 - x) \geqq \varepsilon \cdot \frac{\pi}{2}(1 - x)$$

$$\geqq \varepsilon \sin \frac{\pi}{2}(1 - x) = \varepsilon \cos \frac{\pi x}{2}.$$

This completes the proof of the lemma.

THEOREM 6.4. *If* $f(x)$ *is a minimal c–c* function in* $0 \leqq x \leqq 1$, *then it can be expanded in a convergent modified Abel series* (2.1).

Proof. If $S_n(x)$ denotes the sum of the modified Abel expansion up to n terms, it is clear from the properties of c–c* functions and of the fundamental polynomials $\mu_k(x)$ and $\nu_k(x)$ that

$$S_n(x) \leqq f(x), \qquad 0 \leqq x \leqq 1, \quad n = 0, 1, \cdots,$$

and that $S_n(x)$ is a nondecreasing function of n for each x. Hence, $S_n(x)$ tends to some function as $n \to \infty$. If $\lim_{n\to\infty} S_n(x) = g(x)$, we want to show that $g(x) \equiv f(x)$.

Suppose this is not so; then for some x_0 in $0 \leqq x \leqq 1$,

$$f(x_0) - \lim_{n\to\infty} S_n(x_0) = \Delta > 0,$$

so that

$$f(x_0) - S_{2n}(x_0) = \int_0^1 K_n(x, t) f^{(2n)}(t)\,dt \geqq \Delta, \qquad n = 1, 2, \cdots. \tag{6.2}$$

Since $f(x)$ is minimal c–c*, $f(x) - \varepsilon \cos(\pi x/2)$ fails to be c–c* in $0 \leqq x \leqq 1$ for every $\varepsilon > 0$. Then there exists an integer k, and a number x_1 in (0, 1) such that

$$(-1)^k f^{(2k)}(x_1) - \varepsilon(\pi/2)^{2k} \cos\frac{\pi x_1}{2} < 0.$$

This implies, by virtue of Lemma 6.3, that

$$(-1)^k f^{(2k)}(x) < \varepsilon(\pi/2)^{2k+1},$$

whence, using the estimate (4.5), we have

$$f(x_0) - S_{2n}(x_0) < \varepsilon \cdot \frac{\pi}{2} \cdot C_2 < \Delta,$$

if $\varepsilon < 2\Delta/(C_2\pi)$. This contradicts (6.2) and completes the proof of Theorem 6.4.

We now prove the following theorem.

THEOREM 6.5. *A necessary and sufficient condition that a real function $f(x)$ can be represented by an absolutely convergent modified Abel series is that it should be the difference of two minimal c–c* functions in* $0 \leqq x \leqq 1$.

Proof. Sufficiency. Let $f(x) = g(x) - h(x)$, where $g(x)$ and $h(x)$ are both minimal c–c* functions on $0 \leqq x \leqq 1$. Then, by Theorem 6.4, both can be expanded in the convergent modified Abel series. Since each is a series with positive terms, their difference $f(x)$ can also be expanded in an absolutely convergent modified Abel series.

Necessity. Assume that

$$f(x) = \sum_{n=0}^{\infty} b_n \mu_n(x) + a_n v_n(x), \tag{6.3}$$

the series on the right converging absolutely. Then, setting

$$g(x) = \sum_{n=0}^{\infty} (-1)^n |b_n| \mu_n(x) + (-1)^{n+1} |a_n| v_n(x),$$

and

$$h(x) = \sum_{n=0}^{\infty} (-1)^n [|b_n| - (-1)^n b_n] \mu_n(x) + (-1)^{n+1} [|a_n| - (-1)^{n+1} a_n] v_n(x),$$

we see, on account of Theorem 6.2, that both $g(x)$ and $h(x)$ are minimal c–c* functions. Observing that $f(x) = g(x) - h(x)$ we see that the proof is complete.

7. Representation of c–c* functions. R. P. Boas [1] has given the following representation theorem for completely convex functions, which is analogous to the known theorem of Bernstein for completely monotonic functions.

THEOREM 7.1 (Boas [1]). *A function $f(x)$ defined on $0 \leqq x \leqq 1$ is completely convex if and only if it has the form*

$$f(x) = \sin \pi x \left\{ C + \int_0^{\infty} \frac{\phi(t)}{\cosh \pi t + \cos \pi x} dt + \int_0^{\infty} \frac{\psi(t)}{\cosh \pi t - \cos \pi x} dt \right\}, \tag{7.1}$$

where $C \geqq 0$ *and* ϕ *and* ψ *are entire functions of the form*

$$\phi(t) = \sum_{n=0}^{\infty} \frac{a_n t^{2n}}{(2n)!} \quad \text{and} \quad \psi(t) = \sum_{n=0}^{\infty} \frac{b_n t^{2n}}{(2n)!},$$

where

$$a_n \geqq 0, \quad b_n \geqq 0, \quad \sum_{n=0}^{\infty} a_n \pi^{-2n} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} b_n \pi^{-2n} < \infty. \tag{7.2}$$

Using the known integral representations for Bernoulli polynomials [3], one can write (7.1) in the following known simpler form (see [1]):

$$f(x) = C \sin \pi x - \sum_{k=0}^{\infty} \frac{(-1)^k a_k}{(2k+1)!} 2^{2k+1} B_{2k+1}\left(\frac{1-x}{2}\right) - \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{(2k+1)!} 2^{2k+1} B_{2k+1}\left(\frac{x}{2}\right) \tag{7.3}$$

with conditions in (7.2).

We now give the representation theorem for c–c* functions.

THEOREM 7.2. *A function defined on* $0 \leqq x \leqq 1$ *is c–c* if and only if it has the form*

$$f(x) = C \cos \frac{\pi x}{2} + \sum_{k=0}^{\infty} (-1)^k \{b_k \mu_k(x) - a_k \nu_k(x)\},$$

where $C \geqq 0$, $a_k \geqq 0$, $b_k \geqq 0$, $k = 0, 1, 2, \cdots$, *and*

$$\sum_{k=0}^{\infty} (a_k + b_k)\left(\frac{2}{\pi}\right)^{2k} < \infty. \tag{7.4}$$

Proof. From (7.4) and the nonnegativity of a_k and b_k, we have (4.15), and then reversing the steps in the proof of Theorem 4.7, we see that (7.4) implies the convergence of

$$\sum_{k=0}^{\infty} (-1)^k \{b_k \mu_k(x) - a_k \nu_k(x)\}, \qquad 0 \leqq x \leqq 1.$$

Then from Theorems 6.2 and (6.4) we see that $f(x)$ is minimal c–c* if and only if $f(x)$ has the form

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \{b_k \mu_k(x) - a_k \nu_k(x)\}, \qquad a_k \geqq 0, \quad b_k \geqq 0,$$

where (7.4) holds.

To complete the characterization of c–c* functions it remains to prove that there is a nonnegative number C such that

$$F(x) = f(x) - C \cos(\pi x/2)$$

is minimal c–c*.

To prove this, let E denote the set of all nonnegative numbers t such that the function $f(x) - t \cos(\pi x/2)$ is c–c*. The set E is nonempty and is bounded. Let C

denote its supremum. If $C = 0$, there is nothing to prove. Suppose that $C > 0$, that k is a nonnegative integer, that $0 \leqq s < C$, and that $0 \leqq x \leqq 1$. Thus

$$(-1)^{k+1} F_C^{(2k+1)}(0) = (-1)^{k+1} f^{(2k+1)}(0) \geqq 0.$$

Observe that

$$g_k(s) = (-1)^k F_s^{(2k)}(x) = (-1)^k f^{(2k)}(x) - s(\pi/2)^{2k} \cos(\pi x/2)$$

is a nonincreasing function of s. Therefore,

$$g_k(s) \geqq 0, \quad 0 \leqq s \leqq c,$$

since $g_k(s_0) < 0$, $0 \leqq s_0 < C$, would imply that s_0 is an upper bound for E. Consequently,

$$(-1)^k F_C^{(2k)}(x) \quad \lim_{s \to C} (-1)^k F_s^{(2k)}(x) \geqq 0$$

which establishes that F_C is c–c*. The definition of C guarantees that F_C is minimal c–c*.

Acknowledgment. The above proof has been kindly provided by the referee whom the authors would like to thank for this and other valuable suggestions.

REFERENCES

[1] R. P. Boas, *Representations for completely convex functions*, Amer. J. Math., 81 (1959), pp. 709–714.
[2] ——— *Almost completely convex functions*, Duke Math. J., 24 (1957), pp. 193–195.
[3] N. E. Nörlund, *Vorlesungen über Differenzeurechnung*, Chelsea, New York, 1954.
[4] H. Poritsky, *On certain polynomial and other approximations to analytic functions*, Trans. Amer. Math. Soc., 34 (1932), pp. 274–331.
[5] M. H. Protter, *A generalization of completely convex functions*, Duke Math. J., 24 (1957), pp. 205–213.
[6] I. J. Schoenberg, *On certain two point expansions of integral functions of exponential type*, Bull. Amer. Math. Soc., 42 (1936), pp. 284–288.
[7] ———, *On Hermite-Birkhoff interpolation*, J. Math. Anal. Appl. 16 (1966), pp. 538–543.
[8] J. M. Whittaker, *On Lidstone series and two point expansions of analytic functions*, Proc. London Math. Soc., 36 (1934), pp. 451–469.
[9] D. V. Widder, *Completely convex functions and Lidstone series*, Trans. Amer. Math. Soc., 51 (1942), pp. 387–398.

SIAM J. MATH. ANAL.
Vol. 3, No. 3, August 1972

ERRATA: DISCONJUGACY TESTS FOR SINGULAR LINEAR DIFFERENTIAL EQUATIONS*

D. WILLETT†

1. Add the additional hypothesis to Theorem 1.1 that equation (1.1),

$$Ly \equiv y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0, \qquad p_k \in C(\alpha, \beta),$$

is disconjugate at α, so that Theorem 1.1 is as follows: *If $Ly = 0$ is disconjugate at α, then $Ly = 0$ is disconjugate on $[\alpha, \beta]$ if and only if for any $[\alpha, b] \subset [\alpha, \beta]$, there exists a fundamental principal system on $[\alpha, b]$.*

2. Eliminate the second sentence in the second paragraph of the proof of Theorem 1.1.

3. In the proof of Theorem 3.1, first let $\alpha < c < \beta$ so that c is a nonsingular point. Then the proof as given but adapted to intervals $[b, c]$, $a \leqq b < c$, establishes disconjugacy at α for (2.15). The remainder of the proof is then the proof as given, which uses Theorem 1.1.

4. We note the following theorem, which does not require the hypothesis of disconjugacy at α.

THEOREM. *The equation $Ly = 0$ is disconjugate on $[\alpha, \beta]$ if and only if for any $[a, b] \subset [\alpha, \beta]$, there exists a fundamental principal system on $[a, b]$.*

Proof. If $Ly = 0$ is not disconjugate on $[\alpha, \beta]$ or (α, β), then there exists $[a, b] \subset (\alpha, \beta)$ such that $Ly = 0$ is not disconjugate on $[a, b]$. But this contradicts Theorem 1.1 as given in remark 1 above, since a is not singular and $Ly = 0$ is disconjugate at a. If $Ly = 0$ is disconjugate on (α, β), then it is disconjugate at α and Theorem 1.1 again applies.

Acknowledgment. The author thanks Mr. Euel Kennedy for pointing out the need for the additional hypothesis in Theorem 1.1.

* This Journal, 2 (1971), pp. 536–545. Received by the editors November 23, 1971.

† Department of Mathematics, The University of Alberta, Edmonton 7, Alberta, Canada.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

EVALUATION OF DISTRIBUTIONS USEFUL IN KONTOROVICH-LEBEDEV TRANSFORM THEORY*

G. Z. FORRISTALL† AND J. D. INGRAM‡

Abstract. The transform methods of Kontorovich and Lebedev seem to offer a direct approach to the solution of many physical problems involving the geometry of angular sectors. This promise has often been thwarted in practice by the coupling of boundary conditions which makes a direct inverse transform impossible. In the present work, we offer a method to meet this difficulty, based on the evaluation of the distributions listed below in terms of tabulated functions:

$$\lim_{\varepsilon \searrow 0} \int_0^\infty K_{i\nu}(\lambda r) K_{i\xi}(r) \frac{dr}{r^{1-\varepsilon}},$$

$$\lim_{\varepsilon \searrow 0} \int_0^\infty K_{i\nu}(\lambda r) r^{2m-1-i\xi+\varepsilon}\, dr,$$

$$\lim_{\varepsilon \searrow 0} \int_0^\infty K_{i\xi}(r) I_{i\nu}(\lambda r) \frac{dr}{r^{1-\varepsilon}}.$$

1. Introduction. The transform methods of Kontorovich and Lebedev [7]–[9] seem to offer a direct approach to the solution of many physical problems involving the geometry of angular sectors. Unfortunately, this promise is usually thwarted in practice by a coupling of boundary conditions which makes it impossible to take an inverse transform in a direct manner. For example, let us consider the transform pair:

$$\begin{aligned} \bar{\phi}(r) &= \int_0^\infty X(\nu) K_{i\nu}(kr) \sinh(\nu\pi)\nu\, d\nu, \\ X(\nu) &= \frac{2}{\pi^2} \int_0^\infty \bar{\phi}(r) K_{i\nu}(kr) \frac{dr}{r}, \end{aligned} \tag{1}$$

where $K_{i\nu}(kr)$ is the modified Bessel function. After the application of the transform to boundary conditions, the coupling of the boundary conditions may lead to equations in the transform plane which include both the Bessel functions $K_{i\nu}(k_\alpha r)$ and $K_{i\nu}(k_\beta r)$. In this case the inversion formula in (1) is not applicable, and we need the evaluation of the distribution

$$\lim_{\varepsilon \searrow 0} \int_0^\infty K_{i\nu}(\lambda r) K_{i\xi}(r) \frac{dr}{r^{1-\varepsilon}}, \tag{2}$$

which is discussed in § 4. This evaluation was central to the work of Ingram [6] on wave propagation in acoustic wedges in contact and was also used in the work of Forristall and Ingram [4] on wave propagation in an elastic wedge. In the latter

* Received by the editors November 24, 1970, and in revised form December 13, 1971.

† Fachgruppe fur Mechanik, Eidgenossische Technische Hochschule, Zurich, Switzerland. The work of this author was supported by a National Science Foundation Postdoctoral Fellowship.

‡ Mechanical Engineering Department, Rice University, Houston, Texas 77001.

work and in other problems in which the boundary equations in the transform plane include Bessel functions of different orders it is useful to expand one of the Bessel functions in a series and evaluate the distributions

$$\lim_{\varepsilon \searrow 0} \int_0^\infty K_{i\nu}(kr) r^{2m-1-i\xi+\varepsilon}\, dr, \tag{3}$$

as in § 2.

Similar problems arise in the use of another Kontorovich–Lebedev transform pair:

$$\begin{aligned} \bar{\phi}(r) &= \int_{\delta - i\infty}^{\delta + i\infty} X(\nu) K_\nu(kr) \nu\, d\nu, \\ X(\nu) &= \frac{1}{i\pi} \int_0^\infty \bar{\phi}(r) I_\nu(kr) \frac{dr}{r}, \end{aligned} \tag{4}$$

where $I_\nu(kr)$ is the modified Bessel function of the first kind. In this case, it is useful to have an evaluation of

$$\lim_{\varepsilon \searrow 0} \int_0^\infty I_{i\nu}(\lambda r) K_{i\xi}(r) \frac{dr}{r^{1-\varepsilon}}, \tag{5}$$

which is discussed in § 3.

2. Distribution involving a modified Bessel function of the second kind and a power. Define

$$L(\nu, \xi, m, k) = \lim_{\varepsilon \searrow 0} \int_0^\infty K_{i\nu}(kr) r^{2m-1-i\xi+\varepsilon}\, dr, \tag{6}$$

where ν and ξ are real and nonzero.

For $2m+\varepsilon > 0$, the integral in the expression above has a well-known evaluation [12, p. 388], so for $m > 0$,

$$L(\nu, \xi, m, k) = \frac{1}{4}\left(\frac{2}{k}\right)^{2m-i\xi} \Gamma\left(m - \frac{i}{2}(\nu + \xi)\right) \Gamma\left(m + \frac{i}{2}(\nu - \xi)\right). \tag{7}$$

However, for $m = 0$, we see that we must investigate the singular behavior of the gamma functions in taking the limit:

$$L(\nu, \xi, 0, k) = \frac{1}{4}\left(\frac{2}{k}\right)^{-i\xi} \lim_{\varepsilon \searrow 0} \Gamma\left(\frac{\varepsilon}{2} - \frac{i}{2}(\nu + \xi)\right) \Gamma\left(\frac{\varepsilon}{2} + \frac{i}{2}(\nu - \xi)\right). \tag{8}$$

This could be done by a rather simple contour integration, but it is more convenient to use the apparatus of generalized function theory [5], so that

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \Gamma(\varepsilon + ix) &= \lim_{\varepsilon \searrow 0} \Gamma(\varepsilon + 1 + ix) \frac{1}{\varepsilon + ix} = \Gamma(1 + ix) \frac{-i}{x - i0} \\ &= -i\Gamma(1 + ix) \left\{ \frac{1}{x} + i\pi\delta(x) \right\}, \end{aligned} \tag{9}$$

where $1/x$ should be considered as a principal value. Using this fact in (8), we have

$$L(\nu,\xi,0,k) = -\frac{1}{4}\left(\frac{2}{k}\right)^{-i\xi}\Gamma\left(1-\frac{i}{2}(\nu+\xi)\right)\Gamma\left(1+\frac{i}{2}(\nu-\xi)\right) \cdot\left\{\frac{-2}{\nu+\xi}+2i\pi\delta(\nu+\xi)\right\}\left\{\frac{2}{\nu-\xi}+2i\pi\delta(\nu-\xi)\right\} \tag{10}$$

or

$$L(\nu,\xi,0,k) = \frac{\pi}{2}\left\{\left(\frac{2}{k}\right)^{-i\nu}\Gamma(-i\nu)\delta(\nu-\xi)+\left(\frac{2}{k}\right)^{i\nu}\Gamma(i\nu)\delta(\nu+\xi)\right\} + \frac{1}{\nu^2-\xi^2}\left(\frac{2}{k}\right)^{-i\xi}\Gamma\left(1-\frac{i}{2}(\nu+\xi)\right)\Gamma\left(1+\frac{i}{2}(\nu-\xi)\right), \tag{11}$$

where $1/(\nu^2-\xi^2)$ should be considered as a principal value.

3. Distribution involving modified Bessel functions of the first and second kind. Define

$$H(\nu,\xi;\lambda) = \lim_{\varepsilon\searrow 0}\int_0^\infty I_{i\nu}(\lambda r)K_{i\xi}(r)\frac{dr}{r^{1-\varepsilon}}, \qquad 0<\lambda\leqq 1, \tag{12}$$

where ν and ξ are real and nonzero.

The integral in (12) has been evaluated for positive t [3, vol. 2, p. 93], so that

$$H(\nu,\xi;\lambda) = \lim_{\varepsilon\searrow 0}\frac{1}{4}\frac{1}{\Gamma(1+i\nu)}\lambda^{i\nu}\Gamma\left(\frac{\varepsilon}{2}+\frac{i}{2}(\nu+\xi)\right)\Gamma\left(\frac{\varepsilon}{2}+\frac{i}{2}(\nu-\xi)\right) \cdot {}_2F_1\left(\frac{\varepsilon}{2}+\frac{i}{2}(\nu+\xi),\frac{\varepsilon}{2}+\frac{i}{2}(\nu-\xi);1+i\nu;\lambda^2\right). \tag{13}$$

We note that the singular behavior of (13) as $\varepsilon\searrow 0$ comes, as in the previous section, from the gamma functions, and a similar evaluation is possible:

$$\lim_{\varepsilon\searrow 0}\Gamma\left(\frac{\varepsilon}{2}+\frac{i}{2}(\nu+\xi)\right)\Gamma\left(\frac{\varepsilon}{2}+\frac{i}{2}(\nu-\xi)\right) = -\frac{4}{\nu^2-\xi^2}\Gamma\left(1+\frac{i}{2}(\nu+\xi)\right) \cdot\Gamma\left(1+\frac{i}{2}(\nu-\xi)\right) - \frac{2i\pi}{\nu}\Gamma(1+i\nu)[\delta(\nu+\xi)+\delta(\nu-\xi)]. \tag{14}$$

Using (14) in (13) and noting that

$${}_2F_1(0,b;c;z) = 1, \tag{15}$$

we have

$$H(\nu,\xi;\lambda) = -\frac{i\pi}{2\nu}\lambda^{i\nu}[\delta(\nu+\xi)+\delta(\nu-\xi)] - \frac{1}{\nu^2-\xi^2}\lambda^{i\nu} \cdot\frac{\Gamma(1+i(\nu+\xi)/2)\Gamma(1+i(\nu-\xi)/2)}{\Gamma(1+i\nu)}{}_2F_1\left(\frac{i}{2}(\nu+\xi),\frac{i}{2}(\nu-\xi);1+i\nu;\lambda^2\right), \tag{16}$$

where $1/(\nu^2-\xi^2)$ must again be considered as a principal value.

For $\lambda > 1$, $H(\nu, \xi; \lambda)$ obviously is not defined; in fact, the series used in deriving (13) does not converge. This may be clearly seen by considering the asymptotic expansions for the Bessel functions:

$$I_{i\nu}(\lambda r) \sim e^{\lambda r}/\sqrt{2\pi\lambda r} \tag{17}$$

and

$$K_{i\xi}(r) \sim \sqrt{\frac{\pi}{2r}}\, e^{-r}.$$

However, for $\lambda = 1$, the series still converges, and the hypergeometric function in (16) has a special form [1, p. 556] which gives

$$H(\nu, \xi; 1) = \frac{-i\pi}{2\nu}[\delta(\nu + \xi) + \delta(\nu - \xi)] - \frac{1}{\nu^2 - \xi^2}. \tag{18}$$

4. Distributions involving two modified Bessel functions of the second kind. Define

$$G(\nu, \xi; \lambda) = \lim_{\varepsilon \searrow 0} \int_0^\infty K_{i\nu}(\lambda r) K_{i\xi}(r) \frac{dr}{r^{1-\varepsilon}}, \tag{19}$$

where ν and ξ are real and nonzero.

First we consider the case $0 < \lambda \leqq 1$. From symmetry, the case where $\lambda \geqq 1$ will then be obvious. The derivation could proceed from an integral representation of the product of two Bessel functions [2], but it is more straightforward to use the result of § 3, along with the fact that

$$K_{i\nu}(\lambda r) = \frac{\pi}{2i \sinh \pi\nu}\{I_{-i\nu}(\lambda r) - I_{i\nu}(\lambda r)\}. \tag{20}$$

Thus

$$\begin{aligned} G(\nu, \xi; \lambda) &= \frac{\pi}{2i \sinh \pi\nu}\{H(-\nu, \xi; \lambda) - H(\nu, \xi; \lambda)\} \\ &= \frac{\pi^2}{2} \frac{\cos(\nu \log \lambda)}{\nu \sinh \pi\nu}[\delta(\nu + \xi) + \delta(\nu - \xi)] \\ &\quad - \frac{1}{\nu^2 - \xi^2}\left\{\lambda^{-i\nu} \frac{\Gamma(1 - i(\nu + \xi)/2)\Gamma(1 - i(\nu - \xi)/2)}{\Gamma(1 - i\nu)}\right. \\ &\quad \cdot {}_2F_1\left(\frac{i}{2}(\xi - \nu), -\frac{i}{2}(\nu + \xi); 1 - i\nu; \lambda^2\right) \\ &\quad - \lambda^{i\nu} \frac{\Gamma(1 + i(\nu + \xi)/2)\Gamma(1 + i(\nu - \xi)/2)}{\Gamma(1 + i\nu)} \\ &\quad \left. \cdot {}_2F_1\left(\frac{i}{2}(\nu + \xi), \frac{i}{2}(\nu - \xi); 1 + i\nu; \lambda^2\right)\right\}. \end{aligned} \tag{21}$$

This result may be considerably simplified by using the linear transformation [3, vol. 1, p. 106]:

$$\begin{aligned}\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}{}_2F_1(a,b;c;z) &= e^{i\pi(c-a)}(1-z)^{c-a-b} \\ &\cdot\frac{\Gamma(c-a)\Gamma(1-b)}{\Gamma(c+1-a-b)}{}_2F_1(c-a,c-b;c+1-a-b;1-z) \\ &- e^{i\pi(1-a)}(-z)^{b-c}(1-z)^{c-a-b} \\ &\cdot\frac{\Gamma(1-b)\Gamma(a)}{\Gamma(a+1-b)}{}_2F_1(1-b,c-b;a+1-b;z^{-1})\end{aligned} \tag{22}$$

on the first hypergeometric series in (21), and the related transformation

$$\begin{aligned}\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}{}_2F_1(a,b;c;z) &= e^{i\pi(c-a)}z^{1-c}(1-z)^{c-a-b} \\ &\cdot\frac{\Gamma(c-a)\Gamma(1-b)}{\Gamma(c+1-a-b)}\cdot{}_2F_1(1-a,1-b;c+1-a-b;1-z) \\ &- e^{i\pi(1-a)}(-z)^{b-c}(1-z)^{c-a-b}\frac{\Gamma(1-b)\Gamma(a)}{\Gamma(a+1-b)} \\ &\cdot{}_2F_1(1-b,c-b;a+1-b;z^{-1})\end{aligned} \tag{23}$$

on the second hypergeometric series in (21). Performing the necessary algebra we obtain the result

$$\begin{aligned}G(\nu,\xi;\lambda) &= \frac{\pi^2}{2}\frac{\cos(\nu\log\lambda)}{\nu\sinh\pi\nu}[\delta(\nu+\xi)+\delta(\nu-\xi)] \\ &+\left\{\sinh\frac{\pi}{2}(\nu+\xi)\sinh\frac{\pi}{2}(\nu-\xi)\right\}^{-1}g(\nu,\xi;\lambda),\end{aligned} \tag{24}$$

where

$$g(\nu,\xi;\lambda) = \frac{\pi^2}{8}(\lambda^2-1)\lambda^{-i\nu}{}_2F_1\left(1-\frac{i}{2}(\nu+\xi),1-\frac{i}{2}(\nu-\xi);2;1-\lambda^2\right).$$

Some special cases of (24) are of interest. For $\nu=\xi$, we use an integral representation [1, p. 558],

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \tag{25}$$

to get

$$g(\nu,\nu;\lambda) = \frac{\pi^2}{4\nu}\sin(\nu\log\lambda).$$

For $\lambda=1$, it is obvious that

$$G(\nu,\xi;1) = \frac{\pi^2}{2}\frac{1}{\nu\sinh\pi\nu}[\delta(\nu-\xi)+\delta(\nu+\xi)], \tag{26}$$

which can be seen to be a statement of the Kontorovich–Lebedev transform pair (1).

Acknowledgment. We would like to thank the reviewer for suggesting a simplified method of analysis.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, Dover, New York, 1965.

[2] A. L. DIXON AND W. L. FERRAR, *Integrals for the product of two Bessel functions*, Quart. J. Math., Oxford Ser., 6 (1934), pp. 166–174.

[3] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, Bateman Manuscript Project, Calif. Inst. Tech., 1953.

[4] G. Z. FORRISTALL AND J. D. INGRAM, *Elastodynamics of a wedge*, Bull. Seism. Soc. Amer., to appear.

[5] J. M. GELFAND AND G. E. SHILOV, *Verallgemeinerte Functionen*, V.E.B. Deutscher Verlag Wissenschaften, 1960.

[6] J. D. INGRAM, *Diffraction of scalar waves by a wedge*, J. Acous. Soc. Amer., to appear.

[7] M. J. KONTOROVICH AND N. N. LEBEDEV, Z. Eksper. Ted. Fizike, Moskwa, Leningrad, 8 (1938), pp. 1192–1206.

[8] N. N. LEBEDEV, *Sur un formule d'inversion*, C.R. (Doklady) Acad. Sci. URSS (N.S.), 52 (1946), pp. 655–658.

[9] ———, *On the representation of an arbitrary function by an integral involving Macdonald functions of complex order*, Doklady Akad. Nauk. SSSR (N.S.), 58 (1947), pp. 1007–1010.

[10] M. J. LIGHTHILL, *Fourier Analysis and Generalized Functions*, Cambridge University Press, 1948.

[11] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1941.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

QUANTITATIVE ESTIMATES FOR A NONLINEAR SYSTEM OF INTEGRODIFFERENTIAL EQUATIONS ARISING IN REACTOR DYNAMICS*

T. A. BRONIKOWSKI, J. E. HALL AND J. A. NOHEL†

Abstract. In this paper we obtain precise quantitative estimates for the asymptotic behavior of solutions of a class of nonlinear integrodifferential equations arising in nuclear reactor dynamics. The method uses a Galerkin approximation and certain energy estimates (Lyapunov functions) to obtain suitable bounds for solutions of related ordinary differential equations.

1. Introduction. We study the system

$$(1.1)\qquad \begin{aligned} u'(t) &= -\int_0^c \alpha(x)T(x,t)\,dx, \qquad 0 < x < c, \quad 0 < t < \infty, \\ T_t(x,t) &= (b(x)T_x(x,t))_x - q(x)T(x,t) + \eta(x)\sigma(u(t)), \end{aligned}$$

where $' = d/dt$ and where subscripts denote partial derivatives, subject to the initial conditions

$$(1.2)\qquad u(0) = u_0, \quad T(x,0) = f(x), \qquad 0 < x < c,$$

and to the boundary conditions

$$(1.3)\qquad d_1T(0,t) + d_2T_x(0,t) = 0, \quad d_3T(c,t) + d_4T_x(c,t) = 0.$$

In (1.1), (1.2), (1.3) the real functions α, η, f, b, q are prescribed on $0 \leqq x \leqq c$, the real function $\sigma(u)$ is defined for $-\infty < u < \infty$, and c, u_0, d_i, $i = 1, 2, 3, 4$, are constants with $|d_1| + |d_2| > 0$, $|d_3| + |d_4| > 0$.

Special cases of (1.1) with $b(x) \equiv 1$ and $q(x) \equiv 0$ have been studied by Levin and Nohel [4]–[7] in both the linear case $\sigma(u) = u$ and the nonlinear case, on the finite interval $[0, c]$ and the infinite interval $-\infty < x < \infty$ (for the latter the boundary conditions (1.3) are omitted). The system (1.1) and the various special cases are of interest as dynamic models of a one-dimensional continuous medium nuclear reactor for which $\sigma(u) = -1 + \exp u$. Here, as in earlier papers, the principal interest is in the behavior of solutions as $t \to \infty$, once it has been established that (1.1), (1.2), (1.3) is a properly posed problem.

In [4] it is shown under appropriate conditions that if $\sigma(u) \equiv u$, $b(x) \equiv 1$, $q(x) \equiv 0$, $-\infty < x < \infty$, the solution $u(t)$, $T(x,t)$ of (1.1), (1.2) satisfies $u(t) = O(t^{-3/2})$, $T(x,t) = O(t^{-1/2})$ (uniformly in x) as $t \to \infty$. More recently Miller [8] has shown that under essentially the same hypotheses, but with $\sigma(u) = u + o(u)$ as $|u| \to 0$, the solution $u(t)$, $T(x,t)$ of (1.1), (1.2) with $|u_0|$ and $\|f\|$ sufficiently small ($\|\cdot\|$ is the L_1-norm) approaches zero as $t \to \infty$ (uniformly in x).

In [5] and [6] sufficient conditions are given which ensure that with $b(x) \equiv 1$, $q(x) \equiv 0$ all solutions of the nonlinear system, which need not have $\sigma(u) \equiv u$ as a special case (one assumes only $u\sigma(u) > 0$ $(u \neq 0)$), approach zero as $t \to \infty$ in both

* Received by the editors July 6, 1971. The work of the third author was supported in part by the United States Army Research Office—Durham.

† Department of Mathematics, Marquette University, Milwaukee, Wisconsin 53233.

the finite ($0 \leqq x \leqq c$) and infinite ($-\infty < x < \infty$) cases; i.e., one has global asymptotic stability of the equilibrium state $u \equiv 0$, $T(x, t) \equiv 0$.

In [7] the effect of delayed neutrons is studied in both the finite and infinite cases, in the special case $b(x) \equiv 1$, $q(x) \equiv 0$, $\sigma(u) = -1 + \exp u$, and global asymptotic stability results are obtained. This physically important phenomenon complicates the equation (1.1) considerably and the methods of this paper only yield partial results in this case.

Bronikowski [1] studies (1.1), (1.2), (1.3) in the linear case $\sigma(u) \equiv u$. By applying a suitable tauberian theorem for Laplace transforms (different from the one used in [4]) to solutions of a related linear Volterra equation satisfied by $u(t)$, he obtains precise asymptotic formulas for the solution $u(t)$, $T(x, t)$ as $t \to \infty$ in a variety of cases.

In this paper we obtain precise quantitative estimates for the solution $u(t)$, $T(x, t)$ of (1.1), (1.2), (1.3) in the nonlinear case as $t \to \infty$; several cases are considered, and extensions to boundary conditions more general than (1.3) are indicated. The technique employed combines the Galerkin approximation with suitable Lyapunov functions for related systems of ordinary differential equations using methods developed by Hall [3]. We also remark that the present results require, roughly speaking, less stringent hypotheses regarding α, η, f than in [5], [6], [7], but a somewhat stronger hypothesis on $\sigma(u)$ than was the case in [5], [6]. Moreover, the technique, which appears to be of independent interest, is different; in [5], [6], [7] suitable energy functions were applied to related nonlinear Volterra integrodifferential equations to obtain the behavior of $u(t)$ as $t \to \infty$. It does not appear possible to deduce quantitative estimates concerning the behavior of $u(t)$ and $T(x, t)$ as $t \to \infty$ from those energy functions. However, such quantitative estimates are readily obtainable by elementary methods from the related systems of ordinary differential equations and our analysis makes use of the fact that these estimates do not depend on the number N of approximating ordinary differential equations in the Galerkin procedure.

In the sequel the symbol (m.na) will denote the first expression in relation (m.n) and (m.nb) the second.

2. Summary of results. Intimately connected with (1.1b), (1.3) is the following Sturm–Liouville problem. Let

$$L(y) = -(b(x)y')' + q(x)y, \qquad ' = d/dx, \quad 0 < x < c,$$

and consider the boundary value problem

$$L(y) = \lambda y, \tag{2.1}$$

$$d_1 y(0) + d_2 y'(0) = 0, \quad d_3 y(c) + d_4 y'(c) = 0. \tag{2.2}$$

The asymptotic behavior of solutions of (1.1), (1.2), (1.3) depends heavily on the nature of the spectrum and of the eigenfunctions of (2.1), (2.2). We make the following assumptions:

$$d_1 d_2 \leqq 0, \quad d_3 d_4 \geqq 0; \tag{2.3}$$

(2.4) either b, b', b'' are continuous and $b(x) > 0$ on $[0, c]$ or $b(x) \geqq 0$ on $[0, c]$, $b(x) > 0$ on $(0, c)$, b', b'' exist on $[0, c]$ and b, b^{-1} are integrable on $[0, c]$;

(2.5) q is continuous on $[0, c]$ (more generally q is measurable and integrable) and $q(x) \geqq 0$ on $[0, c]$.

The following result is well known and its proof may be found, for example, in [2, p. 415] or [11, pp. 110–114].

THEOREM A. *Let* (2.3), (2.4), (2.5) *be satisfied. Then the eigenvalues of the boundary value problem* (2.1), (2.2) *are simple, nonnegative, denumerable with no finite limit points. Indexing the eigenvalues so that* $0 \leqq \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ *one has*

$$\lambda_n = n^2\pi^2 L^{-2} + O(1), \quad n \to \infty, \quad \text{where} \quad L = \int_0^c (b(x))^{-1/2}\,dx. \tag{2.6}$$

If $y_n(x)$ *is the eigenfunction corresponding to* λ_n *normalized so that* $\int_0^c y_n^2\,dx = 1$, *then* $\{y_n\}_{n=0}^{\infty}$ *form a complete orthonormal set in* $L^2(0, c)$ *and* $\sup_{0 \leqq x \leqq c} |y_n(x)| \leqq K$, *for some constant* K *independent of* n (actually, precise asymptotic estimates are available, but no use is made of them here).

Moreover, if f *is any real function on* $[0, c]$ *which satisfies the boundary condition* (2.2) *and* $f, (bf')' \in L^2[0, c]$, *then*

$$f(x) = \sum_{n=0}^{\infty} f_n y_n(x), \quad f_n = (f, y_n), \tag{2.7}$$

where the series converges absolutely and uniformly; in fact,

$$\sum_{n=0}^{\infty} |f_n| < \infty.$$

The last statement follows from (2.6) and the elementary calculation using (2.1), (2.2) and integration by parts:

$$f_n = (f, y_n) = \frac{1}{\lambda_n}(f, L(y_n)) = -\frac{1}{\lambda_n}([f'b]', y_n) + \frac{1}{\lambda_n}(fq, y_n);$$

thus $|f_n| = O(1/n^2)$ as $n \to \infty$.

We remark that in the case of more general boundary conditions,

$$\begin{aligned} m_{11}y(0) + m_{12}y'(0) + n_{11}y(c) + n_{12}y'(c) &= 0, \\ m_{21}y(0) + m_{22}y'(0) + n_{21}y(c) + n_{22}y'(c) &= 0, \end{aligned} \tag{2.8}$$

where m_{ij}, n_{ij} are real constants (this includes the case of periodic boundary conditions), the boundary value problem (2.1), (2.8) is self-adjoint if and only if

$$\det\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \det\begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix}.$$

If b, q satisfy (2.4), (2.5) and if it is assumed that m_{ij}, n_{ij} are such that the spectrum of (2.1), (2.8) lies on the nonnegative real axis, the conclusions of Theorem A are unchanged, except for the fact that the eigenvalues are not necessarily simple (they can have multiplicity at most 2; we list double eigenvalues twice); in particular zero can be a double eigenvalue.

In the analysis below involving (2.1), (2.2) it will be important to distinguish when zero is or is not an eigenvalue. If

$$\text{(2.9)} \qquad \text{either} \quad q \not\equiv 0 \text{ on } [0, c] \quad \text{or} \quad |d_1| + |d_3| > 0,$$

then the smallest eigenvalue λ_0 is positive. If

$$\text{(2.10)} \qquad q \equiv 0 \text{ on } [0, c] \quad \text{and} \quad d_1 = d_3 = 0,$$

then $\lambda_0 = 0$ is the smallest eigenvalue. The corresponding normalized eigenfunction is $y_0(x) = c^{-1/2}$, a constant.

In order to state the principal results concerning the problem (1.1), (1.2), (1.3) we make the following assumptions:

(2.11) $\alpha, \eta, f, (b\eta')', (bf')' \in L_2(0, c), f \in C(0, c)$, and f, η satisfy the boundary condition (2.2).

Defining f_n as in (2.7) and similarly α_n, η_n, let

$$\text{(2.12)} \qquad \alpha_n \eta_n \geqq 0, \qquad n = 0, 1, \cdots,$$

and let there exist constants $\hat{c} > 0$ and $\tilde{c}$ such that

$$\text{(2.13)} \qquad \hat{c} \leqq \frac{\alpha_n}{\eta_n} \leqq \tilde{c} \quad \text{for all } n \text{ for which} \quad \alpha_n \eta_n > 0.$$

We shall also be interested in the special case in which

$$\text{(2.14)} \qquad \alpha_m \eta_m = 0 \quad \text{implies} \quad \alpha_m = \eta_m = 0, \qquad m \geqq 0.$$

It may be noted that (2.12), (2.13), (2.14) hold in the physically important case $\alpha(x) = k\eta(x)$ for some constant $k > 0$. We also note that

$$\text{(2.15)} \qquad \sum_{n=0}^{\infty} |f_n| < \infty, \quad \sum_{n=0}^{\infty} |\eta_n| < \infty, \quad \sum_{n=0}^{\infty} |\alpha_n| < \infty$$

in view of (2.11), the calculation following (2.7), and (2.13). Concerning the function $\sigma(u)$ we suppose

$$\text{(2.16)} \qquad \begin{aligned} &\sigma \in C'(-\infty, \infty), \quad x\sigma(x) > 0 \quad \text{if} \quad x \neq 0, \\ &S(x) = \int_0^x \sigma(\tau)\, d\tau \to \infty \quad \text{as} \quad |x| \to \infty, \end{aligned}$$

which is satisfied in the physically important case $\sigma(u) = -1 + \exp u$ as well as in the linear case $\sigma(u) \equiv u$. For some of the conclusions we also need

$$\text{(2.17)} \qquad \sigma'(0) > 0$$

and that there exists a constant Γ so that the growth condition

$$\text{(2.18)} \qquad \sigma^2(x) \leqq \Gamma S(x), \qquad -\infty < x < \infty,$$

is satisfied.

In the results given below, Theorem 1 establishes existence, uniqueness, boundedness and exponential decay of solutions as $t \to \infty$ of (1.1), (1.2), (1.3), in case zero is not an eigenvalue of (2.1), (2.2), i.e., if (2.9) is satisfied. The zero eigenvalue case is treated in Theorem 2. In the zero eigenvalue case the T component of the solution need not decay to zero. We assume throughout that (2.3), (2.4), (2.5), (2.11), (2.12), (2.13), (2.16) are satisfied.

THEOREM 1. *Let* (2.9) *and either* (2.14) *for all* $m \geqq 0$ *or* (2.18) *be satisfied. Then there exists a unique solution* $u(t)$, $T(x,t)$ *of* (1.1) *existing for* $0 < t < \infty, 0 < x < c$ *satisfying condition* (1.2), (1.3); *moreover, there exists a constant* $K > 0$ *such that*

$$(2.19)\qquad |u(t)| \leqq K, \quad \sup_{0 \leqq x \leqq c} |T(x,t)| \leqq K, \qquad 0 \leqq t < \infty.$$

If in addition (2.17) *is satisfied and if the strict inequality holds in* (2.12) *for at least one* n, *then there exist positive constants* K *and* ω *such that*

$$(2.20)\qquad \begin{aligned} |u^{(i)}(t)| &\leqq K e^{-\omega t}, & i = 0,1,2, \\ \sup_{0 \leqq x \leqq c} |T(x,t)| &\leqq K e^{-\omega t}, & 0 \leqq t < \infty. \end{aligned}$$

THEOREM 2. *Let* (2.10) *and either* (2.14) *for all* $m \geqq 0$ *or* (2.18) *as well as* (2.14, $m = 0$) *be satisfied. Then there exists a unique solution* $u(t)$, $T(x,t)$ *of* (1.1), (1.2), (1.3) *existing for* $0 < x < c, 0 < t < \infty$ *such that* $u(t)$ *satisfies* (2.19a) *and such that*

$$(2.21)\qquad \sup_{0 \leqq x \leqq c} |T(x,t)| \leqq K\left(1 + \left|\int_0^t \sigma(u(s))\,ds\right|\right), \qquad 0 \leqq t < \infty.$$

If in addition (2.17) *is satisfied and if the strict inequality holds for at least one* $n \geqq 1$ *in* (2.12), *then* $u(t)$ *satisfies* (2.20a). *If* $\alpha_0 \eta_0 > 0$, *then* $T(x,t)$ *satisfies* (2.20b). *If* $\alpha_0 \eta_0 = 0$ *and if* (2.14, $m = 0$) *is satisfied, then there exist constants* $K > 0$ *and* $\omega > 0$ *such that*

$$(2.22)\qquad \sup_{0 \leqq x \leqq c} |T(x,t) - c^{-1/2} f_0| \leqq K e^{-\omega t}, \qquad 0 \leqq t < \infty.$$

If (2.10), (2.18) *are satisfied, if* $\alpha_0 \eta_0 = 0$ *and* $\alpha_0 = 0$, *but* $\eta_0 \neq 0$, *then* $u(t)$ *again satisfies* (2.20a). *Moreover, there exist constants* $K > 0$ *and* $\omega > 0$ *such that*

$$(2.23)\qquad \sup_{0 \leqq x \leqq c} \left| T(x,t) - c^{-1/2}\left(f_0 + \eta_0 \int_0^t \sigma(u(s))\,ds\right)\right| \leqq K e^{-\omega t}, \quad 0 \leqq t < \infty,$$

where the integral in (2.23) *exists for* $0 \leqq t \leqq \infty$. (It may be noted that if also $\eta_0 = 0$, (2.23) reduces to (2.22).)

We remark that hypothesis (2.11) is not the most general one under which our results can be established. For example, if α, η, f satisfy the hypothesis made in Bronikowski [1] and if one assumes in addition that (2.15) is satisfied, the conclusions of Theorems 1 and 2 hold, interpreting (1.2), (1.3) in a limiting sense.

If the boundary conditions (1.3) are replaced by

$$(\widetilde{1.3})\qquad \begin{aligned} m_{11}T(0,t) + m_{12}T_x(0,t) + n_{11}T(c,t) + n_{12}T_x(c,t) &= 0, \\ m_{12}T(0,t) + m_{22}T_x(0,t) + n_{12}T(c,t) + n_{22}T_x(c,t) &= 0, \end{aligned}$$

where m_{ij}, n_{ij} satisfy the conditions stated following (2.8), the results and proofs of Theorems 1 and 2 are unchanged if either $\lambda = 0$ is not an eigenvalue or $\lambda = 0$ is a simple eigenvalue of the eigenvalue problem (2.1), (2.8). If $\lambda = 0$ is a double eigenvalue of (2.1), (2.8), $u(t)$ satisfies (2.20a); however, the behavior of $T(x, t)$ can be more complicated than (2.20b) or (2.22), (2.23) (see, for example, [7, Theorem 4] where the constant coefficient case of (1.1b) together with delayed neutrons is discussed).

If $\lambda = 0$ is an eigenvalue of (2.1), (2.2), and if $\alpha_0\eta_0 = 0$, but $\alpha_0 \neq 0$, then Theorem 2 does not apply; neither Lemma 3 below nor the Corollary to Lemma 3 below hold. In fact, this case can lead to instability as shown by Bronikowski [1] in the linear case of (1.1) and by Miller [9] in the constant coefficient case of (1.16). Similar results hold here.

3. Related systems of ordinary differential equations. The Galerkin method applied to (1.1), (1.2), (1.3) proceeds as follows. Let $u(t)$, $T(x, t)$ be a solution for $0 < x < c, 0 < t < \infty$, and suppose that

$$T(x, t) = \sum_{n=0}^{\infty} T_n(t)y_n(x), \quad T_n(0) = f_n,$$

where y_n are the eigenfunctions of (2.1), (2.2). Define f_n, α_n, η_n as the Fourier coefficients of f, α, η with respect to the orthonormal system $\{y_n\}$ respectively. Then formally $T(x, t)$ satisfies (1.2b) and (1.3). Substituting in (1.1) and using (2.1) we obtain formally

$$u'(t) = -\sum_{n=0}^{\infty} \alpha_n T_n(t),$$

$$\begin{aligned}\sum_{n=0}^{\infty} T_n'(t)y_n(x) &= \sum_{n=0}^{\infty} (b(x)T_n(t)y_n'(x))_x - q(x)\sum_{n=0}^{\infty} T_n(t)y_n(x) + \eta(x)\sigma(u(t)) \\ &= \sum_{n=0}^{\infty} -L(y_n(x))T_n(t) + \eta(x)\sigma(u(t)) \\ &= -\sum_{n=0}^{\infty} \lambda_n y_n(x)T_n(t) + \eta(x)\sigma(u(t)).\end{aligned}$$

Multiplying the last relation by y_m and integrating from 0 to c yields the infinite system of ordinary differential equations

$$\begin{aligned}u'(t) &= -\sum_{n=0}^{\infty} \alpha_n T_n(t), \\ T_n'(t) &= -\lambda_n T_n(t) + \eta_n\sigma(u(t)), \\ &\qquad 0 < t < \infty, \quad n = 0, 1, \cdots,\end{aligned} \tag{3.1}$$

where $u(0) = u_0$, $T_n(0) = f_n$, $n = 0, 1, \cdots$. It should be noted that if (2.9) is satisfied, then all the $\lambda_n > 0$ in (3.1), while if (2.10) is satisfied, then $\lambda_0 = 0, \lambda_n > 0, n \geqq 1$, and the analysis will be more complicated (see proofs of Lemma 3 and Theorem 2 below). Truncating the system (3.1) we obtain the finite system of ordinary differential equations of the form

$$
\begin{aligned}
x'(t) &= -\sum_{n=0}^{N} \alpha_n z_n(t), \\
z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), \qquad 0 \leqq n \leqq N,
\end{aligned}
\tag{3.2}
$$

with initial conditions $x(0) = u_0, z_n(0) = f_n, n = 0, 1, \cdots, N$.

It is the purpose of this section to show that under various hypotheses (namely those of Theorems 1 and 2) the solutions of (3.2) exist on $0 \leqq t < \infty$, are bounded, and tend to zero exponentially (except for the one case when 0 is an eigenvalue; see Lemma 3 below) and that the estimates involved are independent of N. While such systems were considered by Hall [3], it is not quite possible to apply his results directly to (3.2), although we use entirely his method to derive the results of Lemmas 1, 2, 3 below. The first two of these deal with the case when the smallest eigenvalue λ_0 of (2.1), (2.2) is positive while Lemma 3 is concerned with $\lambda_0 = 0$. Lemma 1 is the most restrictive, but simplest from the point of view of exposition and contains the essential ideas.

LEMMA 1. *Let* $0 < \lambda_0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots$ (actually under hypothesis of Theorem A and (2.9) strict inequalities hold between the λ_i). *Let* α_n, η_n *satisfy* (2.12), (2.13), (2.14). *Define*

$$
c_n = \begin{cases} \alpha_n/\eta_n & \text{if} \quad \alpha_n\eta_n > 0, \\ 1 & \text{if} \quad \alpha_n\eta_n = 0. \end{cases}
\tag{3.3}
$$

Let $\alpha_n = 0$ *if and only if* $\eta_n = 0$. *Let* $\sigma(x)$ *satisfy* (2.16). *Then for any* $u_0, f_n, n = 0, 1, \cdots, N$, *there exists a unique solution* $x(t), z_n(t), n = 0, 1, \cdots, N$, *of* (3.2) *satisfying the initial conditions* $x(0) = u_0, z_n(0) = f_n$, *existing on* $0 \leqq t < \infty$ *and* $x(t), z_n(t)$ *are bounded. If* $\{f_n\}_{n=0}^{\infty} \in l_2$ (ensured by (2.11)) *there exists a constant* $\Omega > 0$, *independent of N, such that*

$$
|x(t)| \leqq \Omega, \quad |z_n(t)| \leqq \Omega, \qquad 0 \leqq n \leqq N, \quad 0 \leqq t < \infty.
\tag{3.4}
$$

If in addition $\sigma(x)$ *satisfies* (2.17), *if* $A = \sum_{n=0}^{\infty} |\alpha_n| < \infty$ (see (2.15)), *if the strict inequality holds in* (2.12) *for at least one n, and if* (2.16) *is satisfied, then there exist constants* $\Omega_0 > 0, \omega > 0$, *independent of N, such that*

$$
|x(t)| \leqq \Omega_0 \exp(-\omega t), \quad |z_n(t)| \leqq \Omega_0 \exp(-\omega t), \qquad 0 \leqq t < \infty, \quad 0 \leqq n \leqq N.
\tag{3.5}
$$

Proof. Define the Lyapunov function

$$
W(x, z_0, \cdots, z_N) = S(x) + \frac{1}{2} \sum_{n=0}^{N} c_n z_n^2.
$$

Clearly W is positive definite and its derivative with respect to the system (3.2) (using (3.3)) is

$$
W'(x, z_0, \cdots, z_N) = -\sum_{n=0}^{N} c_n \lambda_n z_n^2 \leqq 0.
$$

Without loss of generality we may assume $0 < c \leqq 1$ and $\tilde{c} \geqq 1$ in (2.13). By a standard argument one now deduces the existence, uniqueness, and boundedness of the solution $x(t), z_n(t), n = 0, 1, \cdots, N$, of the system (3.2) on $0 \leqq t < \infty$; however, the a priori bound of the solution will depend on $W(u_0, f_0, f_1, \cdots, f_N)$ which depends on N. But letting

$$W_0 = \max\{S(u_0), S(-u_0)\} + \frac{\tilde{c}}{2}\sum_{n=0}^{\infty} f_n^2,$$

which is independent of N, we observe that

$$0 \leqq W(x(t), z_0(t), \cdots, z_N(t)) \leqq W_0, \qquad 0 \leqq t < \infty.$$

Thus defining

$$\Omega = \max[\sqrt{2W_0/\tilde{c}}, \text{l.u.b.}\{|x| : S(x) \leqq W_0\}], \tag{3.6}$$

which exists, since $S(x) \to \infty$ as $|x| \to \infty$, yields the estimate (3.4).

To prove (3.5) we may, without loss of generality, assume $\eta_0 > 0$. For, suppose $\alpha_n \eta_n = 0$ for $0 \leqq n \leqq k, \eta_{k+1} \neq 0$, so that necessarily $N \geqq k + 1$. Then because of the assumption $\alpha_n = 0$ if and only if $\eta_n = 0$, the equations for $z_0, \cdots, z_k$ uncouple from the system (3.2) and $z_0(t), \cdots, z_k(t)$ decay exponentially. The proof given below then applies to the coupled portion of the system with subscript $k + 1$ in place of 0. Further if $\eta_0 < 0$, then $\alpha_0 < 0$ and replacement of z_0 by $-z_0$ in (3.2) yields a system equivalent to (3.2) with $-\alpha_0$ in place of α_0 and $-\eta_0$ in place of η_0.

Define

$$V(x, z_0, z_1, \cdots, z_N) = W(x, z_0, \cdots, z_N) - \beta\sigma(x)z_0,$$

where $\beta > 0$ is a constant to be chosen conveniently below. It follows easily from (2.17) and l'Hospital's rule applied to $\sigma^2(x)/S(x)$ as $x \to 0$ that there exist constants $\varphi = \varphi(\Omega) > 0$ and $\Phi = \Phi(\Omega) > 0$ such that

$$\varphi \leqq \frac{\sigma^2(x)}{S(x)} \leqq \Phi, \qquad |x| \leqq \Omega. \tag{3.7}$$

Using the right-hand inequality of (3.7) it follows that if $|x|, |z_n| \leqq \Omega$ for $0 \leqq n \leqq N$, then

$$\begin{aligned} S(x)\left(1 - \frac{\beta\Phi}{2}\right) &+ \frac{c_0 z_0^2}{2}\left(1 - \frac{\beta}{c_0}\right) + \sum_{n=1}^{N} \frac{c_n z_n^2}{2} \\ &\leqq V(x, z_0, \cdots, z_N) \\ &\leqq S(x)\left(1 + \frac{\beta\Phi}{2}\right) + \frac{c_0 z_0^2}{2}\left(1 + \frac{\beta}{c_0}\right) + \sum_{n=1}^{N} \frac{c_n z_n^2}{2}. \end{aligned}$$

Thus if $\beta < \min(1/\Phi, c_0/2)$ and $|x|, |z_n| \leqq \Omega$ for $0 \leqq n \leqq N$,

$$\begin{aligned} \tfrac{1}{2}W(x, z_0, \cdots, z_N) &\leqq V(x, z_0, \cdots, z_N) \\ &\leqq \tfrac{3}{2}W(x, z_0, \cdots, z_N), \end{aligned} \tag{3.8}$$

and clearly V is positive definite for $|x|, |z_n| \leqq \Omega, 0 \leqq n \leqq N$. Forming the derivative of V relative to the system (3.2) we obtain

$$-V'(x, z_0, \cdots, z_N) = \sum_{n=0}^{N} c_n\lambda_n z_n^2 - \beta\sigma'(x)\sum_{n=0}^{N} \alpha_n z_0 z_n - \beta\lambda_0\sigma(x)z_0 + \beta\eta_0\sigma^2(x).$$

By continuity there exists a constant $D > 0$ such that $|\sigma'(x)| \leqq D$ for $|x| \leqq \Omega$. This together with the inequality $|uv| \leqq \frac{1}{2}\gamma(u^2 + v^2/\gamma^2)$ for any $\gamma > 0$ gives

$$\begin{aligned}-V'(x, z_0, \cdots, z_N) &\geqq \sum_{n=0}^{N} c_n\lambda_n z_n^2 - \frac{\beta D}{2}\sum_{n=0}^{N} |\alpha_n|(z_0^2 + z_n^2) \\ &\quad - \frac{\beta\eta_0}{2}\left(\sigma^2 + \frac{\lambda_0^2 z_0^2}{\eta_0^2}\right) + \beta\eta_0\sigma^2(x) \\ &\geqq \frac{\beta\eta_0}{2}\sigma^2(x) + z_0^2\left[\lambda_0 c_0 - \beta\frac{\lambda_0^2 + 2\eta_0 AD}{2\eta_0}\right] \\ &\quad + \sum_{n=1}^{N} z_n^2\left(\lambda_n c_n - \frac{\beta AD}{2}\right).\end{aligned}$$

Letting the coefficients of z_n^2 be denoted by B_n and $\hat{B} = \lambda_0\hat{c} - \beta AD/2$, we have $B_n \geqq \hat{B}$, where we have used the monotone property of the $\{\lambda_n\}$. Thus

$$-V'(x, z_0, \cdots, z_N) \geqq \frac{\beta\eta_0}{2}\sigma^2(x) + B_0 z_0^2 + \hat{B}\sum_{n=1}^{N} z_n^2.$$

We now choose β so small that both (3.8) and $B_0 > 0, \hat{B} > 0$ are satisfied. This will be the case if

$$0 < \beta < \min\left(\frac{1}{\Phi}, \frac{c_0}{2}, \frac{2\eta_0 c_0\lambda_0}{\lambda_0^2 + 2\eta_0 AD}, \frac{2\hat{c}\lambda_0}{AD}\right).$$

Let $\omega_0 = \min(\beta\eta_0/2, B_0, \hat{B})$. Using the left-hand side of (3.7), we obtain

$$-V'(x, z_0, \cdots, z_N) \geqq \omega_0\left(\sigma^2 + \sum_{n=0}^{N} z_n^2\right) \geqq \omega_0\left(\varphi S + \sum_{n=0}^{N} z_n^2\right).$$

Let $\omega = \min(2\omega_0/3\tilde{c}, \omega_0\varphi/3)$ and apply (3.8), obtaining

$$-V'(x, z_0, \cdots, z_N) \geqq 2\omega V(x, z_0, \cdots, z_N);$$

it may be noted that ω is independent of N. Thus

$$V(x(t), z_0(t), \cdots, z_N(t)) \leqq V(u_0, f_0, \cdots, f_N)\, e^{-2\omega t}, \qquad 0 \leqq t < \infty. \tag{3.9}$$

From (2.17) and l'Hospital's rule as $x \to 0$ applied to $0 < S(x)/x^2, x \neq 0$, we see that there exists a constant $q > 0$ such that $qx^2 \leqq S(x)$ for $|x| \leqq \Omega$. From (3.8) one has first

$$V(u_0, f_0, \cdots, f_N) \leqq \tfrac{3}{2}W(u_0, f_0, \cdots, f_N) \leqq \tfrac{3}{2}W_0.$$

Applying (3.8) once more to the left-hand side of (3.9) one obtains

$$\tfrac{1}{2}W(x(t), z_0(t), \cdots, z_N(t)) \leqq V(u_0, f_0, \cdots, f_N)\, e^{-2\omega t}, \quad 0 \leqq t < \infty,$$

and hence

$$x^2(t) \leqq \frac{2V(u_0, f_0, \cdots, f_N)}{q} e^{-2\omega t} \leqq \frac{3W_0}{q} e^{-2\omega t}, \qquad 0 \leqq t < \infty,$$

$$z_n^2(t) \leqq \frac{4V(u_0, f_0, \cdots, f_N)}{\hat{c}} e^{-2\omega t} \leqq \frac{6W_0}{\hat{c}} e^{-2\omega t}, \qquad 0 \leqq t < \infty.$$

Finally, by defining

$$\Omega_0 = \max\left(\sqrt{\frac{3W_0}{q}}, \sqrt{\frac{6W_0}{\hat{c}}}\right),$$

which is independent of N, we obtain the inequalities (3.5) and complete the proof of Lemma 1.

The condition that $\alpha_n = 0$ if and only if $\eta_n = 0$, which plays an important role in establishing the a priori estimates in Lemma 1, is restrictive. It will be removed at the expense of an additional hypothesis on $\sigma(x)$.

LEMMA 2. *Let the hypotheses of Lemma* 1 *be satisfied, except for the requirement* $\alpha_n = 0$ *if and only if* $\eta_n = 0$. *Let* $\sigma(x)$ *satisfy* (2.18). *Then the conclusions of Lemma* 1 *hold.*

Proof. Since α_n, η_n satisfy (2.12) we may assume without loss of generality that $\alpha_n \geqq 0, \eta_n \geqq 0$ as in Lemma 1. The sequences $\{\alpha_n\}, \{\eta_n\}$ induce a partitioning of the positive integers $\mathscr{N}$ (including 0) into three disjoint classes $\mathscr{A}, \mathscr{B}, \mathscr{I}$ defined as follows:

$$\mathscr{A} = \{n \in \mathscr{N} : \alpha_n > 0, \eta_n = 0\},$$

$$\mathscr{B} = \{n \in \mathscr{N} : \alpha_n = 0, \eta_n > 0\},$$

$$\mathscr{I} = \{n \in \mathscr{N} : \alpha_n \eta_n > 0 \text{ or } \alpha_n = \eta_n = 0\}.$$

For each $N \in \mathscr{N}$ define $\mathscr{A}_N = \mathscr{A} \cap \{0, 1, \cdots, N\}$ and similarly $\mathscr{B}_N, \mathscr{I}_N$. Then the system (3.2) may be written in the form

$$\begin{aligned} x'(t) &= -\sum_{n \in \mathscr{I}_N} \alpha_n z_n(t) - \sum_{n \in \mathscr{A}_N} \alpha_n z_n(t), \\ z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), && n \in \mathscr{I}_N, \\ z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), && n \in \mathscr{B}_N, \\ z_n'(t) &= -\lambda_n z_n(t), && n \in \mathscr{A}_N, \end{aligned} \tag{3.10}$$

subject to the initial conditions $x(0) = u_0, z_n(0) = f_n, n = 0, 1, \cdots, N$. Integrating the last set of equations in (3.10) and substituting in the first equation yields the equivalent system:

$$\begin{aligned} x'(t) &= -\sum_{n \in \mathscr{I}_N} \alpha_n z_n(t) + e_N(t), \\ z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), && n \in \mathscr{I}_N, \\ z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), && n \in \mathscr{B}_N, \end{aligned} \tag{3.11}$$

where $x(0) = u_0$, $z_n(0) = f_n$, $n = 0, 1, \cdots, N$, and where

$$e_N(t) = -\sum_{n \in \mathscr{A}_N} \alpha_n f_n e^{-\lambda_n t}.$$

Since the classes $\mathscr{I}_N$ and $\mathscr{B}_N$ are disjoint the last system in (3.11) is independent of the first two lines. We may therefore restrict our attention to the system

$$\begin{aligned} x'(t) &= -\sum_{n \in \mathscr{I}_N} \alpha_n z_n(t) + e_N(t), \\ z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), \qquad n \in \mathscr{I}_N. \end{aligned} \tag{3.12}$$

The behavior of the components of $z_n(t), n \in \mathscr{B}_N$, is easily deduced from the last line in (3.11) once the appropriate estimates have been obtained for the solution of (3.12). Note that the essential difference between the systems (3.2) and (3.12) is the addition of the forcing term $e_N(t)$ in the latter. We remark also that if for $n \in \mathscr{I}_N$, $\alpha_n = \eta_n = 0$, then the corresponding equations in (3.12b) uncouple and the corresponding $z_n(t)$ decay exponentially. For this reason we may as well assume $\alpha_n \eta_n > 0$.

Since f, α satisfy (2.11) we may assume that the infinite sequences $\{f_n\}$, $\{\alpha_n\} \in l_2$. Thus there exists a constant $K > 0$, independent of N, such that

$$|e_N(t)| \leqq K e^{-\lambda_0 t}, \qquad 0 \leqq t < \infty. \tag{3.13}$$

Note that if $\mathscr{I}_N$ is empty (which it cannot be in the second part of the lemma provided N is sufficiently large) one has trivially the existence, uniqueness and boundedness (independent of N) of solutions of (3.12) by inspection.

Define the Lyapunov function

$$W(x, z) = S(x) + \frac{1}{2} \sum_{n \in \mathscr{I}_N} c_n z_n^2,$$

where z is a vector whose components are z_n, $n \in \mathscr{I}_N$. Relative to the system (3.12) the derivative of W is

$$W'(x, z) = -\sum_{n \in \mathscr{I}_N} \lambda_n c_n z_n^2 + \sigma(x) e_N(t) \leqq \sigma(x) e_N(t).$$

Thus using (2.18) and (3.13) one has easily

$$\begin{aligned} W'(x, z) &\leqq \tfrac{1}{2} |e_N(t)| (1 + \sigma^2(x)) \\ &\leqq \tfrac{1}{2} |e_N(t)| \left(1 + \Gamma S(x) + \frac{\Gamma}{2} \sum_{n \in \mathscr{I}_N} c_n z_n^2\right) \\ &\leqq \tfrac{1}{2} |e_N(t)| + \frac{\Gamma}{2} |e_N(t)| W(x, z) \\ &\leqq \frac{K}{2} e^{-\lambda_0 t} + \frac{\Gamma K}{2} e^{-\lambda_0 t} W(x, z). \end{aligned}$$

Letting $x(t), z_n(t), n \in \mathscr{I}_N$, be a solution of (3.12) one obtains by an elementary argument, on integrating the above differential inequality, the a priori estimate (independent of N)

$$W(x(t), z(t)) \leqq \left(\frac{1}{\Gamma} + W_0\right) \exp\left(\frac{K\Gamma}{2\lambda_0}\right),$$

where W_0 is defined in Lemma 1. Thus defining Ω by (3.6) one obtains the boundedness of solutions of (3.12) and the estimate (3.4) as in Lemma 1.

Now consider the solution $x(t), z_n(t), n \in \mathscr{I}_N$, of (3.12) with $|x(t)| \leqq \Omega$, $|z_n(t)| \leqq \Omega, n \in \mathscr{I}_N, 0 \leqq t < \infty$. Take N sufficiently large that $\mathscr{I}_N$ is not empty (see 2.12). Define, for some constant $\beta > 0$,

$$V(x, z) = W(x, z) - \beta\sigma(x)z_0.$$

If $0 \notin \mathscr{I}_N$ replace z_0 in V by z_k, where k is the smallest integer in $\mathscr{I}_N$. Then by the argument of Lemma 1 one obtains the estimate (3.8) for those z_n having indices $n \in \mathscr{I}_N$, where β, independent of N, has the same meaning as in Lemma 1. Thus $V(x, z)$ is positive definite for $|x| \leqq \Omega$, $|z_n| \leqq \Omega$, $n \in \mathscr{I}_N$. We now compute the derivative of V relative to the system (3.12), and we obtain in much the same manner as in the proof of Lemma 1 the estimate

$$V'(x, z) \leqq -\tau V(x, z) + (\sigma(x) - \beta\sigma'(x)z_0)e_N$$

for $|x| \leqq \Omega, |z_n| \leqq \Omega, n \in \mathscr{I}_N$, where the constant $\tau > 0$ is independent of N and is chosen as follows. Let ω_0 have the same meaning as in Lemma 1. Then define $\tau > 0$ to be any number for which $\tau < \min(2\omega_0\varphi/3, 4\omega_0/3\tilde{c})$ and $\tau \neq \lambda_0$. Since σ, σ' are continuous on $[-\Omega, \Omega]$, there exists a constant $K_0 > 0$ such that

$$V'(x, z) \leqq -\tau V(x, z) + K_0|e_N(t)|, \qquad |x|, |z| \leqq \Omega, \quad n \in \mathscr{I}_N.$$

Thus for the solution $x(t), z_n(t), n \in \mathscr{I}_N$, of (3.12) one has, using (3.13), the differential inequality

$$V'(x(t), z(t)) \leqq -\tau V(x(t), z(t)) + K_0 K \exp(-\lambda_0 t), \qquad 0 \leqq t < \infty. \tag{3.14}$$

Integrating one obtains

$$\begin{aligned} V(x(t), z(t)) \leqq {} & V(u_0, f)\exp(-\tau t) \\ & + \frac{KK_0}{\tau - \lambda_0}(\exp(-\lambda_0 t) - \exp(-\tau t)), \qquad 0 \leqq t < \infty, \end{aligned} \tag{3.15}$$

where f is the vector consisting of those components of $f_0, \cdots, f_N$ for which $n \in \mathscr{I}_N$. The inequality (3.15) now easily yields estimates of the form (3.5) for those components of z_n with $n \in \mathscr{I}_N$. For those components of z_n with $n \in \mathscr{A}_N$ and $n \in \mathscr{B}_N$ one obtains the exponential decay of $|z_n(t)|$ as pointed out at the beginning of the proof. This completes the proof of Lemma 2.

We now consider the case when $\lambda_0 = 0$ while the other eigenvalues $\lambda_n > 0$. The system (3.2) takes the form

$$\begin{aligned} x'(t) &= -\alpha_0 y(t) - \sum_{n=1}^{N} \alpha_n z_n(t), \\ y'(t) &= \eta_0 \sigma(x(t)), \\ z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), \qquad 1 \leqq n \leqq N, \end{aligned} \tag{3.16}$$

with initial conditions $x(0) = u_0, y(0) = f_0, z_n(0) = f_n, n = 1, \cdots, N$, where we have relabeled z_0 by y for emphasis.

LEMMA 3. *Let the hypotheses of Lemma* 1 *or* 2 *be satisfied. Then for any* $u_0, f_0, \cdots, f_N$ *the system* (3.16) *has a unique solution* $x(t), y(t), z_n(t), n = 1, \cdots, N$, $0 \leqq t < \infty$, *satisfying the given initial conditions. There exists a constant* Ω *independent of* N *such that*

$$|x(t)| \leqq \Omega, \quad |y(t)| \leqq \Omega, \quad |z_n(t)| \leqq \Omega, \qquad n = 1, \cdots, N, \quad 0 \leqq t < \infty. \tag{3.17}$$

If the strict inequality in (2.12) *holds for at least one* $n \geqq 1$ (in place of $n \geqq 0$) *and if* $\alpha_0\eta_0 > 0$, *then there exist constants* $\Omega_0 > 0$ *and* $\omega > 0$ *independent of* N *such that*

$$|x(t)| \leqq \Omega_0 \exp(-\omega t), \quad |y(t)| \leqq \Omega_0 \exp(-\omega t), \quad |z_n(t)| \leqq \Omega_0 \exp(-\omega t) \tag{3.18}$$

for $0 \leqq t < \infty$ *and* $n = 1, \cdots, N$. *If* $\alpha_0\eta_0 = 0$ *and if* (2.14) *is satisfied for* $m = 0$, *then* $y(t) \equiv f_0$ *and* $x(t), z_n(t)$ *satisfy* (3.18).

Comments on the proof of Lemma 3. We remark on the proof in the case of the situation analogous to Lemma 2. Introducing classes $\mathscr{A}_N, \mathscr{B}_N, \mathscr{I}_N$ exactly as in Lemma 2, one is led to the system

$$\begin{aligned} x'(t) &= -\alpha_0 y(t) - \sum_{n=1}^{N} \alpha_n z_n(t) + e_N(t), \\ y'(t) &= \eta_0 \sigma(x(t)), \\ z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), \qquad n \in \mathscr{I}_N, \end{aligned} \tag{3.19}$$

where e_N satisfies the estimate

$$|e_N(t)| \leqq K\, e^{-\lambda_1 t}, \qquad 0 \leqq t < \infty,$$

for some constant $K > 0$. One now defines the Lyapunov function

$$W(x, y, z) = S(x) + \frac{c_0 y^2}{2} + \frac{1}{2} \sum_{n \in \mathscr{I}_N} c_n z_n^2,$$

which is used to establish the bounds (3.17). To establish the exponential decay one uses the Lyapunov function

$$V(x, y, z) = W(x, y, z) - \beta\sigma(x)y - \rho\beta\sigma(x)z_1,$$

where it is assumed without loss of generality that $1 \in \mathscr{I}_N$ and where $\beta > 0, \rho > 0$ are constants. To establish the differential inequality of the form (3.14) one proceeds as in the proof of Lemma 2; here one needs to make use of the fact that hypothesis (2.17) together with $|x| \leqq \Omega$ implies the existence of a constant $d > 0$ such that $\sigma'(x) \geqq d$, $-\Omega \leqq x \leqq \Omega$. This is needed because $\sigma'(x)$ occurs as the coefficient of y^2 on the right-hand side of the expression for $V'(x, y, z)$. The exponential bounds (3.18) are then established as in Lemma 2.

If $\alpha_0\eta_0 = 0$ and if (2.14, $m = 0$) is satisfied, then the y dependence is uncoupled from the systems (3.16) and (3.19). In this case $y(t) \equiv f_0$; that $x(t), z_n(t)$ satisfy the exponential decay bounds in (3.18) may then be deduced as in Lemma 2 applied to the system (3.19) with the second equations omitted. This completes the proof of Lemma 3.

If $\alpha_0\eta_0 = 0$ and if $\alpha_0 = 0$, but $\eta_0 \neq 0$, i.e., (2.14, $m = 0$) is not satisfied, the system (3.19) becomes

$$\begin{aligned} x'(t) &= -\sum_{n=1}^{N} \alpha_n z_n(t) + e_N(t), \\ y'(t) &= \eta_0 \sigma(x(t)), \\ z_n'(t) &= -\lambda_n z_n(t) + \eta_n \sigma(x(t)), \qquad n \in \mathscr{I}_N, \end{aligned} \tag{3.20}$$

where $e_N(t)$ is as above. It is evident that the middle equation in (3.20) is uncoupled from the rest of the system. Applying Lemma 2 to the system (3.20) with the y equation omitted, we obtain readily the following corollary.

COROLLARY TO LEMMA 3. *Let* $\alpha_0\eta_0 = 0$ *and let* $\alpha_0 = 0$, *but* $\eta_0 \neq 0$ (i.e., (2.14, $m = 0$) is not satisfied). *If the other assumptions of Lemma* 3 *are satisfied, then there exist constants* $\Omega_0 > 0, \omega > 0$ *independent of N such that*

$$|x(t)| \leqq \Omega_0 e^{-\omega t}, \quad |z_n(t)| \leqq \Omega_0 e^{-\omega t}, \quad y(t) \equiv f_0 + \eta_0 \int_0^t \sigma(x(s))\, ds,$$
$$0 \leqq t < \infty, \quad n = 1, 2, \cdots, N,$$

where the integral exists for $0 \leqq t \leqq \infty$. (Note that if also $\eta_0 = 0$, (2.14, $m = 0$) is satisfied and the corollary reduces to the last assertion of Lemma 3.)

That the integral exists for all $t \geqq 0$ follows from (2.16) and the exponential decay of $x(t)$.

4. A related Volterra equation. We consider first the Volterra equation

$$u(t) = u_0 + K_1(t) + \int_0^t K_2(t - s)\sigma(u(s))\, ds, \tag{4.1}$$

where

$$\begin{aligned} K_1(t) &= \sum_{n=0}^{\infty} \frac{\alpha_n f_n}{\lambda_n}(e^{-\lambda_n t} - 1), \\ K_2(t) &= \sum_{n=0}^{\infty} \frac{\alpha_n \eta_n}{\lambda_n}(e^{-\lambda_n t} - 1), \qquad 0 \leqq t < \infty. \end{aligned} \tag{4.2}$$

In (4.2) it is assumed that all eigenvalues λ_n of the boundary value problem (2.1), (2.2) are strictly positive; this is the case if (2.9) is satisfied. The Volterra equation (4.1) is obtained formally as follows. Solve the second set of equations in (3.1) for the T_n, $n = 0, 1, \cdots$, substitute in the first equation and integrate, again using the initial condition $u(0) = u_0$. It is clear from (2.6), (2.11) that $K_1(t), K_2(t) \in C'[0, \infty)$ and that

$$\begin{aligned} K_1'(t) &= -\sum_{n=0}^{\infty} \alpha_n f_n e^{-\lambda_n t}, \\ K_2'(t) &= -\sum_{n=0}^{\infty} \alpha_n \eta_n e^{-\lambda_n t}, \qquad 0 \leqq t < \infty, \end{aligned} \tag{4.3}$$

in view of the uniform convergence of the series involved.

If the smallest eigenvalue of the boundary value problem (2.1), (2.2) is zero (which is the case if (2.10) rather than (2.9) is satisfied), we consider the Volterra equation (4.1) with, however,

$$(4.4)\qquad \begin{aligned} K_1(t) &= -\alpha_0 f_0 t + \sum_{n=1}^{\infty} \frac{\alpha_n f_n}{\lambda_n}(e^{-\lambda_n t} - 1), \\ K_2(t) &= -\alpha_0 \eta_0 t + \sum_{n=1}^{\infty} \frac{\alpha_n \eta_n}{\lambda_n}(e^{-\lambda_n t} - 1), \end{aligned}$$

for which

$$(4.5)\qquad \begin{aligned} K_1'(t) &= -\alpha_0 f_0 - \sum_{n=1}^{\infty} \alpha_n f_n e^{-\lambda_n t}, \\ K_2'(t) &= -\alpha_0 \eta_0 - \sum_{n=1}^{\infty} \alpha_n \eta_n e^{-\lambda_n t}, \qquad 0 \leqq t < \infty. \end{aligned}$$

We shall establish two results concerning the behavior of solutions of (4.1), first for when (4.2) holds, and then (4.4).

LEMMA 4. *Let* (2.9), (2.11), (2.12), (2.13), (2.16) *be satisfied. Suppose that either* (2.14) *holds for all* $m \geqq 0$ *or that* (2.18) *is satisfied.*

(i) *Then there exist a unique solution* $u(t)$ *of* (4.1), (4.2) *on* $0 \leqq t < \infty$ *and a constant* $K > 0$ *such that* (2.19a) *is satisfied. Moreover,* $u(t) \in C'[0, \infty)$ *and*

$$(4.6)\qquad u'(t) = K_1'(t) + \int_0^t K_2'(t-s)\sigma(u(s))\,ds, \qquad 0 \leqq t < \infty,$$

where K_1', K_2' *are given by* (4.3).

(ii) *If in addition* (2.17) *is satisfied and if the strict inequality in* (2.12) *holds for at least one n, then* $u(t)$ *tends exponentially to zero in the sense of* (2.20a).

Proof. Evidently (4.1) has a unique solution $u(t)$ on $[0, t_0)$ for some $t_0 > 0$. In accordance with a well-known continuation result (see, for example, [10]) this solution $u(t)$ can be extended to the interval $[0, \infty)$ if it satisfies an a priori bound independent of t_0. To establish this property let $0 < t_1 < t_0$, $P = P(t_1) = \sup_{0 \leqq t \leqq t_1} |u(t)|$ and let $M = M(t_1) = \sup_{0 \leqq t \leqq t_1} |\sigma(u(t))|$.

Consider the family of systems for $N = 0, 1, \cdots,$

$$(4.7)\qquad \begin{aligned} x_N' &= -\sum_{n=0}^{N} \alpha_n z_{Nn}, \\ z_{Nn}' &= -\lambda_n z_{Nn} + \eta_n \sigma(x_N), \qquad n = 0, 1, 2, \cdots, N, \end{aligned}$$

where the initial conditions $x_N(0) = u_0$, $z_{Nn}(0) = f_n$, $0 \leqq n \leqq N$, are specified. The system (4.7) of ordinary differential equations is, for each fixed N, of the form discussed in Lemmas 1 and 2 in § 3. The hypotheses of Lemmas 1 or 2 are satisfied and thus by (3.4) there exists a unique solution $x_N(t)$, $z_{Nn}(t)$, $0 \leqq n \leqq N$, of (4.7) and a constant Ω independent of N such that

$$(4.8)\qquad |x_N(t)| \leqq \Omega, \quad |z_{Nn}(t)| \leqq \Omega, \qquad 0 \leqq t < \infty, \quad 0 \leqq n \leqq N, \quad N = 0, 1, \cdots.$$

We next solve the last N equations in (4.7) for the quantities z_{Nn} by variation of constants and substitute in the first equation in (4.7). This yields

$$x_N(t) = u_0 + K_{1N}(t) + \int_0^t K_{2N}(t-s)\sigma(x_N(s))\,ds, \qquad 0 \leqq t < \infty, \tag{4.9}$$

where K_{1N}, K_{2N} are the functions defined by the Nth partial sums of the series in (4.2).

We first show that $x_N(t) \to u(t)$ uniformly on $0 \leqq t \leqq t_1$. From (4.1), (4.2), (4.9) and the mean value theorem one has

$$\begin{aligned} |u(t) - x_N(t)| \leqq{}& \sum_{n=N+1}^{\infty} \frac{|\alpha_n f_n|}{\lambda_n} + \sum_{n=N+1}^{\infty} \frac{\alpha_n \eta_n}{\lambda_n} \int_0^t |\sigma(u(s))|\,ds \\ &+ L \sum_{n=0}^{N} \frac{\alpha_n \eta_n}{\lambda_n} \int_0^t |u(s) - x_N(s)|\,ds, \qquad 0 \leqq t \leqq t_1, \end{aligned} \tag{$4.\tilde{9}$}$$

where $L = \sup_{y \in S} |\sigma'(y)|$ and $S = [-P, P] \cup [-\Omega, \Omega]$. Since $P = P(t_1)$, one also has $L = L(t_1)$. The interchange of summation and integration used in obtaining ($4.\tilde{9}$) is justified by the uniform convergence of the series involved. Let

$$Q(N) = \sum_{n=N+1}^{\infty} \left(\frac{|\alpha_n f_n|}{\lambda_n} + \frac{\alpha_n \eta_n}{\lambda_n} M t_1 \right) \quad \text{and} \quad B = L \sum_{n=0}^{\infty} \frac{\alpha_n \eta_n}{\lambda_n} < \infty .$$

Then ($4.\tilde{9}$) has the form

$$|u(t) - x_N(t)| \leqq Q(N) + B \int_0^t |u(s) - x_N(s)|\,ds, \qquad 0 \leqq t \leqq t_1. \tag{4.10}$$

By the Gronwall inequality one obtains

$$|u(t) - x_N(t)| \leqq Q(N)\, e^{Bt}, \qquad 0 \leqq t \leqq t_1.$$

Since $\lim_{N\to\infty} Q(N) = 0$, it follows that $\lim_{N\to\infty} x_N(t) = u(t)$, uniformly on $0 \leqq t \leqq t_1$. Therefore one also has from (4.8a), $|u(t)| \leqq 2\Omega$, $0 \leqq t \leqq t_1 < t_0$, where Ω is an a priori constant independent of N, t_1 and t_0. Thus $u(t)$ can be continued (uniquely) to the interval $0 \leqq t < \infty$ in such a way that (2.19a) is satisfied with $K = 2\Omega$. That $u(t) \in C'[0, \infty)$ and satisfies (4.6) now follows immediately by differentiation of (4.1) using the uniform convergence of the series in (4.3) on $[0, \infty)$. This completes the proof of (i).

To establish (ii) one notes that the hypotheses imply that Lemma 1 or 2 may be applied to the truncated system (4.7). Thus by (3.5) there exist constants $\Omega_0 > 0$ and $\omega > 0$ independent of N, such that

$$|x_N(t)| \leqq \Omega_0\, e^{-\omega t}, \qquad 0 \leqq t < \infty, \quad N = 0, 1, \cdots .$$

The uniform convergence of $x_N(t)$ to $u(t)$ as $N \to \infty$ on finite t intervals shows that $u(t)$ decays exponentially to zero and that (2.20a) for $i = 0$ is satisfied. To show that (2.20a) holds for $i = 1, 2$, one notes that the continuity of σ', (2.16) and (2.20a), $i = 0$, imply the existence of constants $K_0, \omega_0 > 0$ such that

$$|\sigma(u(t))| \leqq K_0\, e^{-\omega_0 t}, \qquad 0 \leqq t < \infty, \tag{4.11}$$

where, without loss of generality, we may choose $0 < \omega_0 < \lambda_0$. It therefore follows from (4.3), (4.6) and $0 < \lambda_0 < \lambda_1 < \cdots$ that

$$\begin{aligned} |u'(t)| &\leqq \sum_{n=0}^{\infty} |\alpha_n f_n| \, e^{-\lambda_0 t} + K_0 \left(\sum_{n=0}^{\infty} \alpha_n \eta_n \right) \int_0^t e^{-\lambda_0 (t-s)} \, e^{-\omega_0 s} \, ds \\ &\leqq \left(\sum_{n=0}^{\infty} |\alpha_n f_n| \right) e^{-\lambda_0 t} + \frac{K_0}{\lambda_0 - \omega_0} \left(\sum_{n=0}^{\infty} \alpha_n \eta_n \right) e^{-\omega_0 t} \\ &\leqq K \, e^{-\omega_0 t}, \qquad 0 \leqq t < \infty, \end{aligned}$$

which establishes (2.20a), $i = 1$, where

$$K = \max \left(\sum_{n=0}^{\infty} |\alpha_n f_n|, \frac{K_0}{\lambda_0 - \omega_0} \sum_{n=0}^{\infty} \alpha_n \eta_n \right).$$

Differentiating (4.6) one obtains

$$u''(t) = \sum_{n=0}^{\infty} \alpha_n f_n \lambda_n \, e^{-\lambda_n t} - \left(\sum_{n=0}^{\infty} \alpha_n \eta_n \right) \sigma(u(t)) + \int_0^t \sum_{n=0}^{\infty} \alpha_n \eta_n \lambda_n \, e^{-\lambda_n (t-s)} \sigma(u(s)) \, ds$$

for $0 < t < \infty$; the differentiation is justified because of the uniform convergence of the differentiated series for $0 < t_0 \leqq t < \infty$, for any $t_0 > 0$. Now let $0 < \delta < \lambda_0$, $t_0 > 0$, and $K_1 = \sum_{n=0}^{\infty} |\alpha_n f_n \lambda_n| \, e^{-(\lambda_n - \delta) t_0}$. Then for $t \geqq t_0$,

$$\begin{aligned} |u''(t)| &\leqq K_1 \, e^{-\delta t} + K_0 \, e^{-\omega_0 t} \sum_{n=0}^{\infty} \alpha_n \eta_n \\ &\quad + K_0 \sum_{n=0}^{\infty} \alpha_n \eta_n \lambda_n \int_0^t e^{-\lambda_n t} \, e^{(\lambda_n - \omega_0) s} \, ds \\ &\leqq K_1 \, e^{-\delta t} + K_0 \, e^{-\omega_0 t} \sum_{n=0}^{\infty} \alpha_n \eta_n + K_0 \, e^{-\omega_0 t} \sum_{n=0}^{\infty} \frac{\alpha_n \eta_n \lambda_n}{\lambda_n - \omega_0} \\ &\leqq K \, e^{-\omega_1 t}, \end{aligned}$$

where

$$\omega_1 = \min(\delta, \omega_0), \quad K = \max \left(K_1, K_0 \sum_{n=0}^{\infty} \alpha_n \eta_n, K_0 \sum_{n=0}^{\infty} \frac{\alpha_n \eta_n \lambda_n}{\lambda_n - \omega_0} \right).$$

This establishes (2.20a) for $i = 2$. The overall a priori constants K and ω in (2.20a) for $i = 0, 1, 2$ are defined in an obvious way from the estimates just obtained for $|u(t)|$, $|u'(t)|$ and $|u''(t)|$. This completes the proof of Lemma 4.

We now turn to the zero eigenvalue case.

LEMMA 5. (i) *Let* (2.10), (2.11), (2.12), (2.13), (2.16) *be satisfied. Suppose that either* (2.14) *holds for all* $m \geqq 0$ *or that* (2.18) *as well as* (2.14) *with only* $m = 0$ *are satisfied. Then conclusion* (i) *of Lemma* 4 *holds, where* K_1', K_2', *in* (4.6), *are given by* (4.5).

(ii) *If in addition* (2.17) *is satisfied and if the strict inequality in* (2.12) *holds for at least one* $n \geqq 1$, *then conclusion* (ii) *of Lemma* 4 *holds.*

Proof. The proof parallels that of Lemma 4 very closely.

(i) In place of the family of systems (4.7) we now consider the family

$$
(4.12)\qquad
\begin{aligned}
x_N' &= -\alpha_0 y_N - \sum_{n=1}^{N} \alpha_n z_{Nn},\\
y_N' &= \eta_0 \sigma(x_N), \qquad\qquad N = 0, 1, \cdots,\ n = 1, 2, \cdots, N,\\
z_{Nn}' &= -\lambda_n z_{Nn} + \eta_n \sigma(x_N),
\end{aligned}
$$

where the initial conditions $x_N(0) = u_0$, $y_N(0) = f_0$, $z_{Nn}(0) = f_n$, $n = 1, \cdots, N$, are specified. For each N, (4.12) has the form of system (3.16) studied in Lemma 3 of § 3. Using Lemma 3 (in place of Lemmas 1 and 2) and the argument of part (i), Lemma 4, one obtains the result of part (i), Lemma 5, where K_1, K_2, K_1', K_2' are now given by (4.4), (4.5) respectively and where K_{1N}, K_{2N} are now the Nth partial sums of the series in (4.4).

The proof of (ii) is exactly the same as the corresponding proof in Lemma 4 when establishing (2.20a) for $i = 0$, where the result (3.18) of Lemma 3 is applied to the truncated system (4.12), in place of applying Lemmas 1 and 2 in the proof of Lemma 4. To prove (2.20a) for $i = 1$ and 2 we note that from (4.6), (4.5) one has, by differentiation,

$$
(4.13)\qquad u'(t) = -\alpha_0 f_0 - \alpha_0 \eta_0 \int_0^t \sigma(u(s))\, ds + w(t), \qquad 0 \leqq t < \infty,
$$

where

$$
(4.14)\qquad w(t) = -\sum_{n=1}^{\infty} \alpha_n f_n e^{-\lambda_n t} - \sum_{n=1}^{\infty} \alpha_n \eta_n \int_0^t e^{-\lambda_n (t-s)} \sigma(u(s))\, ds,
$$

and

$$
(4.15)\qquad
\begin{aligned}
u''(t) = &-\alpha_0 \eta_0 \sigma(u(t)) + \sum_{n=1}^{\infty} \alpha_n f_n \lambda_n e^{-\lambda_n t} - \left(\sum_{n=1}^{\infty} \alpha_n \eta_n\right) \sigma(u(t))\\
&+ \sum_{n=1}^{\infty} \alpha_n \eta_n \lambda_n \int_0^t e^{-\lambda_n (t-s)} \sigma(u(s))\, ds, \qquad 0 < t < \infty;
\end{aligned}
$$

the differentiation is justified by the uniform convergence of the differentiated series. We observe that (2.20a) for $i = 0$ implies that (4.11) holds. Therefore, (4.15) and the argument employed in the proof of Lemma 4 for the behavior of $u''(t)$ shows that (2.20a) holds for $i = 2$. Since (2.20a) holds for $i = 0$ and $i = 2$, one has that $\lim_{t\to\infty} u'(t) = 0$ by the mean value theorem. On the other hand, (4.14) and the elementary argument employed in proving the exponential decay of $u'(t)$ in Lemma 4 shows that $w(t)$ decays exponentially as $t \to \infty$. Therefore, letting $t \to \infty$ one obtains from (4.13) that

$$
(4.16)\qquad -\alpha_0 f_0 = \alpha_0 \eta_0 \int_0^{\infty} \sigma(u(s))\, ds,
$$

and hence (4.13) yields

$$u'(t) = \alpha_0\eta_0 \int_t^\infty \sigma(u(s))\,ds + w(t).$$

Finally, using (4.11) we obtain

$$|u'(t)| \leqq \alpha_0\eta_0 K_0 \int_t^\infty e^{-\omega_0 s}\,ds + K_1\,e^{-\omega_0 t}$$

$$\leqq \frac{\alpha_0\eta_0 K_0}{\omega_0} e^{-\omega_0 t} + K_1\,e^{-\omega_0 t} = K e^{-\omega_0 t}, \qquad 0 \leqq t < \infty,$$

where

$$K_1 = \max\left(\sum_{n=1}^{\infty} |\alpha_n f_n|, \frac{K_0}{\lambda_1 - \omega_0}\sum_{n=1}^{\infty} \alpha_n\eta_n\right).$$

This establishes (2.20a) for $i = 1$ and completes the proof of Lemma 5.

If $\alpha_0\eta_0 = 0$ and $\alpha_0 = 0$, but $\eta_0 \neq 0$ (i.e., (2.14, $m = 0$) is not satisfied), the system (4.12) is replaced by (see (3.20))

$$\text{(4.17)}\qquad \begin{aligned} x_N' &= -\sum_{n=1}^{N} \alpha_n z_{Nn}, \\ y_N' &= \eta_0\sigma(x_N), \\ z_{Nn} &= -\lambda_n z_{Nn} + \eta_n\sigma(x_N), \end{aligned} \qquad \begin{aligned} N &= 0, 1, \cdots, \\ n &= 1, 2, \cdots, N, \end{aligned}$$

in which the middle equation is uncoupled from the rest of the system. The kernels K_1, K_2 in (4.4) are now given by

$$\text{(4.18)}\qquad \begin{aligned} K_1(t) &= \sum_{n=1}^{\infty} \frac{\alpha_n f_n}{\lambda_n}(e^{-\lambda_n t} - 1), \\ K_2(t) &= \sum_{n=1}^{\infty} \frac{\alpha_n \eta_n}{\lambda_n}(e^{-\lambda_n t} - 1), \end{aligned}$$

which are of the form (4.2) with the sum beginning with $n = 1$ in place of $n = 0$; likewise K_1', K_2' are of the form (4.3) starting with $n = 1$. Hence the argument of Lemma 4 (using also the corollary to Lemma 3) easily yields the following.

COROLLARY TO LEMMA 5. *If* $\alpha_0\eta_0 = 0$ *and* $\alpha_0 = 0$, *but* $\eta_0 \neq 0$ (i.e., if (2.14, $m = 0$) is not satisfied) *and if the other hypotheses of Lemma* 5 *are satisfied, then the unique solution* $u(t)$ *of* (4.1), (4.18) *decays exponentially in the sense of* (2.20a).

5. Proofs of Theorems 1 and 2. If the hypothesis of Theorem 1 is satisfied, let $u(t)$ be the unique solution of the Volterra equation (4.1), (4.2) on $0 \leqq t < \infty$ guaranteed by Lemma 4; if the hypotheses of Theorem 2 are satisfied, let $u(t)$ be the unique solution of (4.1), (4.4) on $0 \leqq t < \infty$ guaranteed by Lemma 5. In either case define

$$\text{(5.1)}\qquad T(x, t) = \int_0^c G(x, \xi; t) f(\xi)\,d\xi + \int_0^t \int_0^c G(x, \xi; t - \tau)\eta(\xi)\sigma(u(\tau))\,d\xi\,d\tau,$$

where

$$\text{(5.2)}\qquad G(x, \xi; t) = \sum_{n=0}^{\infty} y_n(x) y_n(\xi)\,e^{-\lambda_n t},$$

$$0 < x, \xi < c, \quad 0 < t < \infty,$$

where $\lambda_n, y_n(x)$ are the eigenvalues and eigenfunctions of the boundary value problem (2.1), (2.2). The following well-known result for the heat equation may be established by elementary methods; we omit the proof.

LEMMA 6. *Let f, η satisfy* (2.11). *Then $T(x, t)$ has the following properties*:

(i) $T(x, t)$, $T_t(x, t)$, $T_{xx}(x, t)$ *are continuous for* $0 \leqq x \leqq c, 0 < t < \infty$,

(ii) $T_t(x, t) = (b(x)T_x(x, t))_x - q(x)T(x, t) + \eta(x)\sigma(u(t)), 0 < x < c, 0 < t < \infty$,

(iii) $T(x, t)$ *satisfies the boundary conditions* (1.3),

(iv) $T(x, t)$ *is continuous in* (x, t) *for* $0 \leqq x \leqq c, 0 \leqq t < \infty$,

(v) *for each* $t \geqq 0$, $T(x, t) \in L_2(0, c)$, *as a function of* x.

To show that in either Theorem 1 or 2 the pair $u(t)$ (solution of (4.1) in Lemma 4 or 5 respectively) and $T(x, t)$ satisfies (1.1), it is only necessary to show that (1.1a) is satisfied (that (1.1b) is satisfied follows from Lemma 6 (ii)). By the Parseval theorem and (4.6) one readily obtains

$$-\int_0^c \alpha(x)T(x, t)\,dx = K_1'(t) + \int_0^t K_2'(t - s)\sigma(u(s))\,ds = u'(t), \qquad 0 \leqq t < \infty,$$

which is (1.1a); here K_1', K_2' are given by (4.3) if the hypotheses of Theorem 1 are satisfied and by (4.5) if the hypotheses of Theorem 2 are satisfied (by Lemmas 4 and 5 respectively).

The uniqueness of the solution $u(t)$ of (4.1) (in either the case of Theorem 1 or 2) coupled with any standard uniqueness theorem for the inhomogeneous heat equation readily yields the uniqueness of the pair $u(t)$, $T(x, t)$ defined above satisfying (1.1), (1.2), (1.3).

Now suppose that the hypotheses of Theorem 1 are satisfied. Then by Lemma 4 $u(t)$ satisfies (2.19a) and (2.20a) for $i = 0, 1, 2$. To prove (2.19b) we have from (5.1), (5.2) and the Parseval theorem,

$$T(x, t) = \sum_{n=0}^{\infty} f_n y_n(x)\, e^{-\lambda_n t} + \sum_{n=0}^{\infty} \eta_n y_n(x) \int_0^t e^{-\lambda_n(t-s)}\sigma(u(s))\,ds. \tag{5.3}$$

From (2.19a) and the continuity of σ there exists a constant K_0 such that

$$|\sigma(u(t))| \leq K_0, \qquad 0 \leqq t < \infty.$$

From (2.11), Theorem A, one has, $f_n \in l_1$, $\sup_{0 \leqq x \leqq c} |y_n(x)| \leqq K_1$, $0 < \lambda_0 < \lambda_1 < \cdots$, and therefore

$$|T(x, t)| \leqq K_1 \sum_{n=0}^{\infty} |f_n| + K_1 K_0 \sum_{n=0}^{\infty} |\eta_n| \int_0^t e^{-\lambda_0(t-s)}\,ds \leqq K, \tag{5.4}$$

which proves (2.19b). To prove (2.20b), we recall that (2.20a) implies that (4.11) is satisfied and therefore from (5.3),

$$\begin{aligned} |T(x, t)| &\leqq K_1 \sum_{n=0}^{\infty} |f_n|\, e^{-\lambda_n t} + K_1 K_0 \sum_{n=0}^{\infty} |\eta_n| \int_0^t e^{-\lambda_n(t-s)}\, e^{-\omega_0 s}\,ds \\ &\leqq K_1\, e^{-\lambda_0 t} \sum_{n=0}^{\infty} |f_n| + K_1 K_0\, e^{-\omega_0 t} \sum_{n=0}^{\infty} \frac{|\eta_n|}{\lambda_n - \omega_0}, \end{aligned}$$

which proves (2.20b) and completes the proof of Theorem 1.

Now suppose that the hypotheses of Theorem 2 are satisfied. Then by Lemma 5 $u(t)$ satisfies (2.19a) and (2.20a) for $i = 0, 1, 2$. Equation (5.3) now takes the form

$$(5.5)\qquad \begin{aligned} T(x, t) = f_0 c^{-1/2} + \sum_{n=1}^{\infty} f_n y_n(x)\, e^{-\lambda_n t} + \eta_0 c^{-1/2} \int_0^t \sigma(u(s))\, ds \\ + \sum_{n=1}^{\infty} \eta_n y_n(x) \int_0^t e^{-\lambda_n(t-s)} \sigma(u(s))\, ds. \end{aligned}$$

Thus one obtains

$$|T(x, t)| \leqq K_1 \sum_{n=0}^{\infty} |f_n| + K_1 \left| \int_0^t \sigma(u(s))\, ds \right| \sum_{n=0}^{\infty} |\eta_n|,$$

which is (2.21).

We note that (5.5) has the form

$$(5.6)\qquad T(x, t) = \left[f_0 + \eta_0 \int_0^t \sigma(u(s))\, ds \right] c^{-1/2} + T_1(x, t),$$

where, by the argument employed in Theorem 1,

$$(5.7)\qquad |T_1(x, t)| \leqq K_1\, e^{-\lambda_1 t} \sum_{n=1}^{\infty} |f_n| + K_1 K_0\, e^{-\omega_0 t} \sum_{n=1}^{\infty} \frac{|\eta_n|}{\lambda_n - \omega_0}.$$

To establish the remaining conclusions of Theorem 2 suppose first that $\alpha_0 \eta_0 > 0$. Then one has from (4.16),

$$f_0 = -\eta_0 \int_0^{\infty} \sigma(u(s))\, ds;$$

thus from (5.6),

$$T(x, t) = -\eta_0 c^{-1/2} \int_0^{\infty} \sigma(u(s))\, ds + T_1(x, t),$$

and using (4.11),

$$|T(x, t)| \leqq |\eta_0| c^{-1/2} K_0 \int_t^{\infty} e^{-\omega_0 s}\, ds + |T_1(x, t)|,$$

which together with (5.7) yields (2.20b).

If $\alpha_0 \eta_0 = 0$ and (2.14, $m = 0$) is satisfied, then (5.6), with $\eta_0 = 0$, and (5.7) yield (2.22).

If $\alpha_0 \eta_0 = 0$ and $\alpha_0 \neq 0$ (i.e., (2.14, $m = 0$) is not satisfied), the corollary to Lemma 5 shows that $u(t)$, which is now the solution of (4.1), (4.18), and $T(x, t)$, defined again by (5.1), (5.2), satisfy (1.1), (1.2), (1.3) on $0 < x < c$, $0 \leqq t < \infty$ and $u(t)$ satisfies (2.20a). Equations (5.6), (5.7) yield the estimate (2.23) for $T(x, t)$. That the integral in (2.23) exists for $0 \leqq t < \infty$ follows from (2.16), (2.20a). This completes the proof of Theorem 2.

REFERENCES

[1] T. A. BRONIKOWSKI, *An integrodifferential system which occurs in reactor dynamics*, Arch. Rational Mech. Anal., 37 (1970), pp. 363–380.

[2] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. I, Interscience, New York and London, 1953.

[3] J. E. HALL, *Quantitative estimates for nonlinear differential equations by Liapunov functions*, J. Differential Equations, 2 (1966), pp. 173–181.

[4] J. J. LEVIN AND J. A. NOHEL, *On a system of integro-differential equations occurring in reactor dynamics*, J. Math. Mech., 9 (1960), pp. 347–368.

[5] ———, *A system of nonlinear integrodifferential equations*, Michigan Math. J., 13 (1966), pp. 257–270.

[6] ———, *A nonlinear system of integrodifferential equations*, Mathematical Theory of Control, Academic Press, New York, 1967, pp. 398–405.

[7] ———, *The integrodifferential equations of a class of nuclear reactors with delayed neutrons*, Arch. Rational Mech. Anal., 31 (1968), pp. 151–171.

[8] R. K. MILLER, *On the linearization of Volterra integral equations*, J. Math. Anal. Appl., 23 (1968), pp. 198–208.

[9] ———, *An unstable nonlinear integrodifferential system*, Proceedings U.S.-Japan Seminar on Differential and Functional Equations, W. A. Benjamin, New York, 1967, pp. 479–489.

[10] J. A. NOHEL, *Some problems in nonlinear Volterra integral equations*, Bull. Amer. Math. Soc., 68 (1962), pp. 323–329.

[11] K. YOSIDA, *Lectures on Differential and Integral Equations*, Interscience, New York and London, 1960.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

BOUNDEDNESS PROPERTIES OF STATIONARY LINEAR OPERATORS*

JAMES D. BAKER†

Abstract. A characterization of boundedness given by R. E. Lane provides a foundation for studying stationary, linear transformations which map the left-continuous quasi-continuous functions on the line into functions on the line. Properties of these operators are identified in terms of the transforms of $J(t) = \chi_{(0,\infty)}$. It is shown that the norm of an operator T is the variation of TJ, that there is a Stieltjes integral representation of T with this function as the integrator, and that convergence of a sequence of these operators is equivalent to convergence in variation of the sequence of transforms of J. An operator which is continuous under pointwise convergence is shown to be bounded on a closed interval, and is characterized by the left-continuity of TJ.

1. Introduction. A number-valued function on the closed interval $[a, b]$ is said to be quasi-continuous if it is the uniform limit of a sequence of step functions on $[a, b]$; a function on the line is said to be quasi-continuous if it is quasi-continuous on each closed interval. We denote by $Q([a, b])$, the set of quasi-continuous functions on $[a, b]$, and by Q, the set of quasi-continuous functions on the line. The subsets of $Q([a, b])$ and Q which contain the left-continuous functions will be identified by the subscript L.

In [5], Lane considered bounded, stationary, linear transformations on the set Q, and demonstrated that these operators could be represented as the sum of two Stieltjes σ-mean integrals over a closed interval. These operators and their representations were subsequently used to develop a theory for the study of linear physical systems [6]. The subset of Q_L containing the functions of bounded variation on each closed interval and which vanish to the left of the origin were of primary interest in that investigation. Bounded linear functionals on $Q([a, b])$ were first considered by Kaltenborn [4], and it was observed that these functionals have representations as a Stieltjes interior integral plus an infinite sum. Dyer later considered both bounded linear functionals and bounded linear operators on $Q_L([a, b])$ [1], [2]. He provided Stieltjes mean integral representations for both the functionals and operators and gave conditions for the inversion of certain classes of the operators.

In this paper, additional properties of bounded, stationary, linear operators on Q_L are investigated and those operators which are continuous under pointwise convergence are identified. Since Q contains all functions which have length on an interval, it is a natural set for the study of many types of systems. Little is lost in generality by considering Q_L as opposed to Q since analogous results hold for right-continuous functions.

One motivation for studying operators in terms of their integral representations is that it facilitates the study of integral operator equations, and hence, the application of operators to problems. The operator characterization given here is in terms of the left-Cauchy integral, which has been used in studies of integral equations by several investigators, including MacNerney [7] and Dyer [3].

* Received by the editors June 10, 1971, and in revised form October 28, 1971.
†Corporate Research Center, Honeywell, Inc., Hopkins, Minnesota 55343.

2. Preliminaries

DEFINITION 2.1. Suppose A is a nonempty set of real numbers and T is a transformation such that:

(i) if $y \in Q_L$, then Ty is a function on the line;

(ii) if $y_1 \in Q_L$ and $y_2 \in Q_L$, then $T(y_1 + y_2) = Ty_1 + Ty_2$;

(iii) if $y \in Q_L$ and k is a number, then $T(ky) = kTy$;

(iv) if $y \in Q_L$, c is a real number, and $z(t) = y(t + c)$, then $Tz(s) = Ty(s + c)$, for each real number s;

(v) if s is a real number, then there is a positive number B_s such that if $y \in Q_L$ and $|y(s - t)| \leqq M$ for $t \in A$, then $|Ty(s)| \leqq MB_s$.

Then T is said to be a *stationary linear operator on Q_L* which is bounded over the set A, or, equivalently, a *Q_L-operator over A*.

The norm of T at s is the greatest lower bound of the numbers B_s and is denoted by $|T(s)|$. It was shown in [5] that $|T(s)| = |T(0)|$ for each real number s, so the norm of T is independent of s and will be denoted by $\|T\|$.

Definition 2.1 (v) is equivalent to the usual definition of bounded when T is restricted to the subset of Q_L containing the functions which vanish outside the set A. Also, it is a trivial matter to show that if T is a Q_L-operator over A and if $A \subset B$, then T is a Q_L-operator over B.

LEMMA 2.1. *Suppose T is a Q_L-operator over $[a, b]$, $y \in Q_L$, and $[c, d]$ is a closed interval. Then there is a sequence of step functions in Q_L such that $Ty_n(s) \to Ty(s)$ uniformly for $s \in [c, d]$.*

Proof. Any sequence of step functions in Q_L which converges uniformly to y on $[c - b, d - a]$ has the desired property.

If T is a Q_L-operator over $[a, b]$, then T is a mapping from Q_L into Q, and it was shown in [5] that Ty is of bounded variation or continuous accordingly as y is of bounded variation or continuous. Using Lemma 2.1 and the observation that left continuity is preserved under uniform convergence, we have that for T to be a mapping from Q_L into itself, it is necessary and sufficient that $TJ \in Q_L$, where $J = \chi_{(0,\infty)}$.

3. Q_L-Operators. For a given Q_L-operator T, the function $\beta(s)$ will be used to denote $TJ(s)$.

THEOREM 3.1. *Suppose T is a Q_L-operator over $[a, b]$. Then each of the following statements holds:*

(i) *$\beta(s) = 0$ for $s < a$, and $\beta(s) = \beta(b+)$ for $s > b$;*

(ii) *β is of bounded variation on $[a, b]$.*

Proof. If $s < a$, then $J(s - t) = 0$ for $t \in [a, b]$, so $|TJ(s)| = 0$. Suppose $r > b$ and $s > b$. Using an argument like that in [5], it follows that $TJ(r) = TJ(s)$. Property (ii) follows as a result of Theorem 3.3 in [5] since J is of bounded variation.

In Remark 4.1 of [5], an example is given for which $\beta(b) \neq \beta(b+)$. If $y \in Q_L$ and s is a real number, let $Ty(s) = \frac{1}{2}[y(s) + y(s - 1)]$. Then T is a Q_L-operator over $[0, 1]$ with $\beta(s) = 0$ if $s \leqq 0$, $\beta(s) = \frac{1}{2}$ if $s \in (0, 1]$, and $\beta(s) = 1$ if $s > 1$.

THEOREM 3.2. *If T is a Q_L-operator over $[a, b]$ which is not a translation or magnification, then there is a smallest closed interval over which T is a Q_L-operator.*

Proof. Let $X = \bigcap_\alpha [a_\alpha, b_\alpha]$, where T is a Q_L-operator over $[a_\alpha, b_\alpha]$ for each α. If $X = \varnothing$, then $\beta(s) = 0$ for each real number s. If $X = \{c\}$, then $TJ(s)$

$= \beta(c+)J(s-c)$. Since T is stationary, linear, and bounded, it follows that $Ty(s) = \beta(c+)y(s-c)$ for each $y \in Q_L$ and each real number s. The only remaining case is for X to be a closed interval.

It also follows that if T is a Q_L-operator on $[e,f]$ for each $[e,f] \subseteq [a,b]$, then $\beta(s) = 0$ for each real number s.

The properties of β allow one to consider Stieltjes integrals as a representation technique; and of these, the left-Cauchy integral is well-suited for Q_L-operators.

THEOREM 3.3. *Suppose T is a Q_L-operator over $[a,b]$. If $y \in Q_L$ and s is a real number, then*

$$Ty(s) = \mathrm{LC}\int_a^{b+} y(s-t)\,d\beta(t).$$

Conversely, if $y \in Q_L$, s is a real number and

$$Uy(s) = \mathrm{LC}\int_a^{b+} y(s-t)\,d\alpha(t),$$

where α has the properties of Theorem 3.1, then U is a Q_L-operator over $[a,b]$.

Proof. The first part of the theorem follows from a standard argument using the observations that

$$TJ(s) = \beta(s) = \mathrm{LC}\int_a^{b+} J(s-t)\,d\beta(t),$$

that if y is a step function in Q_L and s is a real number, then

$$y(s-t) = \sum_{k=1}^{n} c_k J(s-t-t_k) \quad \text{for } t \in [a,b],$$

and that for a sequence of step functions satisfying Lemma 2.1, we have

$$\mathrm{LC}\int_a^{b+} y(s-t)\,d\beta(t) = \lim_{n\to\infty} \mathrm{LC}\int_a^{b+} y_n(s-t)\,d\beta(t).$$

The second part of the theorem follows directly from properties of the left-Cauchy integral.

THEOREM 3.4. *If T is a Q_L-operator over $[a,b]$, then $\|T\| = V_a^{b+}(\beta)$.*

Proof. Suppose $a = t_0 < t_1 < \cdots < t_{n-1} = b$ is a partition of $[a,b]$ and $t_n > b$. Choose ε_k to be 1 or -1 accordingly as $[\beta(t_k) - \beta(t_{k-1})]$ is nonnegative or negative, and let

$$z(-t) = \sum_{k=1}^{n} \varepsilon_k[J(t_k - t) - J(t_{k-1} - t)].$$

Then

$$\sum_{k=1}^{n} |\beta(t_k) - \beta(t_{k-1})| = |Tz(0)| \leqq \|T\|.$$

The reverse inequality follows directly from properties of the left-Cauchy integral and the definition of $\|T\|$.

As was proved above, the norm of the operator T is equal to the variation of the integrator function in the integral representation. For representations with the Stieltjes mean integral, the integrator function is $g(t) = 2\beta(t) - \beta(t^-)$ (see [1]), and for this case, the strongest result that can be obtained is $\|T\| \leqq V_a^{b+}(g) \leqq 3\|T\|$.

A sequence $\{T_n\}$ of Q_L-operators is said to converge to an operator T if for $\varepsilon > 0$, there is an integer N such that if $y \in Q_L$, s is a real number, and $|y(s - t)| \leqq M$ for $t \in [a, b]$, then $|T_n y(s) - Ty(s)| < M\varepsilon$ for $n \geqq N$. Since

$$|Ty(s)| \leqq |Ty(s) - T_n y(s)| + |T_n y(s)|,$$

there is an integer n_0 and a positive number ε_0 such that

$$|Ty(s)| \leqq M[V_a^{b+}(\beta_{n_0}) + \varepsilon_0].$$

Thus, T is a Q_L-operator over $[a, b]$.

THEOREM 3.5. *Suppose* $\{T_n\}$ *and* T *are* Q_L*-operators on* $[a, b]$. *If* $\beta = TJ$ *and* $\beta_n = T_n J$ *for each* n, *then for* $\lim_{n\to\infty} T_n = T$, *it is necessary and sufficient that* $\lim_{n\to\infty} V_a^{b+}(\beta_n - \beta) = 0$.

Proof. Convergence in variation of $\{\beta_n\}$ assures that $\lim_{n\to\infty} T_n = T$. To prove the converse, suppose $\{\beta_n\}$ does not converge in variation to β. Then there is a number $\delta > 0$ such that for any integer N, there is an $m > M$ such that $V_a^{b+}(\beta_m - \beta) > \delta$. Choose $\varepsilon < \delta$. There is a sequence $\{t_k\}_{k=0}^n$ of real numbers such that $t_0 = a$ and $t_{n-1} = b$, and such that

$$\sum_{k=1}^{n} |(\beta_m - \beta)(t_k) - (\beta_m - \beta)(t_{k-1})| > \delta.$$

Let $z(-t)$ be 1 or -1 for $t \in [t_{k-1}, t_k)$ accordingly as $[(\beta_m - \beta)(t_k) - (\beta_m - \beta)(t_{k-1})]$ is nonnegative or negative. Then $|z(0 - t)| \leqq 1$, and

$$\mathrm{LC}\int_a^{b+} z(0 - t)\, d[\beta_m(t) - \beta(t)] > \delta;$$

hence, $|T_m z(0) - Tz(0)| > \varepsilon$.

4. Continuous operators. Suppose T_c is a transformation on the set Q_L which satisfies (i), (ii), (iii) and (iv) of Definition 2.1. Suppose, also, that if s is a real number and $\{y_n\}$ is a sequence of functions in Q_L such that $y_n(s - t) \to 0$ for $t \in A$, then $T_c y(s) \to 0$. An operator with these properties is said to be *continuous* over the set A. We use β_c to denote $T_c J$.

THEOREM 4.1. *If* T_c *is a continuous operator, then there is a closed interval* $[a, b]$ *such that* T_c *is a* Q_L*-operator over* $[a, b]$.

Proof. Suppose $[c, d]$ is a closed interval and that T_c is continuous over $[c, d]$. Since uniform convergence is stronger than pointwise convergence, we have that if s is a real number and $\{y_n\}$ is a sequence of functions in Q_L such that $y_n(s - t) \to 0$ uniformly for $t \in [c, d]$, then $T_c y_n(s) \to 0$. This property is equivalent to Definition 2.1 (v); hence, T_c is a Q_L-operator over $[c, d]$.

We now show that if T_c is a continuous operator, then there is a closed interval $[a, b]$ such that β_c is zero to the left of a and is constant to the right of b. If no such number b exists, then there is an increasing sequence $\{t_k\}$ of real numbers, which does not converge, and for which $\beta_c(t_{k+1}) - \beta_c(t_k) \neq 0$. Let

$$w_k(-t) = \begin{cases} [\beta_c(t_{k+1}) - \beta_c(t_k)]^{-1} & \text{if } t \in [t_k, t_{k+1}), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$z_n(-t) = \sum_{k=1}^{n} w_k(-t),$$

and let

$$z(-t) = \lim_{n \to \infty} z_n(-t).$$

Then $z \in Q_L$, and

$$\lim_{n \to \infty} (z - z_n) = 0.$$

For $k = 1, 2, \cdots$, let $T_k J(s) = \beta_c(s)$ if $s \in (t_k, t_{k+1}]$, and $T_k J(s) = 0$, otherwise. Then T_k is continuous over $[t_k, t_{k+1}]$, and, hence is a Q_L-operator over $[t_k, t_{k+1}]$. Since

$$T_c z(0) = \lim_{n \to \infty} T_c z_n(0) = \lim_{n \to \infty} \sum_{k=1}^{n} T_k z(0),$$

and since for each integer k,

$$T_k z(0) = \mathrm{LC} \int_{t_k}^{t_{k+1}} w_k(-t)\, d\beta_c(t) = 1,$$

we have that $T_c z(0) = \lim_{n \to \infty} n$. This contradicts the requirement that $T_c z(0)$ be a number. Thus there exists a number b such that $\beta_c(s)$ is constant for $s > b$; a similar argument shows the existence of a number a such that $\beta_c(s)$ is constant for $s \leqq a$. To show that $\beta_c(s) = 0$ for $s \leqq a$, let $J_n(t) = \chi_{(n,\infty)}$. Since $\lim_{n \to \infty} J_n(t) = 0$, we have that

$$\lim_{n \to \infty} T_c J_n(s) = \lim_{n \to \infty} \beta_c(s - n) = 0;$$

and since β_c is constant to the left of a, the theorem is proved.

THEOREM 4.2. *If T_c is a continuous operator over $[a, b]$, then β_c is left continuous.*

Proof. Suppose $r \in (a, b]$. For each positive integer n, let $z_n(-t) = 1$ if $t \in [r - 1/n, r)$, and let $z_n(-t) = 0$ otherwise. Then

$$T_c z_n(0) = [\beta_c(r) - \beta_c(r - 1/n)],$$

and since $\lim_{n \to \infty} T_c z_n(0) = 0$, it follows that $\beta_c(r-) = \beta_c(r)$.

Acknowledgment. The author wishes to acknowledge the very helpful comments and suggestions of Professor James A. Dyer.

REFERENCES

[1] J. A. Dyer, *Concerning the mean-Stieltjes integral representation of a bounded linear transformation*, J. Math. Anal. Appl., 8 (1964), pp. 452–460.

[2] ———, *The inversion of a class of linear operators*, Pacific J. Math., 19 (1966), pp. 57–66.

[3] ———, *The Freedholm and Volterra problems for Stieltjes integral equations*, J. Applicable Anal., to appear.

[4] H. S. Kaltenborn, *Linear functional operations on functions having discontinuities of the first kind*, Bull. Amer. Math. Soc., 40 (1934), pp. 702–708.

[5] R. E. Lane, *Linear operators on quasi-continuous functions*, Trans. Amer. Math. Soc., 89 (1958), pp. 378–394.

[6] ———, *Research on linear operators with applications to systems analysis*, ASD-TDR-62-393, Aeronautical Systems Div., Navigation and Guidance Lab., Wright Patterson AFB, Dayton, Ohio, 1962.

[7] J. S. MacNerney, *A linear initial value problem*, Bull. Amer. Math. Soc., 69 (1963), pp. 314–329.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

CONTINUOUS LINEAR FUNCTIONALS ON CERTAIN $K\{M_p\}$ SPACES*

CHARLES SWARTZ†

Abstract. A distribution T is known to be tempered if and only if there is a positive integer k such that $T * \phi(x)/(1 + |x|^2)^k$ is bounded for any $\phi \in \mathscr{D}$, if and only if there is a positive integer k such that $\{1/(1 + |x|^2)^k \tau_{-x} T : x \in R^n\}$ is bounded in $\mathscr{D}'$. The analogues of these characterizations are established for certain of the test spaces $K\{M_p\}$ of I. M. Gelfand and G. E. Shilov.

In [1, II.4], Gelfand and Shilov obtain a representation for the continuous linear functionals on certain $K\{M_p\}$ spaces. In particular, the representation in [1] contains as a special case the familiar representation of the tempered distributions of L. Schwartz [5, Thm. 25.4]. There are several other well-known characterizations of the tempered distributions [3, Chap. VII, § 4, Thm. VI], and in this note we establish the analogue of some of these characterizations for certain $K\{M_p\}$ spaces. Our results are also applicable to the space of distributions of exponential order [2], [6], [7].

We recall some of the notions pertinent to $K\{M_p\}$ spaces. (These facts are treated in [1, Chaps. II and III].) Let $\{M_p\}_{p=1}^{\infty}$ be a sequence of real-valued continuous functions defined on $\mathbb{R}^n$ and such that $1 \leqq M_1(x) \leqq M_2(x) \leqq \cdots, x \in \mathbb{R}^n$. The vector space $K\{M_p\}$ consists of all infinitely differentiable complex-valued functions ϕ defined on $\mathbb{R}^n$ such that

$$\|\phi\|_p = \sup\{M_p(x)|D^\alpha \phi(x)| : x \in \mathbb{R}^n, |\alpha| \leqq p\} < \infty \tag{1}$$

for all $p \geqq 1$. (The definition of $K\{M_p\}$ spaces given in [1] is more general since the functions M_p are allowed to take on infinite values; we make this restriction so that the resulting $K\{M_p\}$ spaces are closed under translations.) The vector space $K\{M_p\}$ is supplied with the locally convex topology generated by the sequence of norms $\{\|\cdot\|_p\}_{p=1}^{\infty}$. Under this topology, $K\{M_p\}$ is a Frechét space [1, II.2.2].

We shall consider $K\{M_p\}$ spaces which satisfy the conditions (M) and (N) of Gelfand and Shilov [1, II.4.2]. The sequence $\{M_p\}$ satisfies (M) and (N) if:

(M): The functions M_p are quasi-monotonic in each coordinate, i.e., if $|x_j'| \leqq |x_j''|$, then

$$M_p(x_1, \cdots, x_j', \cdots, x_n) \leqq C_p M_p(x_i, \cdots, x_j'', \cdots, x_n)$$

for each fixed point $(x_i, \cdots, x_{j-1}, x_{j+1}, \cdots, x_n)$.

(N): For each p there is $p' > p$ such that the ratio $M_p(x)/M_{p'}(x) = m_{pp'}(x)$ tends to 0 as $|x| \to \infty$ and the function $m_{pp'}$ is Lebesgue summable on $\mathbb{R}^n$.

* Received by the editors July 27, 1971, and in revised form December 17, 1971.

† Department of Mathematics, The University of Arizona, Tucson, Arizona 85721.

In [1, II.4.2], it is shown that if $\{M_p\}$ satisfies (M) and (N), the sequence of norms

$$\|\phi\|'_q = \sup\left\{\int M_q(t)|D^\alpha\phi(t)|\,dt : |\alpha| \leqq q\right\}, \qquad q \geqq 1, \tag{2}$$

generates the same locally convex topology as the sequence of norms defined in (1).

Further, we impose the condition (F) which was employed in [4].

(F): Each M_p is symmetric, i.e., $M_p(x) = M_p(-x)$, and for each p there is a $p' > p$ and $C_{p'} > 0$ such that $M_p(x + h) \leqq C_{p'}M_{p'}(x)M_{p'}(h)$ for all $x, h \in R^n$.

Before stating our first result, we recall that if $T \in \mathscr{D}'$ and $\phi \in \mathscr{D}$, their convolution, $T * \phi \in \mathscr{E}$, is given by $T * \phi(x) = \langle T, \tau_x\phi\rangle$ for $x \in \mathbb{R}^n$, where $\tau_x\phi$ is defined to be $\tau_x\phi(y) = \phi(x + y)$ (see [1, III.3.2]). This definition differs from that given in many of the standard functional analysis texts; see, for example, [5, Def. 27.1]. Also, since the injection $\mathscr{D} \subseteq K\{M_p\}$ is continuous and $\mathscr{D}$ is dense in $K\{M_p\}$, each element of $K\{M_p\}'$ can be identified with a distribution.

THEOREM 1. *Let $\{M_p\}$ satisfy conditions* (M), (N) *and* (F). *For $T \in \mathscr{D}'$, the following are equivalent*:

(i) $T \in K\{M_p\}'$.

(ii) *There exist a positive integer p and bounded measurable functions f_α ($\alpha \in \mathbb{N}^n$, $|\alpha| \leqq p$) such that $T = \sum_{|\alpha| \leqq p} D^\alpha(M_p f_\alpha)$.*

(iii) *There is a positive integer k such that for any $\phi \in \mathscr{D}$, $(T * \phi)/M_k$ is bounded on $\mathbb{R}^n$.*

(iv) *There is a positive integer k such that $\{(1/M_k(h))\tau_{-h}T : h \in \mathbb{R}^n\}$ is bounded in $\mathscr{D}'$.*

Proof. That (i) and (ii) are equivalent is established in [1, Chap. II, § 4.2].

First we show (ii) implies (iii). Assuming the representation in (ii), we have for any $\phi \in \mathscr{D}$,

$$\begin{aligned} |T * \phi(x)| &\leqq \sum_{|\alpha| \leqq p} \int M_p(y - x)|f_\alpha(y - x)D^\alpha\phi(y)|\,dy \\ &\leqq M_{p'}(x)C_{p'}L \sum_{|\alpha| \leqq p} \int M_{p'}(y)|D^\alpha\phi(y)|\,dy \\ &\leqq M_{p'}(x)C_{p'}L\|\phi\|'_{p'}, \end{aligned}$$

where p' is given by condition (F) and L is a common bound for the $\{f_\alpha\}$. Thus, if we set $k = p'$, (iii) is established.

Now we show (iii) implies (iv). Let $\phi \in \mathscr{D}$. By (iii) there exists $B = B_\phi > 0$ such that $|T * \phi(x)| \leqq BM_k(x)$ for $x \in \mathbb{R}^n$. That is, $\sup\{|\langle(1/M_k(x)\tau_{-x}T, \phi\rangle| : x \in \mathbb{R}^n\} \leqq B$ so that $\{(1/M_k(x))\tau_{-x}T : x \in \mathbb{R}^n\}$ is weakly bounded in $\mathscr{D}'$ (and also strongly bounded [3, Chap. III, § 3, Thm. IX]).

We conclude by showing that (iv) implies (ii). Now $\{(1/M_p(h))\tau_{-h}T : h \in \mathbb{R}^n\}$ bounded in $\mathscr{D}'$ implies there is a compact neighborhood K of 0 in $\mathbb{R}^n$ and a positive integer m such that if $\psi \in \mathscr{D}^m_K$, the family of continuous functions

$$\{1/M_p(h)\tau_{-h}T * \psi : h \in \mathbb{R}^n\} \tag{3}$$

is bounded on K [3, Chap. VI, § 7, Thm. XXII]. The elementary solution E of Δ^N is m-times continuously differentiable for large N so if $\gamma \in \mathscr{D}_K$ is such that $\gamma(t) = 1$ for t in some neighborhood of 0, then $\gamma E \in \mathscr{D}_K^m$ and $\delta = \Delta^N(\gamma E) - \phi$, where $\phi \in \mathscr{D}$. Therefore,

$$T = T * \delta = \Delta^N(T * \gamma E) - T * \phi. \tag{4}$$

Now $T * \phi \in \mathscr{E}$ since $T \in \mathscr{D}'$ and $\phi \in \mathscr{D}$, and (3) gives

$$\sup\{1/M_p(h)\langle T, \tau_h\phi\rangle : h \in \mathbb{R}^n\} = \sup\{1/M_p(h)|T * \phi(h)| : h \in \mathbb{R}^n\} < \infty$$

since $0 \in K$. Also $\gamma E \in \mathscr{D}_K^m$ implies

$$\sup\{1/M_p(h)|T * (\gamma E)(h)| : h \in \mathbb{R}^n\} < \infty$$

by (3) since $0 \in K$. Thus (4) yields (ii).

Remark 2. Some parts of Theorem 1 find analogues in parts of [3, Chap. VII, § 4, Thm. VI]. Here $M_p(x) = (1 + |x|^2)^p$ and $\{M_p\}$ satisfies (M), (N) and (F) [4].

Remark 3. Theorem 1 also yields some of the characterizations of $\mathscr{K}'_1$, the space of distributions of exponential growth, as given in Theorem 1 of [7]. In this case, $M_p(x) = \exp(p|x|)$, and $\{M_p\}$ is easily seen to satisfy conditions (M), (N) and (F) [4].

An infinitely differentiable function ψ on $\mathbb{R}^n$ is said to be a multiplier on $K\{M_p\}$ [1, II.3.2] if

(i) $\psi\phi \in K\{M_p\}$ for each $\phi \in K\{M_p\}$ and

(ii) the map $\phi \to \psi\phi$ is continuous from $K\{M_p\}$ into itself. The vector space of all multipliers on $K\{M_p\}$ is denoted by $\mathscr{O}_M(K\{M_p\})$. We have the following result pertaining to multipliers on $K\{M_p\}$ when $\{M_p\}$ satisfies conditions (M), (N) and (F).

THEOREM 4. *Let* $\{M_p\}$ *satisfy conditions* (M), (N) *and* (F). *If a function* $\psi \in \mathscr{E}$ *belongs to* $\mathscr{O}_M(K\{M_p\})$, *then for each* α *there is a positive integer* k *such that* $(D^\alpha\psi)/M_k$ *is bounded on* $\mathbb{R}^n$.

Proof. Suppose $\psi \in \mathscr{O}_M(K\{M_p\})$. We show that there exists a positive integer k such that ψ/M_k is bounded, and since $\mathscr{O}_M(K\{M_p\})$ is closed under differentiation, this will establish the result. If this fails, there are points $\{x_k\}$ in $\mathbb{R}^n$ such that $\|x_k - x_{k+1}\| \geqq 4$ and $|\psi(x_k)/M_k(x_k)| > k$. Pick $h \in \mathscr{D}$ such that $h(x) = 1$ for $\|x\| \leqq 1$ and $h(x) = 0$ for $\|x\| \geqq 2$. Define $\phi_k \in \mathscr{D}$ by $\phi_k(x) = h(x - x_k)/M_k(x_k)$. Now the sequence $\{\phi_k\}$ is bounded in $K\{M_p\}$ since for any positive integer p and $|\beta| \leqq p$,

$$\begin{aligned}\int M_p(t)|D^\beta\phi_k(t)|\,dt &\leqq \int_{\|u\| \leqq 2} \frac{M_p(u + x_k)|D^\beta h(u)|}{M_k(x_k)}\,du \\ &\leqq C_{p'}\frac{M_{p'}(x_k)}{M_k(x_k)}\int M_{p'}(u)|D^\beta h(u)|\,du \\ &\leqq C_{p'}\|h\|'_{p'} \quad \text{for } k \geqq p'.\end{aligned}$$

However, $|(\psi\phi_k)(x_k)| > k$ for each k so that $\{\psi\phi_k\}$ is not bounded in $K\{M_p\}$ and thus $\psi \notin \mathscr{O}_M(K\{M_p\})$. Hence, there is a positive integer k such that ψ/M_k is bounded.

The converse is established in [1, II.3.2] under the additional hypothesis on $\{M_p\}$:

(A): For any two subscripts p, r ($p \geqq r$) there exists $s \geqq p$ such that $M_p(x)M_r(x) \leqq C_{pr}M_s(x)$ for $x \in \mathbb{R}^n$.

Combining this result with Theorem 4 we obtain the following corollary.

COROLLARY 5. *Let* $\{M_p\}$ *satisfy conditions* (M), (N), (F) *and* (A). *A function* $\psi \in \mathscr{E}$ *is a multiplier on* $K\{M_p\}$ *if and only if for each* α *there is a positive integer* k *such that* $(D^\alpha\psi)/M_k$ *is bounded on* $\mathbb{R}^n$.

Remark 6. This gives the characterization of multipliers on $\mathscr{S}$ as in Theorem 25.5 of [5] since in this case $\mathscr{S} = K\{M_p\}$ with $M_p(x) = (1 + |x|^2)^p$ and $\{M_p\}$ satisfies conditions (M), (N), (F) and (A).

Remark 7. The corollary also gives the characterization of multipliers on $\mathscr{K}_1$ [7] as in Proposition 4, parts (1) and (2) of [6]. In this case, $M_p(x) = \exp(p\gamma(x))$, where $\gamma(x) = \sqrt{1 + \|x\|^2}$, and $\{M_p\}$ is easily seen to satisfy (M), (N), (F) and (A).

Using the corollary, if $\{M_p\}$ satisfies (M), (N), (F) and (A), we may replace condition (iii) in Theorem 1 by (iii)′: for each $\phi \in \mathscr{D}$, $T * \phi \in \mathscr{O}_M(K\{M_p\})$.

COROLLARY 8. *Suppose* $\{M_p\}$ *satisfies* (M), (N), (F) *and* (A). *Then for* $T \in \mathscr{D}'$, *conditions* (i), (ii), (iii), (iii)′ *and* (iv) *are equivalent*.

Proof. Clearly (iii)′ implies (iii) by Corollary 5, and to see that (iii) implies (iii)′, we note that for $\phi \in \mathscr{D}$, $\alpha \in \mathbb{N}^n$, $D^\alpha(T * \phi) = T * D^\alpha\phi$.

Remark 9. The equivalence of (i) and (iii)′ in the case where $\mathscr{S} = K\{M_p\}$ is given in the remarks following Theorem IX of [3, Chap. VII, § 5]. For the case where $K\{M_p\} = \mathscr{K}_1$ see [6, Prop. 9].

REFERENCES

[1] I. GELFAND AND G. SHILOV, *Generalized Functions, II*, Academic Press, New York, 1968.

[2] M. HASUMI, *Note on the n-dimensional tempered ultradistributions*, Tôhoku Math. J., 13 (1961), pp. 99–104.

[3] L. SCHWARTZ, *Théorie des distributions*, Hermann, Paris, 1966.

[4] C. SWARTZ, *Convolution in* $K\{M_p\}$ *spaces*, Rocky Mountain J. Math., to appear.

[5] F. TREVES, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.

[6] K. YOSHINAGA, *On spaces of distributions of exponential growth*, Bull. Kyushu Inst. Tech. Math Natur. Sci., 6 (1960), pp. 1–16.

[7] Z. ZIELEZNY, *On the space of convolution operators in* $\mathscr{K}'_1$, Studia Math., 31 (1968), pp. 111–124.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

ON OSCILLATORY SOLUTIONS OF CERTAIN FOURTH ORDER LINEAR DIFFERENTIAL EQUATIONS*

M. S. KEENER†

Abstract. We consider the fourth order linear homogeneous differential equation

$$y^{(4)} + p_3(x)y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0$$

with continuous coefficients. Our studies center on the oscillatory behavior of solutions of the above equation under the separate disconjugacy conditions $r_{22} = r_{31} = \infty$ and $r_{22} = r_{13} = \infty$.

1. Introduction. In this paper we consider the differential equation

$$y^{(4)} + p_3(x)y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0, \tag{1.1}$$

where $p_i(x) \in C[\varepsilon, \infty)$, $i = 0, 1, 2, 3$, for some real number ε. We are interested in the behavior of solutions of (1.1) under certain types of disconjugacy conditions. The following definitions are therefore needed.

DEFINITION 1. A nontrivial solution $y(x)$ of (1.1) has an $i_1 - i_2 - \cdots - i_n$ *distribution of zeros* on an interval $I \subset [\varepsilon, \infty)$, where $n > 1$, provided there exist n points $x_1 < x_2 < \cdots < x_n$ in I such that $y(x)$ has a zero at x_k of order at least i_k for $k = 1, 2, 3, \cdots, n$.

DEFINITION 2. For equation (1.1) and $t \in [\varepsilon, \infty)$, $r_{i_1 i_2 \cdots i_n}(t)$ is the infimum of the numbers $b > t$ such that there exists a nontrivial solution of (1.1) having an $i_1 - i_2 - \cdots - i_n$ distribution of zeros on $[t, b]$. If no such number b exists, we write

$$r_{i_1 i_2 \cdots i_n}(t) = \infty. \tag{1.2}$$

If (1.2) holds for all $t \in [\varepsilon, \infty)$, we write

$$r_{i_1 i_2 \cdots i_n} = \infty.$$

For $t \in [\varepsilon, \infty)$ the numbers $r_{i_1 i_2 \cdots i_n}(t)$, $\sum_{k=1}^{n} i_k = 4$, have been of considerable interest recently. In a paper by Ridenhour and Sherman [5] it was shown that $r_{121} = \infty$ implies $r_{13} = r_{31} = \infty$. In his recent dissertation, Schneider [6] considered self-adjoint fourth order differential equations for which $r_{121} = \infty$. For $t \in [\varepsilon, \infty)$, we define $\eta(t)$ by

$$\eta(t) \equiv r_{1111}(t).$$

Peterson [4] has shown that

$$\eta(t) = \min \{r_{112}(t), r_{121}(t), r_{211}(t)\}$$

and that at most one of the numbers in the brace is infinite if $\eta(t) < \infty$. It can also be shown (see [1]) that

$$\eta(t) = \min \{r_{31}(t), r_{22}(t), r_{13}(t)\}.$$

* Received by the editors September 8, 1971, and in revised form December 10, 1971.

† Department of Mathematics and Statistics, Oklahoma State University, Stillwater, Oklahoma 74074.

In Leighton and Nehari's [2] classical paper on self-adjoint fourth order differential equations, the authors considered the behavior of solutions of

$$[r(x)y'']'' - p(x)y = 0 \tag{1.3}$$

and

$$[r(x)y'']'' + p(x)y = 0, \tag{1.4}$$

where $r(x)$ and $p(x)$ are positive and continuous on an interval $[\varepsilon, \infty)$. For (1.3) it was shown that $\eta(t) = r_{22}(t)$ and $r_{31} = r_{13} = r_{121} = \infty$. Others have considered the behavior of solutions of more general equations than (1.3) and (1.4) under the assumption $\eta(t) = r_{22}(t)$. Either [1] or [8] furnishes a good list of references.

The conditions imposed on the equation in this study force the equation to be nonself-adjoint. Such equations may arise in the study of a vibrating string in a medium where some frictional force is considered. The purpose of this paper is to study the behavior of solutions of (1.1) under the conditions

$$r_{22} = r_{31} = \infty, \qquad \eta(t) = r_{13}(t),$$

and

$$r_{22} = r_{13} = \infty, \qquad \eta(t) = r_{31}(t).$$

We shall be interested in the oscillatory behavior of solutions. A nontrivial solution of (1.1) is said to be *oscillatory* on $[\varepsilon, \infty)$ provided it has unbounded zeros on $[\varepsilon, \infty)$.

The following lemma is well known and is fundamental in the study of the behavior of solutions of (1.1). The proof is omitted.

LEMMA 1.1. *Let $u(x)$ be a nontrivial solution of* (1.1) *with a zero at $x = a \in [\varepsilon, \infty)$ of order $n \geqq 1$ and a zero of order $m \geqq 1$ at $x = b > a$ and $u(x) \neq 0$ on (a, b). Let $v(x)$ be a solution of* (1.1) *which is not zero on (a, b) and which does not have zeros at $x = a$ and $x = b$ of orders $\geqq n$ and $\geqq m$ respectively. There then exists a nontrivial linear combination of $u(x)$ and $v(x)$ which has a double zero on (a, b).*

In this paper it will be convenient to refer to the fundamental set of solutions $\{y_i(a; x)\}$, $i = 0, 1, 2, 3$, of (1.1) defined by the initial conditions

$$y_i^{(j)}(a; a) = \delta_{ij},$$

where $i, j = 0, 1, 2, 3$. Note that if $y(x)$ is a solution of (1.1) which vanishes at $x = a$, then $y(x)$ is a linear combination of $y_i(a; x)$, $i = 1, 2, 3$.

2. $r_{22} = r_{31} = \infty$. In this section we shall consider the behavior of oscillatory solutions of the system

$$\begin{gathered} y^{(4)} + p_3(x)y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0, \\ r_{22} = r_{31} = \infty. \end{gathered} \tag{2.1}$$

A motivating example for this case is

$$y^{(4)} - y' = 0. \tag{2.2}$$

A fundamental set of solutions of (2.2) is

$$\{1, e^x, e^{px} \sin qx, e^{px} \cos qx\},$$

where $p = -1/2$ and $q = \sqrt{3}/2$. Throughout this section it is understood (though not always stated) that $r_{22} = r_{31} = \infty$ for $[\varepsilon, \infty)$.

It has been reported [3] that Aliev has shown that for $t \in [\varepsilon, \infty)$,

$$r_{31}(t) \geqq r_{211}(t) \geqq \min\{r_{22}(t), r_{31}(t)\}. \tag{2.3}$$

It then follows that under our standing hypotheses, $r_{211} = \infty$. The following lemma is then immediate.

LEMMA 2.1. *For equation* (2.1) *we have* $r_{211} = \infty$, *and all zeros of oscillatory solutions are simple zeros.*

LEMMA 2.2. *If two nontrivial solutions of* (2.1) *have three zeros* (*counting multiplicities*) *in common, they are constant multiples of each other.*

Proof. Suppose $u(x)$ and $v(x)$ are nontrivial solutions of (2.1) each having zeros at $x = \alpha$, $x = \beta$, and $x = \gamma$ with

$$\alpha \leqq \beta \leqq \gamma. \tag{2.4}$$

If equality holds throughout (2.4), then both $u(x)$ and $v(x)$ are constant multiples of $y_3(\alpha; x)$. If $\alpha = \beta < \gamma$, then $u''(\alpha)v''(\alpha) \neq 0$ and the solution

$$w_1(x) = u''(\alpha)v(x) - v''(\alpha)u(x)$$

has a 3–1 distribution of zeros on $[\alpha, \infty)$. Hence $w_1(x) \equiv 0$. If $\alpha < \beta = \gamma$, then $u'(\alpha)v'(\alpha) \neq 0$ and the solution

$$w_2(x) = u'(\alpha)v(x) - v'(\alpha)u(x)$$

has a 2–2 distribution of zeros on $[\alpha, \infty)$. Accordingly, $w_2(x) = 0$. Finally, suppose $\alpha < \beta < \gamma$. It follows that $u'(\alpha)v'(\alpha) \neq 0$, and the solution

$$w_3(x) = u'(\alpha)v(x) - v'(\alpha)u(x)$$

has a 2–1–1 distribution of zeros on $[\alpha, \infty)$. This concludes the proof of the lemma.

In the example mentioned above (2.2) there are two linearly independent oscillatory solutions whose zeros separate each other. Theorems 2.1 and 2.2 are motivated by that observation. Before proceeding with these theorems, we prove the following lemma.

LEMMA 2.3. *Suppose $u(x)$ and $v(x)$ are two nontrivial linearly independent solutions of* (2.1) *such that $u(\alpha) = v(\alpha) = u(\beta) = v(\beta) = 0$, where $a \leqq \beta$. Then the zeros of $u(x)$ and $v(x)$ separate on (ε, α). If $\alpha = \beta$, we mean $u(\alpha) = v(\alpha) = u'(\alpha) = v'(\alpha) = 0$.*

Proof. Assume the conclusion is false. To fix the ideas, suppose that there exist two consecutive zeros $x = a$ and $x = b$ of $u(x)$ such that $v(x) \neq 0$ on (a, b) and $a < b < \alpha$. By Lemma 2.2, $v(x) \neq 0$ on $[a, b]$. Applying Lemma 1.1, there exists a nontrivial linear combination $w(x)$ of $u(x)$ and $v(x)$ such that $w(x)$ has a double zero on (a, b). Hence $w(x)$ has either a 2–2 or a 2–1–1 distribution of zeros on (ε, ∞) according as $\alpha = \beta$ or $\alpha < \beta$, respectively. This contradiction concludes the proof of the lemma.

THEOREM 2.1. *Suppose there exists an oscillatory solution $y(x)$ of* (2.1). *There then exists an oscillatory solution $z(x)$ of* (2.1) *such that $z(x)$ and $y(x)$ are linearly independent.*

Proof. Let $x = b$ be a point for which $y(b) \neq 0$, and let $\{a_n\}_{n=1}^{\infty}$ denote the increasing sequence of zeros of $y(x)$ with $b < a_1$. Define a sequence of solutions $\{z_n(x)\}_{n=1}^{\infty}$ of (2.1) by the boundary conditions

$$z_n(a_n) = z_n(a_{n+1}) = z_n(b) = 0. \tag{2.5}$$

Without loss of generality for each n, let c_{1n}, c_{2n}, c_{3n} be constants such that

$$c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$$

and

$$z_n(x) = c_{1n}y_1(b;x) + c_{2n}y_2(b;x) + c_{3n}y_3(b;x). \tag{2.6}$$

If we consider the sequence of vectors

$$\{(c_{1n}, c_{2n}, c_{3n})\}_{n=1}^{\infty}$$

on the unit ball, it follows that there exists a subsequence

$$\{(c_{1n_k}, c_{2n_k}, c_{3n_k})\}_{k=1}^{\infty}$$

which converges to a vector (c_1, c_2, c_3) on the unit ball. Define

$$z(x) \equiv c_1y_1(b;x) + c_2y_2(b;x) + c_3y_3(b;x).$$

Since $c_1^2 + c_2^2 + c_3^2 = 1$, $z(x)$ is a nontrivial solution of (2.1). Furthermore, $\{z_{n_k}(x)\}_{k=1}^{\infty}$ converges uniformly to $z(x)$ on compact subsets of (ε, ∞) by (2.6). That $y(x)$ and $z(x)$ are linearly independent follows immediately from

$$z(b) = 0 \neq y(b).$$

It remains to show that $z(x)$ is oscillatory. Recall from (2.5) that $z_n(x)$ and $y(x)$ have two zeros in common. By Lemma 2.3, the zeros of $z_n(x)$ and $y(x)$ separate each other on the interval (ε, a_n). If $\alpha < \beta$ are consecutive zeros of $y(x)$, then $z_{n_k}(x)$ has a zero on (α, β) for all k large enough. It follows then that $z(x)$ is oscillatory. This concludes the proof of the theorem.

It is clear from the proof of Theorem 2.1 that $z(x)$ has a zero on the interval $[\alpha, \beta]$, where $y(\alpha) = y(\beta) = 0$ and $y(x) \neq 0$ on (α, β). Suppose for some choice of constants c_1 and c_2 the solution

$$w(x) = c_1z(x) + c_2y(x) \not\equiv 0$$

is nonoscillatory. Note $c_1c_2 \neq 0$ and assume without loss of generality that $w(x) > 0$ for large x. Then

$$c_1z(x) > -c_2y(x) \tag{2.7}$$

for large x. By Lemma 2.1 we may choose n large enough so that

$$\operatorname{sgn} c_1 \neq \operatorname{sgn} y(x)$$

on the intervals (a_{n+2k}, a_{n+2k+1}), $k = 0, 1, 2, \cdots$. Then $z(x)$ does not vanish on $[a_{n+2k}, a_{n+2k+1}]$ by (2.7). These observations constitute a proof of the following theorem.

THEOREM 2.2. *Every linear combination of* $y(x)$ *and* $z(x)$ *is oscillatory.*

It should be noted that

$$W[y(x), z(x)] = y'(x)z(x) - z'(x)y(x)$$

does not vanish. If $W[y(x), z(x)] = 0$ for $x = b$, then there would exist a nontrivial linear combination $w(x)$ of $y(x)$ and $z(x)$ having a double zero at $x = b$. By Lemma 2.1, $w(x)$ would be nonoscillatory contradicting Theorem 2.2. Since $W[y(x), z(x)] \neq 0$, it follows easily from Lemma 1.1 that the zeros of $y(x)$ and $z(x)$ separate each other on (ε, ∞).

In the above example (2.2), the reader may note that all oscillatory solutions of (2.1) are linear combinations of $y(x) = e^{px} \sin q(x)$ and $z(x) = e^{px} \cos q(x)$. The question that presents itself is whether or not a similar statement can be made concerning the solutions $y(x)$ of $z(x)$ of Theorem 2.1. Theorem 2.2 provides only a partial answer to this question. The converse of Theorem 2.2 remains an open question.

3. $r_{22} = r_{13} = \infty$. We now consider the system

$$y^{(4)} + p_3(x)y''' + p_2(x)y'' + p_1(x)y' + p_0(x)y = 0, \qquad r_{22} = r_{13} = \infty. \tag{3.1}$$

The motivating example for this case is

$$y^{(4)} + y' = 0 \tag{3.2}$$

which has

$$\{1, e^{-x}, e^{px} \sin qx, e^{px} \cos qx\}$$

($p = 1/2$, $q = \sqrt{3}/2$) as a fundamental set.

The following inequality is known (see [3]):

$$r_{112}(t) \geqq \min [r_{22}(t), r_{13}(t)]. \tag{3.3}$$

We state the following lemma for reference; it follows immediately from (3.3).

LEMMA 3.1. $r_{22} = r_{13} = \infty$ *implies* $r_{112} = \infty$.

A proof similar to that of Lemma 2.2 may be constructed for the following lemma.

LEMMA 3.2. *Two nontrivial solutions of* (3.1) *with three zeros* (*counting multiplicities*) *in common are constant multiples of each other.*

LEMMA 3.3. *If equation* (3.1) *admits an oscillatory solution* $y(x)$, *then any nontrivial solution* $z(x)$ *with two zeros* (*counting multiplicities*) *in common with* $y(x)$ *is also oscillatory. Furthermore, if* $z(x)$ *and* $y(x)$ *are not constant multiples of each other, their zeros separate each other for large* x.

Proof. Suppose $z(x)$ has two zeros in common with $y(x)$. If $\alpha < \beta$ are consecutive zeros of $y(x)$ and α is large enough, then $z(x)$ must vanish on (α, β). Otherwise, Lemma 1.1 would imply the existence of a nontrivial solution with either a 2–2 or a 1–1–2 distribution of zeros on (ε, ∞) contradicting either the standing hypotheses or Lemma 3.1. A similar argument shows that $y(x)$ has a zero between consecutive zeros of $z(x)$. The proof is then complete.

In the previous section of this paper it was shown that $r_{22} = r_{31} = \infty$ implied that the zeros of oscillatory solutions are all simple zeros. The following theorem states that this is not the case under the assumptions $r_{22} = r_{13} = \infty$. Whereas $r_{22} = r_{31} = \infty$ implies that solutions with multiple zeros are nonoscillatory, the opposite is true if $r_{22} = r_{13} = \infty$.

THEOREM 3.1. *If* $y(x)$ *is an oscillatory solution of* (3.1), *then any nontrivial solution which vanishes twice* (*counting multiplicities*) *is oscillatory.*

Proof. Let $z(x)$ be a nontrivial solution of (3.1) which vanishes twice. If $z(x)$ has two zeros in common with $y(x)$, we may apply Lemma 3.3. Assume then $z(x)$ has two zeros at $x = \alpha$ and $x = \beta$, $\alpha \leqq \beta$, and at most one zero in common with $y(x)$. There then exist points $x = a$ and $x = b$ for which $y(a) = y(b) = 0$ and $z(a)z(b) \neq 0$. Let $w(x)$ be a nontrivial solution of (3.1) which satisfies the boundary conditions

$$w(\beta) = w(a) = w(b) = 0.$$

By Lemma 3.3, $w(x)$ is oscillatory. Let $u(x)$ be a nontrivial solution of (3.1) satisfying the boundary conditions

$$u(\alpha) = u(\beta) = u(a) = 0.$$

Accordingly, $u(x)$, hence $z(x)$, is oscillatory by Lemma 3.3. This completes the proof of the theorem.

In view of the previous theorem it is not clear whether or not nonoscillatory solutions exist. Clearly, solutions with multiple zeros will be oscillatory if (3.1) is oscillatory. The following theorem shows that nonoscillatory solutions always exist if $r_{22} = r_{13} = \infty$.

THEOREM 3.2. *Suppose there exists an oscillatory solution* $y(x)$ *of* (3.1). *Given* $a \in (\varepsilon, \infty)$ *there then exists a nonoscillatory solution* $z(x)$ *such that* $z(a) = 0$.

Proof. Let $a \in (\varepsilon, \infty)$, and let $\{a_n\}_{n=1}^{\infty}$ be the increasing sequence of distinct zeros of $y(x)$ with $a \leqq a_1$. Define a sequence $\{z_n(x)\}_{n=1}^{\infty}$ of solutions of (3.1) by the boundary conditions

$$z_n(a) = z_n(b_n) = z_n'(b_n) = 0, \tag{3.4}$$

where $b_n = (a_n + a_{n+1})/2$, $n = 1, 2, 3, \cdots$. Then $z_n(x) \neq 0$ if $x < b_n$ and $x \neq a$. Furthermore, we may choose $z_n(x)$ so that there exist constants c_{1n}, c_{2n}, c_{3n} such that

$$c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$$

and

$$z_n(x) = c_{1n}y_1(a; x) + c_{2n}y_2(a; x) + c_{3n}y_3(a; x). \tag{3.5}$$

Considering the sequence of vectors $\{(c_{1n}, c_{2n}, c_{3n})\}_{n=1}^{\infty}$ on the unit sphere in three-space, there exists a subsequence

$$\{(c_{1n_k}, c_{2n_k}, c_{3n_k})\}_{k=1}^{\infty}$$

which converges to a vector (c_1, c_2, c_3) on the unit sphere. Consider the solution

$$z(x) \equiv c_1 y_1(a; x) + c_2 y_2(a; x) + c_3 y_3(a; x).$$

By (3.5), the sequence $\{z_{n_k}(x)\}_{k=1}^{\infty}$ converges uniformly to $z(x)$ on compact subsets of (ε, ∞). Note $z(a) = 0$.

To complete the proof of the theorem it is necessary and sufficient to show that $z(x)$ has no zeros other than $x = a$. Suppose $z(b) = 0, a \neq b$. Then by Theorem 3.1 there exists a point $x = c > \max\{a, b\}$ such that $z(c) = 0$. Since $r_{112} = \infty$, it follows that $z'(c) \neq 0$, and hence $z(x)$ must change sign as x passes through the point $x = c$. In order to fix the ideas, for x sufficiently close to $x = c$, let

$$z(x) > 0 \quad \text{if } x > c,$$
$$z(x) < 0 \quad \text{if } x < c.$$

Choose x_1 and x_2 sufficiently close to $x = c$ and $x_1 < c < x_2$. Then there exist K_1 and K_2 such that $z_{n_k}(x_1) < 0$ for all $k > K_1$ and $z_{n_k}(x_2) > 0$ for all $k > K_2$. For $k > \max\{K_1, K_2\}$, we have

$$z_{n_k}(x_1) < 0 < z_{n_k}(x_2).$$

Accordingly, $z_{n_k}(x)$ vanishes on the interval (x_1, x_2) for all $k > \max\{K_1, K_2\}$. But $z_{n_k}(x)$ also satisfies the boundary conditions (3.4). For k large enough, $z_{n_k}(x)$ would then have a 1–1–2 distribution of zeros. This contradiction completes the proof of the theorem.

REFERENCES

[1] J. Barrett, *Oscillation theory of ordinary linear differential equations*, Advances in Math., 3 (1969), pp. 415–509.

[2] W. Leighton and Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc., 89 (1958), pp. 325–377.

[3] A. Peterson, *Distribution of zeros of solutions of a fourth order differential equation*, Pacific J. Math., 30 (1969), pp. 751–764.

[4] ———, *A theorem of Aliev*, Proc. Amer. Math. Soc., 23 (1969), pp. 364–366.

[5] J. Ridenhour and T. Sherman, *Conjugate points for fourth order linear differential equations*, to appear.

[6] L. Schneider, *Oscillatory properties of the fourth order linear homogeneous formally self-adjoint differential equation*, Doctoral thesis, Case Western Reserve Univ., Cleveland, Ohio, 1971.

[7] ———, *Oscillation properties of the 2–2 disconjugate fourth order self-adjoint differential equation*, to appear.

[8] C. Swanson, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

ON THE SYMMETRIZED KRONECKER POWER OF A MATRIX AND EXTENSIONS OF MEHLER'S FORMULA FOR HERMITE POLYNOMIALS*

DAVID SLEPIAN†

Abstract. Mehler's formula expresses the exponential of a quadratic form in two variables as a series of products of Hermite polynomials. We give several useful generalizations of this formula to the case of n variables, being guided in this work by interpretations in terms of Gaussian variates. Along the way we encounter the symmetrized Kronecker power of a matrix and we present a new generating function and recipe for calculating this quantity.

1. Introduction, notation, and main results. For a probabilist, the natural way to define the Hermite polynomials is in terms of the all-important standard Gaussian density

$$\varphi(z) = e^{-z^2/2}/\sqrt{2\pi}. \tag{1}$$

In this connection, it is convenient to denote derivatives by subscripts,

$$\varphi_l(z) \equiv \frac{d^l}{dz^l}\varphi(z), \qquad l = 0, 1, \cdots, \tag{2}$$

with $\varphi_0 = \varphi$. The Hermite polynomials are then defined[1] by

$$H_\nu(z) = (-1)^\nu \varphi_\nu(z)/\varphi(z), \qquad \nu = 0, 1, 2, \cdots. \tag{3}$$

Mehler's formula, the subject of this paper, asserts that

$$\frac{1}{\sqrt{1-\gamma^2}} \exp\left\{-\frac{\gamma^2(x^2+y^2) - 2\gamma xy}{2(1-\gamma^2)}\right\} = \sum_{\nu=0}^{\infty} \frac{\gamma^\nu}{\nu!} H_\nu(x) H_\nu(y). \tag{4}$$

On using (3) and (2), (4) can be written as

$$p(x, y) \equiv \frac{1}{2\pi\sqrt{1-\gamma^2}} \exp\left\{-\frac{x^2+y^2-2\gamma xy}{2(1-\gamma^2)}\right\} = \sum_{\nu=0}^{\infty} \frac{\gamma^\nu}{\nu!} \varphi_\nu(x)\varphi_\nu(y). \tag{5}$$

This formula shows that the probability density $p(x, y)$ for two jointly Gaussian variates, each of unit variance, has a simple Maclaurin series in their correlation coefficient, γ. It is this notion that generalizes easily to higher dimensions.

To present our results, we must first introduce some notation. Boldface lower-case Greek letters, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, etc., will be used to denote matrices; boldface lower-case Latin letters, $\mathbf{l}$, $\mathbf{m}$, etc., will denote column vectors. If $\boldsymbol{\nu}$ is a matrix with n_1

* Received by the editors August 10, 1971.

† Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey 07974.

[1] The definition given coincides with that of Cramér [1, p. 133]. Another normalization in common use ([2, p. 193], for example) calls $2^{\nu/2}H_\nu(\sqrt{2}z)$ the νth Hermite polynomial.

rows and n_2 columns, we write

$$\mathbf{r}(\mathbf{v}) = (r_1, r_2, \cdots, r_{n_1}), \tag{6}$$
$$r_i = \sum_{j=1}^{n_2} v_{ij}, \qquad i = 1, \cdots, n_1,$$

for the vector whose components are the row sums of $\mathbf{v}$, and we write

$$\mathbf{c}(\mathbf{v}) = (c_1, c_2, \cdots, c_{n_2}), \tag{7}$$
$$c_j = \sum_{i=1}^{n_1} v_{ij}, \qquad j = 1, \cdots, n_2,$$

for the vector whose components are the column sums of $\mathbf{v}$. Throughout we adopt the convenient abbreviations

$$\boldsymbol{\mu}^{\mathbf{v}} \equiv \prod_{i,j} \mu_{ij}^{v_{ij}}, \qquad \mathbf{r}^{\mathbf{l}} \equiv \prod_i r_i^{l_i},$$
$$\boldsymbol{\mu}! \equiv \prod_{i,j} \mu_{ij}!, \qquad \mathbf{l}! \equiv \prod_i l_i!, \tag{8}$$
$$\sum_{\mathbf{v}=0}^{\infty} \equiv \sum_{v_{11}=0}^{\infty} \sum_{v_{12}=0}^{\infty} \cdots \sum_{v_{n_1 n_2}=0}^{\infty}, \qquad \sum_{\mathbf{l}=0}^{\infty} \equiv \sum_{l_1=0}^{\infty} \cdots \sum_{l_n=0}^{\infty},$$

where the entries of $\boldsymbol{\mu}$ are μ_{ij}, the components of $\mathbf{l}$ are l_i, etc. We call a matrix of nonnegative integers, such as $\mathbf{v}$ in the last line of (8), an *index matrix*. If

$$\mathbf{l} = (l_1, l_2, \cdots, l_n) \tag{9}$$

is an n-vector, we write

$$[\mathbf{l}] = l_1 + l_2 + \cdots + l_n \tag{10}$$

and we call $[\mathbf{l}]$ the *weight* of $\mathbf{l}$. Finally, we introduce the function

$$\varphi(\mathbf{z}; \boldsymbol{\rho}) \equiv \frac{1}{(2\pi)^{n/2}|\boldsymbol{\rho}|^{1/2}} \exp\{-\tfrac{1}{2}\tilde{\mathbf{z}}\boldsymbol{\rho}^{-1}\mathbf{z}\} \tag{11}$$

and, in analogy with (2), its derivatives

$$\varphi_{\mathbf{l}}(\mathbf{z}; \boldsymbol{\rho}) \equiv \frac{\partial^{[\mathbf{l}]}}{\partial z_1^{l_1} \partial z_2^{l_2} \cdots \partial z_n^{l_n}} \varphi(\mathbf{z}; \boldsymbol{\rho}). \tag{12}$$

In these definitions, $\boldsymbol{\rho}$ is a positive definite $n \times n$ matrix with determinant $|\boldsymbol{\rho}|$, and $\mathbf{z}$ and $\mathbf{l}$ are n-vectors, the latter having nonnegative integers as components. The tilde denotes transpose.

With these preliminaries out of the way, we can state our main result. Let the $n \times n$ positive definite symmetric matrix $\boldsymbol{\rho}$ have the partitioned structure

$$\boldsymbol{\rho} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \tilde{\boldsymbol{\gamma}} & \boldsymbol{\beta} \end{pmatrix}. \tag{13}$$

Here $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are square matrices of n_1 and n_2 rows respectively, $\boldsymbol{\gamma}$ is a matrix of n_1 rows and n_2 columns, and $n = n_1 + n_2$. Let $\mathbf{x}$ be an n_1-vector and $\mathbf{y}$ an n_2-vector and denote by

$$\mathbf{z} = \mathbf{x} \oplus \mathbf{y} \tag{14}$$

the n-vector whose first n_1 components are $x_1, x_2, \cdots, x_{n_1}$ and whose last n_2 components are $y_1, y_2, \cdots, y_{n_2}$. Then

$$p(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) \equiv \varphi(\mathbf{z}; \boldsymbol{\rho}) = \sum_{\mathbf{v}=0}^{\infty} \frac{\boldsymbol{\gamma}^{\mathbf{v}}}{\mathbf{v}!} \varphi_{\mathbf{r}(\mathbf{v})}(\mathbf{x}; \boldsymbol{\alpha}) \varphi_{\mathbf{c}(\mathbf{v})}(\mathbf{y}; \boldsymbol{\beta}) \tag{15}$$

in analogy with (5). When $n = 2$ and $\alpha = \beta = 1$, (15) does indeed yield (5). Equation (15) gives the power series expansion for the probability density of a jointly Gaussian n_1-vector and an n_2-vector in terms of the correlation coefficients γ_{ij} between the vectors, and is a natural extension of (5).

In § 3, we show how a formula closely related to (15) can be used to generate identities in Hermite polynomials.

In § 4 we study further the functions $\varphi_{\mathbf{l}}(\mathbf{z}; \boldsymbol{\rho})$ of (12) and evaluate certain integrals of products of these functions. In § 6 we reconsider (15) and show how to find the eigenvalues and eigenfunctions of the equation

$$\int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_{n_1} p(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) \psi(\mathbf{y}) w(\mathbf{y})\, dy_1 \cdots dy_{n_1} = \lambda \psi(\mathbf{x}) \tag{16}$$

in the important case $\boldsymbol{\alpha} = \boldsymbol{\beta}$, $w(\mathbf{y}) = 1/\varphi(\mathbf{y}; \boldsymbol{\alpha})$. Here certain irreducible representations of the real general linear group unexpectedly make an appearance, and we exhibit in § 5 a simple generating function for the matrices of these representations that does not seem to be noted in the literature. The Mercer expansion of the kernel of (16) is yet another generalization of the Mehler formula (5).

2. Derivation of formula (15). The multivariate Gaussian density has the well-known Fourier representation [1, p. 118]

$$\varphi(\mathbf{z}; \boldsymbol{\rho}) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} dt_1 \cdots \int_{-\infty}^{\infty} dt_n \exp\{i \textstyle\sum t_j z_j\} \exp\left\{-\frac{1}{2} \sum \rho_{jk} t_j t_k\right\}. \tag{17}$$

By differentiating behind the integral signs, we see that for $j \neq k$,

$$\frac{\partial \varphi(\mathbf{z}; \boldsymbol{\rho})}{\partial \rho_{jk}} = \frac{\partial^2 \varphi(\mathbf{z}; \boldsymbol{\rho})}{\partial z_j \partial z_k}, \tag{18}$$

where we take account of the fact that $\rho_{kj} = \rho_{jk}$ and consider $\varphi(\mathbf{z}; \boldsymbol{\rho})$ as a function of the $n(n-1)/2$ quantities ρ_{jk} for $j > k$.

Now if $\boldsymbol{\rho}$ has the form (13), the multiple power series for $\varphi(\mathbf{z}; \boldsymbol{\rho})$ in the quantities $\gamma_{11}, \gamma_{12}, \cdots, \gamma_{n_1 n_2}$ is

$$\varphi(\mathbf{z}; \boldsymbol{\rho}) = \sum_{\mathbf{v}=0}^{\infty} \frac{\boldsymbol{\gamma}^{\mathbf{v}}}{\mathbf{v}!} \frac{\partial^{v_{11} + \cdots + v_{n_1 n_2}}}{\partial \gamma_{11}^{v_{11}} \cdots \partial \gamma_{n_1 n_2}^{v_{n_1 n_2}}} \varphi(\mathbf{z}; \boldsymbol{\rho}) \Bigg|_{\boldsymbol{\gamma}=0}. \tag{19}$$

But (18) in the present case gives

$$\frac{\partial \varphi(\mathbf{z};\boldsymbol{\rho})}{\partial \gamma_{jk}} = \frac{\partial^2 \varphi(\mathbf{z};\boldsymbol{\rho})}{\partial x_j \partial y_k} \tag{20}$$

when (13) and (14) are kept in mind. Replacing the derivatives with respect to the γ's in (19) by derivatives with respect to x and y by use of formula (20) yields

$$\varphi(\mathbf{z};\boldsymbol{\rho}) = \sum_{\mathbf{v}=0}^{\infty} \frac{\boldsymbol{\gamma}^{\mathbf{v}}}{\mathbf{v}!} \frac{\partial^{[\mathbf{r}(\mathbf{v})]}}{\partial x_1^{r_1} \cdots \partial x_{n_1}^{r_{n_1}}} \frac{\partial^{[\mathbf{c}(\mathbf{v})]}}{\partial y_1^{c_1} \cdots \partial y_{n_2}^{c_{n_2}}} \varphi(\mathbf{z};\boldsymbol{\rho}) \Big|_{\boldsymbol{\gamma}=0}, \tag{21}$$

where we have used the definitions (6), (7) and (10). But since no differentiation with respect to $\boldsymbol{\gamma}$ is indicated in (21), the evaluation at $\boldsymbol{\gamma} = 0$ shown can be effected before differentiation. Since

$$\varphi(\mathbf{z};\boldsymbol{\rho})|_{\boldsymbol{\gamma}=0} = \varphi(\mathbf{x};\boldsymbol{\alpha})\varphi(\mathbf{y};\boldsymbol{\beta}),$$

(15) results when (12) is used, and our derivation is complete.

3. Hermite polynomial identities. Equation (18) and the technique of the last section allow one to expand $\varphi(\mathbf{z};\boldsymbol{\rho})$ in a power series in the off-diagonal elements of $\boldsymbol{\rho}$. One finds immediately

$$\varphi(\mathbf{z};\boldsymbol{\rho}) = \sum_{\nu_{12}=0}^{\infty} \cdots \sum_{\nu_{n-1,n}=0}^{\infty} \prod_{i<j} \frac{\rho_{ij}^{\nu_{ij}}}{\nu_{ij}!} \prod_{k=1}^{n} \frac{\partial^{s_k}}{\partial z_k^{s_k}} \varphi\left(\frac{z_k}{\sqrt{\rho_{kk}}}\right), \tag{22}$$

where

$$\begin{aligned} s_k &= \sum_{j\neq k} \nu_{jk}, & k &= 1,2,\cdots,n, \\ \nu_{jk} &\equiv \nu_{kj}, & j,k &= 1,2,\cdots,n. \end{aligned} \tag{23}$$

If now

$$\rho_{ii} = 1, \qquad i = 1,\cdots,n, \tag{24}$$

equation (22) becomes

$$\varphi(\mathbf{z};\boldsymbol{\rho}) = \sum_{\mathbf{v}} \frac{\boldsymbol{\rho}^{\mathbf{v}}}{\mathbf{v}!} \prod_{1}^{n} \varphi_{s_i}(z_i), \tag{25}$$

in a symbolic notation. Dividing by $\prod \varphi(z_i)$, we find finally

$$\frac{\exp\{-\frac{1}{2}\sum(\rho_{ij}^{-1} - \delta_{ij})z_i z_j\}}{|\boldsymbol{\rho}|^{1/2}} = \sum_{\mathbf{v}} \frac{\boldsymbol{\rho}^{\mathbf{v}}}{\mathbf{v}!} H_{s_1}(z_1) \cdots H_{s_n}(z_n), \tag{26}$$

where ρ_{ij}^{-1} is the element in the ith row and jth column of $\boldsymbol{\rho}^{-1}$ and δ_{ij} is the usual Kronecker symbol. Note that if an off-diagonal element ρ_{ij} is zero, the corresponding summation index ν_{ij} can be omitted from (26) and from the definition (23) of the s's.

Equation (26) can be used to generate Hermite polynomial identities indefinitely. Every symmetric positive definite matrix for which the inverse can be written explicitly gives rise to such a formula. We illustrate with a few simple examples.

Example 1. Let

$$\rho_{ii} = 1, \quad i = 1, 2, \cdots, n,$$
$$\rho_{1j} = t_j, \quad j = 2, 3, \cdots, n,$$
$$\rho_{ij} = 0, \quad 1 < i < j.$$

The inverse is readily found to have elements $\rho_{ij}^{-1} = a_{ij}/d$, where

$$d = |\boldsymbol{\rho}| = 1 - \sum_2^n t_j^2,$$
$$a_{11} = 1,$$
$$a_{1j} = -t_j, \qquad j = 2, \cdots, n,$$
$$a_{ii} = d + t_i^2, \qquad i = 2, \cdots, n,$$
$$a_{ij} = t_i t_j, \qquad 1 < i < j,$$

and, of course, $\rho_{ij}^{-1} = \rho_{ji}^{-1}$. In (26), let $\nu_{1j} = k_j$ for $j = 2, 3, \cdots, n$ and set the remaining ν's equal to zero. We then have

$$\sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \frac{t_2^{k_2} \cdots t_n^{k_n}}{k_2! \cdots k_n!} H_{k_2+k_3+\cdots+k_n}(z_1) H_{k_2}(z_2) H_{k_3}(z_3) \cdots H_{k_n}(z_n)$$
$$= \frac{1}{\sqrt{d}} \exp\left\{-\frac{1}{2d}\left[\sum_2^n t_j^2(z_1^2 + z_j^2) - 2z_1 \sum_2^n t_j z_j + 2 \sum_{2<i<j} t_i t_j z_i z_j\right]\right\}$$

which is the general case of equation (1.3) of [3].

Example 2. For the general 3×3 matrix satisfying (24),

$$\rho = \begin{pmatrix} 1 & t_3 & t_2 \\ t_3 & 1 & t_1 \\ t_2 & t_1 & 1 \end{pmatrix},$$
$$\Delta = |\boldsymbol{\rho}| = 1 - t_1^2 - t_2^2 - t_3^2 + 2t_1t_2t_3,$$
$$\rho^{-1} = \frac{1}{\Delta}\begin{pmatrix} 1 - t_1^2 & t_1t_2 - t_3 & t_1t_3 - t_2 \\ t_1t_2 - t_3 & 1 - t_2^2 & t_2t_3 - t_1 \\ t_1t_3 - t_2 & t_2t_3 - t_1 & 1 - t_3^2 \end{pmatrix}.$$

In (26) set $\nu_{12} = k_3$, $\nu_{13} = k_2$, $\nu_{23} = k_1$. Then

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} \frac{t_1^{k_1} t_2^{k_2} t_3^{k_3}}{k_1! k_2! k_3!} H_{k_2+k_3}(z_1) H_{k_3+k_1}(z_2) H_{k_1+k_2}(z_3)$$
$$(27) \qquad = \frac{1}{\sqrt{\Delta}} \exp\left\{-\frac{1}{2\Delta}\left[z_1^2(t_2^2 + t_3^2) + z_2^2(t_1^2 + t_3^2) + z_3^2(t_1^2 + t_2^2) - 2t_1t_2t_3 \sum z_i^2\right.\right.$$
$$\left.\left. + 2\sum_{i<j} t_i t_j z_i z_j - 2t_3 z_1 z_2 - 2t_2 z_1 z_3 - 2t_1 z_2 z_3\right]\right\}$$

which is (1.4) of [3].

4. The n-dimensional Hermite functions. We shall call $\varphi_{\mathbf{l}}(\mathbf{z};\boldsymbol{\rho})$ as given by (12) an n-dimensional *Hermite function of weight* $[\mathbf{l}]$. In terms of these functions, and in analogy with (3), we can define homogeneous polynomials of total degree $[\mathbf{l}]$ in n variables by the formula

$$H_{\mathbf{l}}(\mathbf{z}) \equiv (-1)^{[\mathbf{l}]}\varphi_{\mathbf{l}}(\mathbf{z};\boldsymbol{\rho})/\varphi(\mathbf{z};\boldsymbol{\rho}). \tag{28}$$

Except for scaling, these are identical with those first studied by Hermite. Many details of their properties can be found in [2, pp. 283–291] or in [4].

The polynomials (28) for different $\mathbf{l}$'s are linearly independent, so the same is true of the Hermite functions. There are

$$N(n,p) = \binom{n+p-1}{p} \tag{29}$$

such functions of weight p since this is just the number of n-vectors $\mathbf{l}$ with non-negative integer components that sum to $p = [\mathbf{l}]$.

The Hermite functions have a simple generating function

$$\varphi(\mathbf{z}+\mathbf{t};\boldsymbol{\rho}) = \sum_{\mathbf{l}=0}^{\infty} \frac{\mathbf{t}^{\mathbf{l}}}{\mathbf{l}!}\varphi_{\mathbf{l}}(\mathbf{z};\boldsymbol{\rho}), \tag{30}$$

which is just Taylor's theorem in many variables. By means of this formula we can evaluate an integral that will be of use to us later.

Consider

$$I = \int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_n \varphi(\mathbf{z}+\mathbf{s};\boldsymbol{\rho})\varphi(\mathbf{z}+\mathbf{t};\boldsymbol{\rho})w(\mathbf{z};\boldsymbol{\rho}), \tag{31}$$

where the positive weight function w is

$$w(\mathbf{z};\boldsymbol{\rho}) = 1/\varphi(\mathbf{z};\boldsymbol{\rho}). \tag{32}$$

Using (30), we find at once that

$$I = \sum_{\mathbf{l},\mathbf{m}=0}^{\infty} \frac{\mathbf{s}^{\mathbf{l}}}{\mathbf{l}!}\frac{\mathbf{t}^{\mathbf{m}}}{\mathbf{m}!}\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_n \varphi_{\mathbf{l}}(\mathbf{z};\boldsymbol{\rho})\varphi_{\mathbf{m}}(\mathbf{z};\boldsymbol{\rho})w(\mathbf{z};\boldsymbol{\rho}). \tag{33}$$

On the other hand, (31) can be evaluated directly using the definitions of $\varphi(\mathbf{z};\boldsymbol{\rho})$ as given by (11). From this formula we find the algebraic identity

$$\frac{\varphi(\mathbf{z}+\mathbf{s};\boldsymbol{\rho})\varphi(\mathbf{z}+\mathbf{t};\boldsymbol{\rho})}{\varphi(\mathbf{z};\boldsymbol{\rho})} = \exp\{\textstyle\sum \rho_{ij}^{-1}s_i t_j\}\varphi(\mathbf{z}+\mathbf{s}+\mathbf{t};\boldsymbol{\rho})$$

so that

$$\begin{aligned} I &= \exp\{\textstyle\sum \rho_{ij}^{-1}s_i t_j\}\displaystyle\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_n \varphi(\mathbf{z}+\mathbf{s}+\mathbf{t};\boldsymbol{\rho}) \\ &= \exp\{\textstyle\sum \rho_{ij}^{-1}s_i t_j\}\displaystyle\int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_n \varphi(\mathbf{z};\boldsymbol{\rho}) = \exp\{\textstyle\sum \rho_{ij}^{-1}s_i t_j\} \end{aligned} \tag{34}$$

since $\varphi(\mathbf{z};\boldsymbol{\rho})$ integrates to unity over the whole space.

Now if $\boldsymbol{\alpha}$ is any $n \times n$ matrix, we have the expansion

$$\exp\{\sum \alpha_{ij} s_i t_j\} = \sum_{\boldsymbol{\mu}=0}^{\infty} \frac{\boldsymbol{\alpha}^{\boldsymbol{\mu}}}{\boldsymbol{\mu}!} \mathbf{s}^{\mathbf{r}(\boldsymbol{\mu})} \mathbf{t}^{\mathbf{c}(\boldsymbol{\mu})}.$$

Let $\mathbf{l} = \mathbf{r}(\boldsymbol{\mu})$, $\mathbf{m} = \mathbf{c}(\boldsymbol{\mu})$ in this sum and observe that $[\mathbf{l}] = [\mathbf{m}]$. There results

$$\exp\{\sum \alpha_{ij} s_i t_j\} = \sum_{p=0}^{\infty} \sum_{\substack{\mathbf{l},\mathbf{m} \\ [\mathbf{l}]=[\mathbf{m}]=p}}^{\infty} \boldsymbol{\sigma}_p(\boldsymbol{\alpha})_{\mathbf{lm}} \frac{\mathbf{s}^{\mathbf{l}}}{\sqrt{\mathbf{l}!}} \frac{\mathbf{t}^{\mathbf{m}}}{\sqrt{\mathbf{m}!}}, \tag{35}$$

where

$$\boldsymbol{\sigma}_p(\boldsymbol{\alpha})_{\mathbf{lm}} = \sqrt{\mathbf{l}!}\sqrt{\mathbf{m}!} \sum_{\substack{\boldsymbol{\mu} \\ \mathbf{r}(\boldsymbol{\mu})=\mathbf{l} \\ \mathbf{c}(\boldsymbol{\mu})=\mathbf{m} \\ [\mathbf{l}]=[\mathbf{m}]=p}} \frac{\boldsymbol{\alpha}^{\boldsymbol{\mu}}}{\boldsymbol{\mu}!}. \tag{36}$$

As indicated, the sum here is over all index matrices $\boldsymbol{\mu}$ with row-sum vector $\mathbf{l}$ and column-sum vector $\mathbf{m}$, these latter being of weight p.

Now use (35) to develop (34) in a power series in the s's and t's. Comparison with (33) gives the desired formula

$$\frac{1}{\sqrt{\mathbf{l}!}\sqrt{\mathbf{m}!}} \int_{-\infty}^{\infty} dz_1 \cdots \int_{-\infty}^{\infty} dz_n \varphi_{\mathbf{l}}(\mathbf{z};\boldsymbol{\rho}) \varphi_{\mathbf{m}}(\mathbf{z};\boldsymbol{\rho}) w(\mathbf{z};\boldsymbol{\rho}) = \delta_{[\mathbf{l}][\mathbf{m}]} \boldsymbol{\sigma}_{[\mathbf{l}]}(\boldsymbol{\rho}^{-1})_{\mathbf{lm}}, \tag{37}$$

where δ_{ij} is the usual Kronecker symbol. This formula shows that Hermite functions of different weights are orthogonal with respect to w and that the scalar product of any two functions of the same weight p can be expressed simply in terms of the elements of the $N(n, p) \times N(n, p)$ matrix $\boldsymbol{\sigma}_p(\boldsymbol{\rho}^{-1})$ defined by (36).

5. The symmetrized Kronecker product. We digress to establish the relationship between the matrix $\boldsymbol{\sigma}_p(\boldsymbol{\alpha})$ and certain well-known representations of the general linear group.

Let $\boldsymbol{\alpha}$ be an $n \times n$ matrix and $\mathbf{t}$ an n-vector. Let

$$\mathbf{t}' = \boldsymbol{\alpha}\mathbf{t}. \tag{38}$$

Then each of the $N(n, p)$ normalized homogeneous products

$$\mathbf{t}'^{\mathbf{l}}/\sqrt{\mathbf{l}!} \equiv \prod_1^n \frac{t_i'^{l_i}}{\sqrt{l_i!}}$$

of total degree $p = [\mathbf{l}]$ in the components of $\mathbf{t}'$ is transformed under (38) into a linear combination of the $N(n, p)$ normalized homogeneous products $\mathbf{t}^{\mathbf{m}}/\sqrt{\mathbf{m}!}$, $[\mathbf{m}] = [\mathbf{l}] = p$, of total degree p in the components of $\mathbf{t}$. The $N(n, p) \times N(n, p)$ matrix specifying these linear transformations between the normalized homogeneous products $\mathbf{t}'^{\mathbf{l}}/\sqrt{\mathbf{l}!}$ and $\mathbf{t}^{\mathbf{m}}/\sqrt{\mathbf{m}!}$ is known as the *symmetrized Kronecker p-th power of* $\boldsymbol{\alpha}$, [5, p. 77], which we here denote by $\boldsymbol{\kappa}_p(\boldsymbol{\alpha})$, so that

$$\frac{\mathbf{t}'^{\mathbf{l}}}{\sqrt{\mathbf{l}!}} = \sum_{\substack{\mathbf{m} \\ [\mathbf{m}]=[\mathbf{l}]=p}} \boldsymbol{\kappa}_p(\boldsymbol{\alpha})_{\mathbf{lm}} \frac{\mathbf{t}^{\mathbf{m}}}{\sqrt{\mathbf{m}!}}. \tag{39}$$

Now let $\boldsymbol{\beta}$ be another $n \times n$ matrix. Consider the transformation $\mathbf{t}'' = \boldsymbol{\beta}\mathbf{t}'$. The quantities $\mathbf{t}''^{\mathbf{l}}/\sqrt{\mathbf{l}!}$ are then linear combinations of the $\mathbf{t}^{\mathbf{m}}/\sqrt{\mathbf{m}!}$ and it is clear that

$$\boldsymbol{\kappa}_p(\boldsymbol{\beta\alpha}) = \boldsymbol{\kappa}_p(\boldsymbol{\beta})\boldsymbol{\kappa}_p(\boldsymbol{\alpha}). \tag{40}$$

Thus if $\boldsymbol{\alpha}$ is restricted to lie in a matrix group, $\boldsymbol{\kappa}_p(\boldsymbol{\alpha})$ furnishes a representation of the group. Representations of the general linear group obtained in this way have been frequently studied [5], [6].

Let us now return to (35) and there write $\mathbf{t}' = \boldsymbol{\alpha}\mathbf{t}$. We have

$$\begin{aligned} \exp\{\textstyle\sum \alpha_{ij}s_i t_j\} &= \sum_{p=0}^{\infty} \sum_{\substack{\mathbf{l},\mathbf{m} \\ [\mathbf{l}]=[\mathbf{m}]=p}} \boldsymbol{\sigma}_p(\boldsymbol{\alpha})_{\mathbf{lm}} \frac{\mathbf{s}^{\mathbf{l}}}{\sqrt{\mathbf{l}!}} \frac{\mathbf{t}^{\mathbf{m}}}{\sqrt{\mathbf{m}!}} \\ &= \exp\{\textstyle\sum s_i t'_i\} = \sum_{\mathbf{l}=0}^{\infty} \frac{\mathbf{s}^{\mathbf{l}}\mathbf{t}'^{\mathbf{l}}}{\mathbf{l}!} = \sum_{p=0}^{\infty} \sum_{\substack{\mathbf{l} \\ [\mathbf{l}]=p}} \frac{\mathbf{s}^{\mathbf{l}}}{\sqrt{\mathbf{l}!}} \frac{\mathbf{t}'^{\mathbf{l}}}{\sqrt{\mathbf{l}!}} \\ &= \sum_{p=0} \sum_{\substack{\mathbf{l} \\ [\mathbf{l}]=p}} \sum_{\substack{\mathbf{m} \\ [\mathbf{m}]=p}} \boldsymbol{\kappa}_p(\boldsymbol{\alpha})_{\mathbf{lm}} \frac{\mathbf{s}^{\mathbf{l}}}{\sqrt{\mathbf{l}!}} \frac{\mathbf{t}^{\mathbf{m}}}{\sqrt{\mathbf{m}!}}, \end{aligned} \tag{41}$$

where the last equality comes from the use of (39). Comparison of the second and last members of (41) shows that

$$\boldsymbol{\sigma}_p(\boldsymbol{\alpha}) = \boldsymbol{\kappa}_p(\boldsymbol{\alpha}) \tag{42}$$

so that (35) is a generating function for the symmetrized Kronecker powers of $\boldsymbol{\alpha}$ and (36) gives a convenient recipe for finding the matrix elements themselves. Neither formula seems to have been previously noted in the literature.

The following basic properties of the matrices $\boldsymbol{\sigma}_p(\boldsymbol{\alpha})$ that we shall need later can be derived readily from (35), (36), (40) and (42):

$$\begin{aligned} \boldsymbol{\sigma}_p(\tilde{\boldsymbol{\alpha}}) &= \tilde{\boldsymbol{\sigma}}_p(\boldsymbol{\alpha}); \\ \boldsymbol{\sigma}_p(\boldsymbol{\alpha\beta}) &= \boldsymbol{\sigma}_p(\boldsymbol{\alpha})\boldsymbol{\sigma}_p(\boldsymbol{\beta}); \\ \boldsymbol{\sigma}_p(\boldsymbol{\alpha}^{-1}) &= \boldsymbol{\sigma}_p(\boldsymbol{\alpha})^{-1}; \end{aligned} \tag{43}$$

if $\boldsymbol{\alpha} = \operatorname{diag}(a_1, a_2, \cdots, a_n)$, then $\boldsymbol{\sigma}_p(\boldsymbol{\alpha})_{\mathbf{lm}} = \mathbf{a}^{\mathbf{l}}\delta_{\mathbf{lm}}$.

6. Another extension of Mehler's formula. When $n = 1$ and $p_{11} = 1$, equation (37) shows that the functions $\psi_\nu(x) \equiv \varphi_\nu(x)/\sqrt{\nu!}$ are orthonormal on the infinite interval with respect to the weight function $w(x) \equiv 1/\varphi(x)$. Equation (5) in this notation,

$$p(x, y) = \sum_{\nu=0}^{\infty} \gamma^\nu \psi_\nu(x)\psi_\nu(y), \tag{44}$$

is then the Mercer expansion of the integral operator $\int_{-\infty}^{\infty} dy p(x, y)w(y)$. Thus the eigenvalues and eigenfunctions of

$$\int_{-\infty}^{\infty} p(x, y)\Phi(y)w(y)\, dy = \lambda\Phi(x) \tag{45}$$

are

$$\lambda_n = \gamma^n, \qquad \Phi_n(x) = \varphi_n(x)/\sqrt{n!}. \tag{46}$$

Let us see how this interpretation of Mehler's equation generalizes to n-dimensions.

In this section

$$\boldsymbol{\rho} = \begin{pmatrix} \boldsymbol{\alpha} & \boldsymbol{\gamma} \\ \tilde{\boldsymbol{\gamma}} & \boldsymbol{\alpha} \end{pmatrix} \tag{47}$$

is a $(2n) \times (2n)$ positive definite matrix, $\boldsymbol{\alpha}$ is an $n \times n$ positive definite matrix and $\boldsymbol{\gamma}$ is an $n \times n$ symmetric matrix. The basic result (15) becomes in this case

$$p(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) = \sum_{\mathbf{v}=0}^{\infty} \frac{\boldsymbol{\gamma}^{\mathbf{v}}}{\mathbf{v}!} \varphi_{\mathbf{r}(\mathbf{v})}(\mathbf{x}; \boldsymbol{\alpha}) \varphi_{\mathbf{c}(\mathbf{v})}(\mathbf{y}; \boldsymbol{\alpha}).$$

Now set $\mathbf{r}(\mathbf{v}) = \mathbf{l}$, $\mathbf{c}(\mathbf{v}) = \mathbf{m}$ and remember that therefore $[\mathbf{l}] = [\mathbf{m}]$. We find

$$p(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) = \sum_{p=0}^{\infty} \sum_{\substack{\mathbf{l},\mathbf{m} \\ [\mathbf{l}]=[\mathbf{m}]=p}} \boldsymbol{\sigma}_p(\boldsymbol{\gamma})_{\mathbf{lm}} \frac{\varphi_{\mathbf{l}}(\mathbf{x}; \boldsymbol{\alpha})}{\sqrt{\mathbf{l}!}} \frac{\varphi_{\mathbf{m}}(\mathbf{y}; \boldsymbol{\alpha})}{\sqrt{\mathbf{m}!}}, \tag{48}$$

where $\boldsymbol{\sigma}_p(\boldsymbol{\gamma})$ is the symmetrized Kronecker pth power of $\boldsymbol{\gamma}$ as given by (36).

Equation (48) is nearly in the diagonal form required of a Mercer expansion, for the Hermite functions $\varphi_{\mathbf{l}}(\mathbf{x}; \boldsymbol{\alpha})$ of different weights are orthogonal with respect to the weight function $w(\mathbf{x}; \boldsymbol{\alpha})$ of (32). To complete the diagonalization, we therefore seek new basis functions that will be linear combinations of the Hermite functions of a given weight, that will be orthonormal with respect to $w(\mathbf{x}; \boldsymbol{\alpha})$, and that will reduce (48) to diagonal form.

For each p we set

$$\psi_{\mathbf{k}}(\mathbf{x}; \boldsymbol{\alpha}) \equiv \sum_{\substack{\mathbf{l} \\ [\mathbf{l}]=p}} (\boldsymbol{\theta}_p^{-1})_{\mathbf{kl}} \frac{\varphi_{\mathbf{l}}(\mathbf{x}; \boldsymbol{\alpha})}{\sqrt{\mathbf{l}!}} \tag{49}$$

to hold for all index vectors $\mathbf{k}$ of weight p. We assume the matrix $\boldsymbol{\theta}_p$ has an inverse. Then (37) shows that the ψ's will be orthonormal with respect to w if

$$\boldsymbol{\theta}_p^{-1} \boldsymbol{\sigma}_p(\boldsymbol{\alpha}^{-1}) \tilde{\boldsymbol{\theta}}_p^{-1} = \mathbf{I}$$

with $\mathbf{I}$ the unit matrix, or equivalently, because of the properties of $\boldsymbol{\sigma}_p(\boldsymbol{\alpha})$ shown in (43),

$$\tilde{\boldsymbol{\theta}}_p \boldsymbol{\sigma}_p(\boldsymbol{\alpha}) \boldsymbol{\theta}_p = \mathbf{I}. \tag{50}$$

The condition that (48) be diagonal in the ψ's is seen to be equivalent to

$$\tilde{\boldsymbol{\theta}}_p \boldsymbol{\sigma}_p(\boldsymbol{\gamma}) \boldsymbol{\theta}_p = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_{N(n,p)}). \tag{51}$$

Thus we seek a matrix $\boldsymbol{\theta}_p$ that will simultaneously diagonalize the symmetric matrices $\boldsymbol{\sigma}_p(\boldsymbol{\alpha})$ and $\boldsymbol{\sigma}_p(\boldsymbol{\gamma})$.

Solutions of the diagonalization problem just mentioned are well known [7, p. 58], [8, p. 171, also Chap. 22] and these techniques could be applied directly to $\boldsymbol{\sigma}_p(\boldsymbol{\alpha})$ and $\boldsymbol{\sigma}_p(\boldsymbol{\gamma})$. However, the representational nature (43) of the Kronecker power permits us to find $\boldsymbol{\theta}_p$ by carrying out the diagonalization for the smaller matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Thus let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ and $e_1, e_2, \cdots, e_n$ be the eigenvectors and eigenvalues of $\boldsymbol{\alpha}^{-1}\boldsymbol{\gamma}$, so that

$$\boldsymbol{\gamma}\mathbf{v}_i = e_i\boldsymbol{\alpha}\mathbf{v}_i, \qquad i = 1, 2, \cdots, n. \tag{52}$$

The $\mathbf{v}$'s can be normalized to give

$$\tilde{\mathbf{v}}_i\boldsymbol{\alpha}\mathbf{v}_j = \delta_{ij}, \qquad i, j = 1, 2, \cdots, n. \tag{53}$$

Next let $\boldsymbol{\mu}$ be the $n \times n$ matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$. Then

$$\begin{aligned} \tilde{\boldsymbol{\mu}}\boldsymbol{\alpha}\boldsymbol{\mu} &= \mathbf{I}, \\ \tilde{\boldsymbol{\mu}}\boldsymbol{\gamma}\boldsymbol{\mu} &= \operatorname{diag}(e_1, e_2, \cdots, e_n). \end{aligned} \tag{54}$$

Finally, set

$$\boldsymbol{\theta}_p = \boldsymbol{\sigma}_p(\boldsymbol{\mu}). \tag{55}$$

Equation (43) shows that (50) and (51) now hold with the λ's of (51) being the homogeneous products $\mathbf{e}^{\mathbf{l}}$.

The preceding considerations show that

$$p(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) = \sum_{\mathbf{l}=0}^{\infty} \mathbf{e}^{\mathbf{l}}\psi_{\mathbf{l}}(\mathbf{x}; \boldsymbol{\alpha})\psi_{\mathbf{l}}(\mathbf{y}; \boldsymbol{\alpha}), \tag{56}$$

where the orthonormal functions $\psi_{\mathbf{l}}$ are defined in terms of Hermite functions by (49) and (55) and the components of $\mathbf{e}$ are the eigenvalues of $\boldsymbol{\alpha}^{-1}\boldsymbol{\gamma}$. This expansion is another natural analogue of Mehler's formula (44). In analogy with (45)–(46), we see that the eigenvalues and eigenfunctions of

$$\int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n p(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho})\Phi(\mathbf{y})w(\mathbf{y}; \boldsymbol{\alpha}) = \lambda\Phi(\mathbf{x}) \tag{57}$$

are

$$\lambda_{\mathbf{l}} = \mathbf{e}^{\mathbf{l}}, \qquad \Phi_{\mathbf{l}}(\mathbf{x}) = \psi_{\mathbf{l}}(\mathbf{x}; \boldsymbol{\alpha}). \tag{58}$$

In closing we note that since

$$\int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n p(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho}) = \varphi(\mathbf{x}; \boldsymbol{\alpha}) = \frac{1}{w(\mathbf{x}, \boldsymbol{\alpha})},$$

the probabilist would write

$$\begin{aligned} p(\mathbf{x}, \mathbf{y}; \boldsymbol{\rho})w(\mathbf{y}; \boldsymbol{\alpha}) &= p(\mathbf{x}|\mathbf{y}), \\ \varphi(\mathbf{x}; \boldsymbol{\alpha}) &= p(\mathbf{x}), \end{aligned}$$

and would call these quantities respectively the conditional density of $\mathbf{x}$ given $\mathbf{y}$, and the density of $\mathbf{x}$. Equation (57) written in these terms for the eigenvalue $\lambda_{\mathbf{0}} = \mathbf{e}^{\mathbf{0}} = 1$ is

$$\int_{-\infty}^{\infty} dy_1 \cdots \int_{-\infty}^{\infty} dy_n p(\mathbf{y})p(\mathbf{x}|\mathbf{y}) = p(\mathbf{x}),$$

the familiar Chapman–Kolmogorov equation for the stationary density for the vector Markov process generated by the transition probabilities $p(\mathbf{x}|\mathbf{y})$. Equations (57)–(58) permit explicit answers to be given to many questions about this process.

REFERENCES

[1] H. CRAMÉR, *Mathematical Methods of Statistics*, Princeton Univ. Press, Princeton, N.J., 1946.

[2] A. ERDÉLYI, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.

[3] L. CARLITZ, *Some extensions of the Mehler formula*, Seminario Matematico de Barcelona, Coll. Math., V., XXI, Fasc. 2, 1970, pp. 117–130.

[4] P. APPELL AND J. KAMPÉ DE FÉRIET, *Fonctions hypergéométriques et hypersphériques, Polynomes d'Hermite*, Gauthier-Villars, Paris, 1926.

[5] F. D. MURNAGHAN, *The Theory of Group Representations*, Johns Hopkins Press, Baltimore, 1938.

[6] D. E. LITTLEWOOD, *The Theory of Group Characters*, Oxford Univ. Press, London, 1950.

[7] R. BELLMAN, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.

[8] M. BÔCHER, *Introduction to Higher Algebra*, Macmillan, New York, 1936.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

LOWER BOUNDS FOR EIGENVALUES WITH DISPLACEMENT OF ESSENTIAL SPECTRA*

DAVID W. FOX†

Abstract. New constructions of comparison operators for rigorous lower bounds to eigenvalues of a class of self-adjoint operators are presented. The formulation uses noncompact finite perturbations to displace eigenvalues and essential spectra, and leads to workable numerical procedures. These methods make possible for the first time lower bound calculations for the lower eigenvalues of the Schrödinger operators for atoms and ions having three or more electrons.

1. Introduction. This article gives a new construction of comparison operators for lower bounds to eigenvalues of a class of self-adjoint operators that includes Schrödinger operators for atomic systems. Although methods for calculation of lower bounds have enjoyed success in some applications to Schrödinger operators, a serious limitation shows up for nearly all but the simplest. This limitation is rooted in the stability of essential spectra under compact perturbations, and it has blocked the calculation of rigorous lower bounds for atoms more complicated than helium.

To overcome this difficulty it is necessary to use perturbing operators that are not of finite rank, but it is equally necessary to ensure that the resulting formulation leads to workable numerical procedures.

In the following sections the construction of suitable families of such operators and the resolution of their spectral problems are sketched. To avoid complications we shall give the construction in its simplest form and only indicate generalizations. Essential use is made of tensor products of Hilbert spaces and of abstract separation of variables.

2. Construction of comparison operators. Let A be an operator self-adjoint on its domain $\mathfrak{D}$ in a complex separable Hilbert space $\mathfrak{H}$. Suppose that A is bounded below and that the lowest part of its spectrum is made up of isolated eigenvalues λ_ν of finite multiplicity. These eigenvalues are enumerated starting with the lowest and accounting for multiplicity. With the corresponding orthonormal eigenvectors u_ν they satisfy

$$(A - \lambda_\nu)u_\nu = 0, \quad \nu = 1, 2, \cdots, \quad \text{and} \quad \lambda_1 \leqq \lambda_2 \leqq \cdots < \lambda_*,$$

where λ_* is the lowest point of the essential spectrum of A.

The object of the construction is to obtain families of increasing operators that are smaller than A, that have the lowest part of their spectra made up of isolated eigenvalues of finite multiplicity, and that are resolvable, at least to the extent that their lowest eigenvalues can be determined with any desired precision.

* Received by the editors March 1, 1971.

† Applied Physics Laboratory, The Johns Hopkins University, Silver Spring, Maryland 20910. This work was supported in part by the Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, under Contract AR-8-0001 and in part by the Department of the Navy under Contract NOw-62-0604-c.

The result of this construction is a means of determining improvable lower bounds to the lowest eigenvalues of A. The path to be followed for a short distance is that of Weinstein–Aronszajn intermediate operators (see [1]–[4], [8], [13]).

Suppose A satisfies

$$A = A^0 + \hat{A},$$

where $\hat{A}$ is nonnegative, symmetric and closeable and A^0 is self-adjoint. The lowest part of the spectrum of A^0 is supposed to be made up of isolated eigenvalues λ_ν^0 of finite multiplicity. These are indexed in nondecreasing order, and with their corresponding orthonormal eigenvectors u_ν^0 they satisfy

$$(A^0 - \lambda_\nu^0)u_\nu^0 = 0, \quad \nu = 1, 2, \cdots, \quad \text{and} \quad \lambda_1^0 \leqq \lambda_2^0 \leqq \cdots < \lambda_*^0.$$

Since $\hat{A}$ is positive, the values $\lambda_1^0, \lambda_2^0, \cdots, \lambda_*^0$ give rudimentary lower bounds,

$$\lambda_\nu^0 \leqq \lambda_\nu, \quad \nu = 1, 2, \cdots, \quad \text{and} \quad \lambda_*^0 \leqq \lambda_*.$$

However, whenever it happens,[1] as in atomic systems, that λ_*^0 lies below an eigenvalue λ_μ of A for which an improved lower bound is desired, the methods based on the approximation of $\hat{A}$ by an operator of finite rank cannot yield useful information, since the best information obtained in this way is $\lambda_*^0 \leqq \lambda_\mu$, which is supposed already known. What is needed is a modification that pushes up limit points of the spectrum of A^0 in a suitable way.

To proceed further it is necessary to leave the beaten path and to make a number of special assumptions concerning the operators that are involved. Before entering into the details it is appropriate to say that the operator A^0 will be supposed to be resolved by elementary separation of variables and $\hat{A}$ will be assumed to couple the part operators in pairs by positive terms.

2.1. Elementary separation of variables. Let $\mathfrak{H}$ be a tensor product $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \cdots \otimes \mathfrak{H}_m$ of m separable Hilbert spaces $\mathfrak{H}_i$ with inner products $(\cdot,\cdot)_i$ and let A^0 separate[2] with resolvable self-adjoint part operators A_i in $\mathfrak{H}_i$. This means that on elementary tensor products $u = u_1 \otimes u_2 \otimes \cdots \otimes u_m$ with $u_i \in \mathfrak{D}_i$, where $\mathfrak{D}_i$ is the domain of A_i, A^0 is given by

$$A^0 u = A_1 u_1 \otimes u_2 \otimes \cdots \otimes u_m + u_1 \otimes A_2 u_2 \otimes \cdots \otimes u_m + \cdots + u_1 \otimes u_2 \otimes \cdots \otimes A_m u_m.$$

Since A^0 is bounded below and has the lowest part of its spectrum discrete, each part operator has these properties as well. The lowest eigenvalues of A_i are designated λ_i^ν and the corresponding orthonormal eigenvectors are u_i^ν, so that for each part operator A_i in $\mathfrak{H}_i$,

$$(A_i - \lambda_i^\nu)u_i^\nu = 0, \quad \nu = 1, 2, \cdots, \quad \text{and} \quad \lambda_i^1 \leqq \lambda_i^2 \leqq \cdots < \lambda_i^*.$$

[1] For example, in the case of the lithium atom there is evidence that λ_*^0 for the lower comparison operator corresponding to three independent electrons about the nucleus lies below even the lowest eigenvalue of the operator for the completed atom!

[2] For details on separation of variables in Hilbert space and the corresponding spectral theory, see [5]–[7].

An orthonormal system of eigenvectors of A^0 is given by the products of eigenvectors of the part operators, and the eigenvalues are sums of eigenvalues of the part operators; in particular, the lowest eigenvalues and corresponding eigenvectors of A^0 take the form

$$\lambda_\nu^0 = \lambda_1^{\nu_1} + \lambda_2^{\nu_2} + \cdots + \lambda_m^{\nu_m} \quad \text{and} \quad \mu_\nu^0 = u_1^{\nu_1} \otimes u_2^{\nu_2} \otimes \cdots \otimes u_m^{\nu_m},$$

where the indices ν_i are such that the eigenvalues λ_ν^0 are ordered.

2.2. Coupling of pairs of operators. What is required of $\hat{A}$ is that the separation of variables $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \cdots \otimes \mathfrak{H}_m$ that works for A^0 also allows $\hat{A}$ to be written as the sum of *pairwise coupling* terms $\hat{A}_{ij}$, that is,

$$\hat{A} = \sum_{i<j} \hat{A}_{ij}.$$

The sense of *pairwise coupling* is now to be made exact.[3] For $\hat{A}_{12}$ it means that $\hat{A}_{12}$ has the expression $A_{12} \otimes (I_3 \otimes I_4 \otimes \cdots \otimes I_m)$, where A_{12} is symmetric, closeable, and nonnegative on a dense domain $\mathfrak{D}_{12}$ in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. To describe the other terms $\hat{A}_{ij}$ it is convenient to use the fact that $\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \cdots \otimes \mathfrak{H}_m$ is isomorphic to $(\mathfrak{H}_i \otimes \mathfrak{H}_j) \otimes (\mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \cdots \otimes \mathfrak{H}_m)'$, where the prime on the parenthesis means that the factors $\mathfrak{H}_i$ and $\mathfrak{H}_j$ are not included. Such an isomorphism V_{ij} is determined by

$$V_{ij}(u_1 \otimes u_2 \otimes \cdots \otimes u_m) = (u_i \otimes u_j) \otimes (u_1 \otimes u_2 \otimes \cdots \otimes u_m)'$$

and amounts to an (abstract) interchange of variables. Thus $\hat{A}_{ij}$ is supposed to be expressed as

$$\hat{A}_{ij} = V_{ij}^{-1}[A_{ij} \otimes (I_1 \otimes I_2 \otimes \cdots \otimes I_m)']V_{ij}$$

with A_{ij} nonnegative, symmetric, and closeable on a dense domain $\mathfrak{D}_{ij} \subset \mathfrak{H}_i \otimes \mathfrak{H}_i$.

2.3. Example. To clarify the notation and exemplify the situation we have in mind, consider the operator[4] corresponding to the nonrelativistic fixed-nucleus model for the lithium atom without spin interaction,

$$A = \left(-\frac{\Delta}{2} - \frac{3}{|x_1|} - \frac{3}{|x_2|} - \frac{3}{|x_3|}\right) + \left(\frac{1}{|x_1 - x_2|} + \frac{1}{|x_1 - x_3|} + \frac{1}{|x_2 - x_3|}\right)$$

in $\mathfrak{H} = \mathfrak{L}^2(R^9) = \mathfrak{H}_1 \otimes \mathfrak{H}_2 \otimes \mathfrak{H}_3 = \mathfrak{L}^2(R^3) \otimes \mathfrak{L}^2(R^3) \otimes \mathfrak{L}^2(R^3)$. Here Δ is the 9-dimensional Laplacian, x_1, x_2, and x_3 are vectors in R^3, and $|\cdot|$ means the Euclidean distance in R^3. The operator A is expressible as the sum $A^0 + \hat{A}$ with A^0 given by the quantity in the first parenthesis and $\hat{A}$ by the second. Further, A^0 separates under the factorization of $\mathfrak{H}$ indicated above into three resolvable "hydrogenic" operators.[5] The positive operator $\hat{A}$ is given by

$$\hat{A} = \hat{A}_{12} + \hat{A}_{13} + \hat{A}_{23} = \sum_{i<j} \hat{A}_{ij}$$

[3] Here and in the following suppose $m \geqq 3$ (see also Jauch [9, p. 284]).

[4] For details on such operators, see T. Kato [10], [11]. The additional complications that arise when the Pauli principle is taken into account will be discussed elsewhere.

[5] See [12] for details on the resolution of these part operators by further separation of variables.

with $\hat{A}_{ij}$ given in $\mathfrak{H}$ by the operation of multiplication with $|x_i - x_j|^{-1}$. These operators are of the required form, for if A_{12} means multiplication by $|x_1 - x_2|^{-1}$ in $\mathfrak{L}^2(R^6)$, it is clear that $\hat{A}_{12} = A_{12} \otimes I_3$, and thus $\hat{A}_{12}$ is a pairwise coupling; similarly (using isomorphisms) for $\hat{A}_{13}$ and $\hat{A}_{23}$.

2.4. Approximation of $\hat{A}$. Under the assumptions made earlier in this section it is possible, as we shall show, to generate families of resolvable comparison operators that have substantial promise of yielding useful lower bounds when limit points must be moved up. The essential device is to approximate the positive operators[6] A_{ij} in $\mathfrak{H}_i \otimes \mathfrak{H}_j$ by operators of finite rank instead of attacking $\hat{A}$ or $\hat{A}_{ij}$ in $\mathfrak{H}$ directly. The key is that even when A^k_{12} is an approximation to A_{12} of finite rank, the operator $\hat{A}^k_{12}$ defined by $\hat{A}^k_{12} = A^k_{12} \otimes (I_3 \otimes I_4 \otimes \cdots \otimes I_m)$ is noncompact and potentially able to move limit points about; and similarly for the other terms $\hat{A}_{ij}$.

Suppose $\{p^l_{ij}\}_l$ is a family of linearly independent vectors in $\mathfrak{D}_{ij}$, and let P^k_{ij} be the orthogonal projection with respect to the inner product generated by A_{ij} on the linear span of the first k vectors. It is well known that $\{A^k_{ij}\}_k$ with $A^k_{ij} = A_{ij}P^k_{ij}$ is a family of bounded operators of finite rank, increasing with k and bounded above by A_{ij}, and from this it follows that $\{\hat{A}^k_{ij}\}_k$ with $\hat{A}^k_{ij} = V^{-1}_{ij}[A^k_{ij} \otimes (I_1 \otimes I_2 \otimes \cdots \otimes I_m)']V_{ij}$ is also bounded, increasing with k, and bounded above by $\hat{A}_{ij}$. Let K be the family of multi-index vectors k with $m(m-1)/2$ components $k_{ij}, i < j$; and let $k^1 \leqq k^2$ mean $k^1_{ij} \leqq k^2_{ij}$. Now define $\{\hat{A}^k\}_{k\in K}$ by $\hat{A}^k = \sum_{i<j} \hat{A}^{k_{ij}}_{ij}$. Clearly, this family is bounded, increasing with k, and bounded above by $\hat{A}$.

When the vectors $\{p^l_{ij}\}_l$ can be chosen so that the resulting spectral problems are resolvable, the operators $\{A^k\}_k$ defined by

$$A^k = A^0 + \hat{A}^k$$

form a suitable family of comparison operators, since they satisfy

$$A^0 \leqq A^{k^1} \leqq A^{k^2} \leqq A \quad \text{for} \quad k^1 \leqq k^2,$$

and the approximations introduced do not preclude the desired displacement of limit points of the spectrum of A^0. The resolution of the spectral problems is discussed in the next section.

2.5. Approximation of the operator A^0. When it is difficult to find vectors $\{p^l_{ij}\}_l$ that make the spectral problem for A^k easy to resolve, there is still an avenue open through modification of A^0. This procedure is an adaptation of one used in [3] and [4].

Recall that the nth order truncation A^n_i of A_i is the bounded symmetric operator on $\mathfrak{H}_i$ defined by

$$A^n_i = A_iE^n_i + \lambda^{n+1}_i(I_i - E^n_i),$$

where $E^n_i = E_i(\lambda^n_i)$ and E_i is the resolution of the identity associated with A_i.

[6] A similar device with the same kinds of properties can be used to approximate quadratic forms. This parallel is shown by comparing [4] with [3].

The family $\{A_i^n\}_n$ is increasing with n and bounded above by A_i. Let N be the ordered family of m-tuples $n = (n_1, n_2, \cdots, n_m)$, where $n^1 \leqq n^2$ means $n_i^1 \leqq n_i^2$. It is easy to show that the operator $A^{n,0}$ defined by

$$A^{n,0} = A_1^{n_1} \otimes I_2 \otimes \cdots \otimes I_m + I_1 \otimes A_2^{n_2} \otimes \cdots \otimes I_m + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_m^{n_m}$$

is bounded, symmetric, increasing with n, and bounded above by A^0. Observe that the definitions of $A^{n,0}$ and $A_i^{n_i}$ show that $A^{n,0}$ can be written as the sum of two operators, one that separates with part operators of finite rank and the other equal to $(\sum_i \lambda_i^{n_i+1}) \cdot I$.

Since $A^{n,0}$ is bounded, it makes sense to write

$$A = A^{n,0} + (A - A^{n,0}),$$

and to consider the simpler operator $A^{n,0}$ as a starting point with $A - A^{n,0}$ as a positive perturbation. Note that $A - A^{n,0}$ has the same domain as A and that it is a sum of pairwise coupling positive terms whenever $\hat{A}$ is, so that the technique of approximation of $\hat{A}$ introduced in § 2.4 can be applied to it.

3. Resolution of spectral problems. This section is devoted to working out the resolution of the spectral problems for two applications of the approximations given in § 2.

3.1. Special choice. This technique extends ideas discussed in [2] and [3]. It depends on the assumption that families $\{p_{ij}^l\}_l$ can be chosen so that each operator[7] A_{ij}^k has its range contained in the tensor product of one subspace spanned by a finite number of eigenvectors of A_i with another finite-dimensional space spanned by eigenvectors of A_j. This is accomplished whenever each vector $A_{ij}P_{ij}^l$ is a finite linear combination of elementary products of eigenvectors of A_i with eigenvectors of A_j. Let $k \in K$ be fixed and let $\mathfrak{M}_r$ be the linear span of those eigenvectors of A_r needed to express at least one of the vectors $A_{ij}P_{ij}^l$. Clearly the range of A_{ij}^k is contained in $\mathfrak{M}_i \otimes \mathfrak{M}_j$. The subspaces $\mathfrak{M} = \mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes \cdots \otimes \mathfrak{N}_m$ with $\mathfrak{N}_i$ equal either to $\mathfrak{M}_i$ or to the orthogonal complement $\mathfrak{M}_i^\perp$ are a complete family of orthogonal reducing spaces for the operator A^k introduced in § 2.4; and, as will soon be clear, the resolution of the spectral problem for A^k on each of these spaces is either immediate or equivalent to the diagonalization of a (finite) symmetric matrix.

The reduction of A^k by the spaces $\mathfrak{M}$ is quite easy to demonstrate. First, observe that each space $\mathfrak{M}_i$ is a reducing space for A_i, and hence each $\mathfrak{M}$ reduces A^0. Further, the range of A_{ij}^k is contained in $\mathfrak{M}_i \otimes \mathfrak{M}_j$, hence $\mathfrak{M}_i \otimes \mathfrak{M}_j$ reduces A_{ij}^k; and, indeed, A_{ij}^k vanishes orthogonal to $\mathfrak{M}_i \otimes \mathfrak{M}_j$. From this it follows that each $\mathfrak{M}$ reduces every term in the defining sum for $\hat{A}^k$, hence $\hat{A}^k$ as well. Since each $\mathfrak{M}$ reduces A^0 and $\hat{A}^k$, each reduces A^k, too.

The resolution of the spectral problem for A^k depends on the behavior of A^k on the spaces $\mathfrak{M}$, and this depends very much on how many factors $\mathfrak{M}_i$ appear in $\mathfrak{M}$.

[7] We write A_{ij}^k in place of the cumbersome notation $A_{ij}^{k_{ij}}$.

A subspace $\mathfrak{M}$ will be said to be of type r if it has r factors $\mathfrak{M}_i$. Evidently there are $\binom{m}{r}$ different subspaces of type r, and $\mathfrak{H}$ is thus decomposed into 2^m orthogonal reducing spaces.

The subspace of type 0 and the m spaces of type 1 are trivial; for on them each term of $\hat{A}^k$ vanishes, and consequently $A^k = A^0$. Since each $\mathfrak{M}_i$ is a reducing subspace for the part operator A_i of A^0, the restrictions of A^0 to the reducing subspaces $\mathfrak{M}$ of types 0 and 1 can be resolved by restricting[8] the part operators and identities to the appropriate $\mathfrak{M}_i$ or $\mathfrak{M}_i^\perp$ that appear in $\mathfrak{M}$. Thus the spectral problems for A^k are resolved in the spaces of type 0 and 1 by separation of variables as for A^0.

Beginning with r equal to 2, the spectral problems yield new information. Evidently, for any fixed r the analysis of each case is essentially the same, so it is enough to examine what happens for the one in which the notation is the easiest. Thus, for $r = 2$ consider $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2 \otimes \mathfrak{M}_3^\perp \otimes \cdots \otimes \mathfrak{M}_m^\perp$. In this space A^k coincides with $A^0 + \hat{A}_{12}$ since the other terms in $\hat{A}^k$ vanish, and separation of variables can be applied neatly to obtain the spectral resolution. In fact, A^k can be written

$$A^k = (A^0_{12,1} + A^k_{12}) \otimes I_{12,2} + I_{12,1} \otimes A^0_{12,2},$$

where $A^0_{12,1}$ stands for $A_1 \otimes I_2 + I_1 \otimes A_2$, $I_{12,1}$ for $I_1 \otimes I_2$, $I_{12,2}$ for $I_3 \otimes I_4 \otimes \cdots \otimes I_m$, and $A^0_{12,2}$ for $A_3 \otimes I_4 \otimes \cdots \otimes I_m + I_3 \otimes A_4 \otimes \cdots \otimes I_m + \cdots + I_3 \otimes I_4 \otimes \cdots \otimes A_m$. Now observe that on the finite-dimensional space $\mathfrak{M}_1 \otimes \mathfrak{M}_2$, the resolution of the spectral problem for $A^0_{12,1} + A^k_{12}$ is equivalent to the diagonalization of a symmetric matrix. Further, on $\mathfrak{M}_3^\perp \otimes \mathfrak{M}_4^\perp \otimes \cdots \otimes \mathfrak{M}_m^\perp$ the part operator $A^0_{12,2}$ is resolvable by additional separation of variables and restriction of the part operators, just as A^0. Thus the spectral problem for A^k is easily resolvable on the spaces $\mathfrak{M}$ of type 2.

When r is larger than two but smaller than m, similar analyses can be made, and each case is essentially the same for any given r. For $\mathfrak{M} = \mathfrak{M}_1 \otimes \mathfrak{M}_2 \otimes \cdots \otimes \mathfrak{M}_r \otimes \mathfrak{M}_{r+1}^\perp \otimes \cdots \otimes \mathfrak{M}_m^\perp$, the operator A^k is expressed by

$$A^k = (A^0_{12\cdots r,1} + \hat{A}^k_{12\cdots r}) \otimes I_{12\cdots r,2} + I_{12\cdots r,1} \otimes A_{12\cdots r,2},$$

where $A^0_{12\cdots r,1} = A_1 \otimes I_2 \otimes \cdots \otimes I_r + I_1 \otimes A_2 \otimes I_3 \otimes \cdots \otimes I_r + \cdots + I_1 \otimes I_2 \otimes \cdots \otimes A_r$, $I_{12\cdots r,1} = I_1 \otimes I_2 \otimes \cdots \otimes I_r$, $I_{12\cdots r,2} = I_{r+1} \otimes I_{r+2} \otimes \cdots \otimes I_m$, and $A^0_{12\cdots r,2} = A_{r+1} \otimes I_{r+2} \otimes \cdots \otimes I_m + I_{r+1} \otimes A_{r+2} \otimes \cdots \otimes I_m + \cdots + I_{r+1} \otimes I_{r+2} \otimes \cdots \otimes A_m$, and $\hat{A}^k_{12\ldots r}$ contains the nonvanishing contribution of $\hat{A}^k$. The part operator $A^0_{12\cdots r,1} + \hat{A}^k_{12\cdots r}$ on the finite-dimensional space $\mathfrak{M}_1 \otimes \mathfrak{M}_2 \otimes \cdots \otimes \mathfrak{M}_r$ is equivalent to a symmetric matrix, and it is resolved by the diagonalization of this matrix; in $\mathfrak{M}_{r+1}^\perp \otimes \mathfrak{M}_{r+2}^\perp \otimes \cdots \otimes \mathfrak{M}_m^\perp$ the part operator $A^0_{12\cdots r,2}$ is resolved by further separation of variables.

When r equals m, the space $\mathfrak{M}$ itself is finite-dimensional, and A^k is resolved by diagonalization of the symmetric matrix that represents it.

Since each of the reducing spaces of type r with $r \geqq 2$ leads to a matrix diagonalization, the complete resolution of the spectral problem for A^k under the

[8] See [6, Proposition 2].

hypothesis of a special choice is achieved by the diagonalization of $2^m - m - 1$ symmetric matrices. From the smaller ordered eigenvalues λ_ν^k of A^k and the smallest limit point λ_*^k of the spectrum of A^k come the lower bounds

$$\lambda_\nu^k \leqq \lambda_\nu, \quad \nu = 1, 2, \cdots ; \quad \lambda_*^k \leqq \lambda_* .$$

3.2. Truncation. When a special choice is not known or is inconvenient, it is usually possible to determine useful lower bounds by employing the operator $A^{n,0}$ introduced in § 2.5. As pointed out at the end of that section, it is quite feasible to approximate $A - A^{n,0}$ as has been done for $\hat{A}$; however, for the sake of clarity we give a somewhat simpler procedure that approximates only $\hat{A}$.

Consider the family $\{A^{n,k}\}_{n,k}$ of bounded operators defined by

$$A^{n,k} = A^{n,0} + \hat{A}^k.$$

Clearly from the properties of $A^{n,0}$ and $\hat{A}^k$ given in §§ 2.4 and 2.5, the family $\{A^{n,k}\}$ is increasing with n and with k, and it is bounded above by A. Since the lowest eigenvalues and eigenvectors of $A^{n,0}$ are the same as those of A^0, the lowest eigenvalues of $A^{n,k}$ will generally show some increase over those of A^0.

Some restriction on the vectors $\{p_{ij}^l\}_l$ used to define $\hat{A}^k$ seems to be needed in order to get manageable resolutions of the spectral problems for $A^{n,k}$. For this reason suppose that k is fixed and that the $\{p_{ij}^l\}_l$ are such that each vector $A_{ij}p_{ij}^l$ is a finite sum of elementary tensor products in $\mathfrak{H}_i \otimes \mathfrak{H}_j$. Note that these products do not need to have any relation to the spectral families of A_i and A_j as is the case when a special choice is made.

Let $\mathfrak{M}_r'$ be the span of those vectors in $\mathfrak{H}_r$ that are used to express at least one of the vectors $A_{ij}p_{ij}^l$, and observe that A_{ij}^k is reduced by $\mathfrak{M}_i' \otimes \mathfrak{M}_j'$ and vanishes on the orthogonal complement. However, since the spaces $\mathfrak{M}_i'$ are not necessarily reducing spaces for the operators $A_i^{n_i}$, it is necessary to augment them. To this end let $\mathfrak{M}_i$ be the linear span of $\mathfrak{M}_i'$ with the first n_i eigenvectors of A_i. The importance of the spaces $\mathfrak{M}_i$ is that simultaneously they reduce $A_i^{n_i}$ and their pairwise products $\mathfrak{M}_i \otimes \mathfrak{M}_j$ still reduce A_{ij}^k. This is what is needed to obtain a complete family of manageable orthogonal reducing spaces for $A^{n,k}$. In fact, the spaces $\mathfrak{M}$ expressed by $\mathfrak{M} = \mathfrak{N}_1 \otimes \mathfrak{N}_2 \otimes \cdots \otimes \mathfrak{N}_m$, where $\mathfrak{N}_i$ is either $\mathfrak{M}_i$ or its orthogonal complement, are the required spaces.

Formally, the analysis of $A^{n,k}$ proceeds now just as that for A^k when a special choice is made. The classification of the spaces $\mathfrak{M}$ according to the number r of factors $\mathfrak{M}_i$ in its expression is used, and separation of variables is employed to split the problems as needed into one part with a known spectral resolution and another on a finite-dimensional space. This similarity is not at all surprising since it can be observed that *any* vectors $\{p_{ij}^l\}$ such that $A_{ij}p_{ij}^l$ are finite sums of elementary products are, indeed, a special choice for $A^{n,0}$ in the sense that these vectors can always be expressed in terms of a finite number of eigenvectors of $A_i^{n_i}$ and of $A_j^{n_j}$.

Acknowledgments. My interest in problems arising from the essential spectra of operators of the type that describe atomic systems owes much to stimulating discussions with N. W. Bazley and to encouragement by Professor P.-O. Löwdin.

REFERENCES

[1] N. ARONSZAJN, *Approximation methods for eigenvalues of completely continuous symmetric operators*, Proc. Symposium on Spectral Theory and Differential Problems, Stillwater, Okla., 1951, pp. 179–202.

[2] N. W. BAZLEY, *Lower bounds for eigenvalues with applications to the helium atom*, Proc. Nat. Acad. Sci., 45 (1959), pp. 850–853.

[3] N. W. BAZLEY AND D. W. FOX, *Truncations in the method of intermediate problems for lower bounds to eigenvalues*, J. Res. Nat. Bur. Standards, 65B (1961), pp. 105–111.

[4] ———, *Lower bounds to eigenvalues using operator decompositions of the form B^*B*, Arch. Rational Mech. Anal., 10 (1962), pp. 352–360.

[5] JU. M. BEREZANSKIĬ, *Expansions in Eigenfunctions of Self-adjoint Operators*, Trans. Math. Monographs, vol. 17, American Mathematical Society, Providence, R.I., 1968.

[6] D. W. FOX, *Separation of variables and spectral theory for self-adjoint operators in Hilbert space*, Informal Report, Applied Mathematics Group, Applied Physics Laboratory, The Johns Hopkins University, Silver Spring, Md., 1968.

[7] ———, *Spectral measures and separation of variables*, to appear.

[8] S. H. GOULD, *Variational Methods for Eigenvalue Problems*, 2nd ed., University of Toronto Press, Toronto, 1966.

[9] J. M. JAUCH, *Foundations of Quantum Mechanics*, Addison-Wesley, Reading, Mass., 1968.

[10] T. KATO, *Fundamental properties of Hamiltonian operators of Schrödinger type*, Trans. Amer. Math. Soc., 70 (1951), pp. 195–211.

[11] ———, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1966.

[12] E. C. KEMBLE, *The Fundamental Principles of Quantum Mechanics*, McGraw-Hill, New York, 1937.

[13] A. WEINSTEIN, *Étude des spectres des équations aux dérivées partielles*, Mém. Sci. Math., fasc. no. 88, Gauthier-Villars, Paris, 1937.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

EXPONENTIAL STABILITY OF SOLUTIONS OF DIFFERENTIAL EQUATIONS OF SOBOLEV TYPE*

JOHN LAGNESE†

Abstract. Let $\mathscr{M}(x, D)$ and $\mathscr{L}(x, D)$ be linear partial differential operators of order $2m$ with complex-valued coefficients defined on a bounded region Ω in R^n and suppose $\mathscr{M}$ is elliptic in Ω. Necessary and sufficient conditions are given in order that solutions of $\mathscr{M}(x, D)\partial u/\partial t - \mathscr{L}(x, D)u = 0$ in the cylinder $\Omega \times [0, \infty)$ which satisfy general boundary conditions on the wall of the cylinder satisfy inequalities of the form $\|u(t)\|_{2m} \leqq Ce^{-at}\|u(0)\|_{2m}$ and $|u(t)|_{2m+\rho} \leqq Ce^{-at}|u(0)|_{2m+\rho}$, $t > 0$, with positive constants a and C independent of u. $\|\cdot\|_{2m}$ and $|\cdot|_{2m+\rho}$ denote the customary norms in the spaces $H^{2m,2}(\Omega)$ and $C^{2m+\rho}(\bar{\Omega})$, $0 < \rho < 1$, respectively.

1. Introduction. The present note is concerned with exponential stability of solutions of

$$\mathscr{M}(x, D)\frac{\partial u}{\partial t} - \mathscr{L}(x, D)u = 0 \tag{1.1}$$

in a cylinder $\Omega \times R^+$, where $R^+ = [0, \infty)$ and Ω is a bounded open set in R^n, which satisfy on the wall of the cylinder the conditions

$$B_j(x, D)u = 0, \qquad j = 1, 2, \cdots, m. \tag{1.2}$$

In (1.1), $\mathscr{M}$ and $\mathscr{L}$ are linear partial differential operators of order $2m$ with complex-valued coefficients defined in $\bar{\Omega}$ and $\mathscr{M}$ is assumed to be elliptic there. The boundary operators $\{B_j\}_1^m$ form a normal system of respective orders $m_j \leqq 2m - 1$.

Let $H^{2m}(\Omega)$ be the Sobolev space of order $2m$ with norm $\|\cdot\|_{2m}$, and $C^{2m+\rho}(\bar{\Omega})$ be the space of functions having continuous derivatives in $\bar{\Omega}$ to order $2m$ and whose derivatives of order $2m$ satisfy a uniform Hölder condition in $\bar{\Omega}$ with exponent ρ, $0 < \rho < 1$. Denote by $|\cdot|_{2m+\rho}$ the norm in $C^{2m+\rho}(\bar{\Omega})$. In what follows we obtain necessary and sufficient conditions in order that solutions of (1.1), (1.2) in $C'(R^+, H^{2m}(\Omega))$ (respectively, in $C'(R^+, C^{2m+\rho}(\bar{\Omega}))$) satisfy the inequality

$$\|u(t)\|_{2m} \leqq Ce^{-at}\|u(0)\|_{2m}, \qquad t > 0,$$

(respectively,

$$|u(t)|_{2m+\rho} \leqq Ce^{-at}|u(0)|_{2m+\rho}, \qquad t > 0)$$

with positive constants a and C independent of u. Roughly speaking, the necessary and sufficient conditions are that, whenever $\operatorname{Re}\lambda \geqq 0$, $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ is a regular elliptic boundary value problem and that the eigenvalues of each of the eigenvalue problems

$$\mathscr{L}u - \lambda\mathscr{M}u = 0 \quad \text{in } \Omega, \qquad \{B_ju = 0\}_1^m \quad \text{on } \partial\Omega, \tag{1.3}$$

$$\mathscr{L}^*u - \lambda\mathscr{M}^*u = 0 \quad \text{in } \Omega, \qquad \{C_ju = 0\}_1^m \quad \text{on } \partial\Omega, \tag{1.4}$$

* Received by the editors June 22, 1971, and in revised form December 13, 1971.

† Department of Mathematics, Georgetown University, Washington, D.C. 20007.

lie in the open left half-plane. ($\mathscr{L}^*$ and $\mathscr{M}^*$ denote the formal adjoints of $\mathscr{L}$ and $\mathscr{M}$, respectively, and $\{C_j\}_1^m$ is the system of boundary operators adjoint to $\{B_j\}_1^m$ relative to $\mathscr{L} - \lambda\mathscr{M}$ with respect to Green's formula. See, e.g., [3], [8].) The conditions on the eigenvalues of the problems (1.3), (1.4) are in turn shown to be equivalent to the conditions

$$\text{(1.5)} \qquad \operatorname{Re}(\mathscr{M}u, \mathscr{L}u) < [\|\mathscr{M}u\|^2\|\mathscr{L}u\|2 - |\operatorname{Im}(\mathscr{M}u, \mathscr{L}u)|^2]^{1/2}$$

and

$$\operatorname{Re}(\mathscr{M}^*u, \mathscr{L}^*u) < [\|\mathscr{M}^*u\|^2\|\mathscr{L}^*u\|^2 - |\operatorname{Im}(\mathscr{M}^*u, \mathscr{L}^*u)|^2]^{1/2}$$

for all $u \not\equiv 0$ satisfying on $\partial\Omega$ the conditions $\{B_ju = 0\}_1^m$ and $\{C_ju = 0\}_1^m$, respectively. In the above inequalities, $\|\cdot\|$ and $(\cdot,\cdot)$ respectively denote the $L^2(\Omega)$ norm and scalar product.

If for $\operatorname{Re}\lambda \geqq 0$, $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ is an *absolutely* elliptic problem and $\mathscr{L} - \lambda\mathscr{M}$ satisfies a certain algebraic condition, it turns out that (1.5) alone is necessary and sufficient for exponential stability.

The term "Sobolev equation" refers to linear partial differential equations in which mixed space and time derivatives occur in the terms with highest order time derivative. Some authors call these equations "pseudoparabolic" [11], [12], although this term is also applied to other types of equations. The literature on such equations is extensive; we refer to [10] for a fairly complete bibliography and for information on the physical origins of these equations. In particular, equations of the form (1.1) appear in the theories of flows of second order fluids, soil mechanics and the seepage of fluid through fissured rocks. A general integration theory for (1.1), (1.2) may be found in [6]. The question of exponential stability of solutions of various special cases of (1.1), (1.2) has been studied in [11], [12] where some sufficient, but not necessary, conditions for exponential stability are given. These results can be deduced as easy consequences of the theory to be developed here. For other Sobolev equations, including those with higher order t-derivatives and especially the equation $\Delta u_{tt} + u_{yy} = 0$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, various asymptotic properties of solutions have been obtained in [7], [9], [13]–[15]. Additional references may be found in the bibliographies of the cited papers.

The outline of this paper is as follows. In § 2 we present the requisite preliminary material and briefly develop an existence and uniqueness theory of solutions of (1.1), (1.2), both in the space $H^{2m}(\Omega)$ and in $C^{2m+\rho}(\bar{\Omega})$. The main results of the paper are presented in § 3 and § 4. In the third section exponential stability with respect to the $\|\cdot\|_{2m}$-norm is studied while § 4 deals with exponential stability with respect to the $|\cdot|_{2m+\rho}$-norm. There the principal result is that exponential stability with respect to this norm is equivalent to exponential stability with respect to the $\|\cdot\|_{2m}$-norm. In the last section we apply our results to the case where the boundary conditions (1.2) are the Dirichlet conditions.

2. Preliminaries and existence of solutions. Let Ω be a bounded open set in R^n with smooth boundary (to be made precise) and x denote a variable point in Ω. We write $D_i = \partial/\partial x_i$, $D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}$ for any multi-integer $\alpha = (\alpha_1,$

$\alpha_2, \cdots, \alpha_n$), $\alpha_i \geqq 0$, and $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $\mathscr{M}$ be the differential operator

$$\mathscr{M}(x, D) = \sum_{|\alpha| \leqq 2m} m_\alpha(x) D^\alpha$$

with complex-valued coefficients defined in $\bar{\Omega}$ and let $\{B_j\}_1^m$ be the boundary system defined by

$$B_j(x, D) = \sum_{|\alpha| \leqq m_j} b_\alpha^j(x) D^\alpha, \qquad j = 1, \cdots, m,$$

with complex-valued coefficients defined on $\partial\Omega$. We shall always assume $\{B_j\}_1^m$ is a *normal system* of respective orders $m_j \leqq 2m - 1$. Thus $m_j \neq m_k$ if $j \neq k$ and $\partial\Omega$ is noncharacteristic to B_j at each point. We impose the following smoothness conditions.

(S_0) Ω is a bounded domain of class C^{2m}. The coefficients in $\mathscr{M}$ are of class $C^0(\bar{\Omega})$ and those in B_j of class $C^{2m-m_j}(\partial\Omega)$.

Following Agmon [1], we call the boundary value problem ($\mathscr{M}$, $\{B_j\}$, Ω) a *regular elliptic boundary value problem* if the above conditions hold and if:

(i) $\mathscr{M}$ is elliptic in $\bar{\Omega}$ and satisfies the *roots condition* there.

(ii) At each point of $\partial\Omega$, the operators $\{B_j\}_1^m$ satisfy the *complementing condition* with respect to the operator $\mathscr{M}$.

Throughout this paper we shall assume ($\mathscr{M}$, $\{B_j\}$, Ω) is a regular elliptic boundary value problem.

$H^{2m}(\Omega)$ will denote the Hilbert space consisting of the subclass of functions in $L^2(\Omega)$ whose distributional derivatives of orders $\leqq 2m$ belong to $L^2(\Omega)$, with the norm

$$\|u\|_{2m} = \left(\sum_{|\alpha| \leqq 2m} \int_\Omega |D^\alpha u|^2 \, dx \right)^{1/2}.$$

The $L^2(\Omega)$-norm is denoted by $\| \cdot \|$. Let $H^{2m}(\Omega; \{B_j\})$ be the closed subspace of $H^{2m}(\Omega)$ which is the closure in $H^{2m}(\Omega)$ of functions u in $C^{2m}(\bar{\Omega})$ which satisfy

$$B_j u = 0 \quad \text{on } \partial\Omega, \qquad 1 \leqq j \leqq m. \tag{2.1}$$

For regular elliptic boundary value problems the following a priori estimates are valid for all $u \in H^{2m}(\Omega; \{B_j\})$:

$$\|u\|_{2m} \leqq C(\|\mathscr{M}u\| + \|u\|), \tag{2.2}$$

where C does not depend on u. If in addition $\|\mathscr{M}u\| > 0$ for all $u \in H^{2m}(\Omega; \{B_j\})$, $u \not\equiv 0$, then (2.2) holds even if the term $\|u\|$ is omitted.

Conversely, if $\{B_j\}_1^m$ is a normal system of boundary operators of respective orders $m_j \leqq 2m - 1$, if the smoothness conditions (S_0) are assumed and if (2.2) is known to hold for all $u \in C^{2m}(\bar{\Omega})$ which satisfy (2.1), then necessarily the ellipticity, roots and complementing must hold; that is, ($\mathscr{M}$, $\{B_j\}$, Ω) must be a regular elliptic boundary value problem [2, § 10]. We shall have use for this fact in § 3.

We denote by M the unbounded linear operator in $L^2(\Omega)$ defined by $D(M) = H^{2m}(\Omega; \{B_j\})$ and $Mu = \mathscr{M}(\cdot, D)u(\cdot)$ for $u \in D(M)$. M is densely defined and it follows from (2.2) that M is closed, has finite-dimensional null space and closed range. If the spectrum of M is not the whole complex plane, i.e., if the resolvent $R(\lambda, M)$ exists for some $\lambda = \lambda_0$, then since $R(\lambda_0, M)$ is compact it follows that

$R(\lambda, M)$ exists for all λ except a discrete sequence of eigenvalues of M. In general, however, one cannot exclude the possibility that the spectrum of M is the whole complex plane. We therefore assume the following condition.

Spectrum condition. Zero lies in the resolvent set of M.

If the spectrum of M is discrete, the spectrum condition can be satisfied by replacing M by $M + k$ with some suitable constant k. Necessary and sufficient conditions in order that the spectrum of M be discrete are given in [1].

Next we define the differential operator $\mathscr{L}$ by

$$\mathscr{L}(x, D) = \sum_{|\alpha| \leqq 2m} \ell_\alpha(x) D^\alpha$$

with complex-valued coefficients of class $C^0(\bar{\Omega})$. The corresponding realization of $\mathscr{L}$ in $L^2(\Omega)$ is denoted by L. Thus

$$D(L) = H^{2m}(\Omega; \{B_j\}), \qquad Lu = \mathscr{L}(\cdot, D)u(\cdot) \quad \text{for } u \in D(L).$$

It is now easy to prove existence and uniqueness of *strong solutions* of (1.1), (1.2), satisfying a prescribed initial condition $u(x, 0) = u_0(x)$, that is, functions of class $C'(R^+, H^{2m}(\Omega; \{B_j\})$ satisfying $Mu_t - Lu = 0$ for all $t \geqq 0$ and $u(0) = u_0$. In fact, since zero is in the resolvent set of M the estimates (2.2) hold without the term $\|u\|$. Applying these estimates to $M^{-1}Lu$ for $u \in D(L)$ we obtain

$$\|M^{-1}Lu\|_{2m} \leqq C\|Lu\| \leqq C_1\|u\|_{2m}.$$

Thus $A = M^{-1}L$ is a bounded linear operator on the space $H^{2m}(\Omega; \{B_j\})$ and therefore generates the group $\{e^{tA}: -\infty < t < +\infty\}$ of bounded linear operators on that space. The existence of a unique strong solution of (1.1), (1.2) with a prescribed initial value $u(0) = u_0 \in H^{2m}(\Omega; \{B_j\})$ follows immediately by setting $u(t) = e^{tA}u_0$.

The existence of solutions of (1.1), (1.2) in $C^{2m+\rho}(\bar{\Omega})$ can be deduced in a similar way under slightly stronger smoothness assumptions. For $0 < \rho < 1$, $C^{2m+\rho}(\bar{\Omega})$ is the Banach space of functions u of class $C^{2m}(\bar{\Omega})$ whose derivatives $D^\alpha u$ of order $2m$ satisfy a uniform Hölder condition in $\bar{\Omega}$ with exponent ρ. The norm in $C^{2m+\rho}(\bar{\Omega})$ is

$$|u|_{2m+\rho} = \sum_{|\alpha| \leqq 2m} |D^\alpha u|_0 + \sup \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x - y|^\rho},$$

where $|v|_0 = \max_{\bar{\Omega}} |v(x)|$ and the supremum is taken over $|\alpha| = 2m$ and x, y in $\bar{\Omega}$ with $x \neq y$. Let $C^{2m+\rho}(\bar{\Omega}; \{B_j\})$ be the closed subspace of $C^{2m+\rho}(\bar{\Omega})$ consisting of those functions which satisfy (2.1). Suppose the smoothness condition (S_0) is replaced by the following one.

(S_ρ) Ω is a bounded domain of class $C^{2m+\rho}$. The coefficients in $\mathscr{M}$ are of class $C^\rho(\bar{\Omega})$ and those in B_j of class $C^{2m-m_j+\rho}(\partial\Omega)$.

Then the following a priori estimates hold for all $u \in C^{2m+\rho}(\bar{\Omega}; \{B_j\})$:

$$|u|_{2m+\rho} \leqq C(|\mathscr{M}u|_\rho + |u|_0) \tag{2.3}$$

with C independent of u.

Now let $\tilde{L}$ and $\tilde{M}$ denote the restrictions of L and M, respectively, to $C^{2m+\rho}(\bar{\Omega}; \{B_j\})$ and assume the coefficients of $\mathscr{L}$ are of class $C^\rho(\bar{\Omega})$. $\tilde{L}$ and $\tilde{M}$

are unbounded operators in $C^{\rho}(\bar{\Omega})$. Moreover $\tilde{M}u = 0$ implies $u = 0$ since zero is in the resolvent set of M. Thus (2.3) holds with the term $|u|_0$ omitted. If we show that the range of $\tilde{M}$ is all of $C^{\rho}(\bar{\Omega})$, it will then follow from (2.3) that $\tilde{A} = \tilde{M}^{-1}\tilde{L}$ is a bounded linear operator on $C^{2m+\rho}(\bar{\Omega}; \{B_j\})$ and, therefore, $u(t) = e^{t\tilde{A}}u_0$ provides the unique solution in class $C'(R^+, C^{2m+\rho}(\bar{\Omega}))$ of (1.1), (1.2) with the prescribed initial value $u(0) = u_0 \in C^{2m+\rho}(\bar{\Omega}; \{B_j\})$. ($e^{t\tilde{A}}$ is therefore the restriction of e^{tA} to $C^{2m+\rho}(\bar{\Omega}; \{B_j\})$.)

To show that $\tilde{M}$ maps onto $C^{\rho}(\bar{\Omega})$ we employ the following regularity result of N. Ikebe [5] (cf. [2, Appendix 5]).

THEOREM 2.1. *Suppose* $(\mathscr{M}, \{B_j\}, \Omega)$ *is a regular elliptic boundary value problem satisfying the smoothness conditions* (S_ρ). *If* $u \in H_{2m}(\Omega; \{B_j\})$ *and* $\mathscr{M}u \in C^{\rho}(\bar{\Omega})$, *then* $u \in C^{2m+\rho}(\bar{\Omega}; \{B_j\})$.

It follows from this result that M^{-1} maps $C^{\rho}(\bar{\Omega})$ into $C^{2m+\rho}(\bar{\Omega}; \{B_j\})$, that is, $\tilde{M}$ maps onto $C^{\rho}(\bar{\Omega})$.

We summarize the existence theory in the following.

THEOREM 2.2. *Suppose that* $(\mathscr{M}, \{B_j\}, \Omega)$ *is a regular elliptic boundary value problem satisfying the spectrum condition and that the coefficients* ℓ_α *of* $\mathscr{L}$ *belong to* $C^0(\bar{\Omega})$. *Then* $A = M^{-1}L$ *is a bounded linear operator on* $H^{2m}(\Omega; \{B_j\})$ *and, for any* u_0 *in this space,* $u(t) = e^{tA}u_0$ *is the unique solution in* $C'(R^+, H^{2m}(\Omega))$ *of* (1.1), (1.2) *satisfying* $u(0) = u_0$.

Suppose in addition (S_ρ) *holds and* $\ell_\alpha \in C^{\rho}(\bar{\Omega})$. *Then the restriction to* $C^{2m+\rho}(\bar{\Omega}; \{B_j\})$ *of the group* $\{e^{tA}: -\infty < t < +\infty\}$ *is a group* $\{e^{t\tilde{A}}: -\infty < t < +\infty\}$ *of bounded linear operators on* $C^{2m+\rho}(\bar{\Omega}; \{B_j\})$ *and, for any* u_0 *in this space,* $u(t) = e^{t\tilde{A}}u_0$ *is the unique solution of* (1.1), (1.2) *in* $C'(R^+, C^{2m+\rho}(\bar{\Omega}))$ *satisfying* $u(0) = u_0$.

3. Exponential stability of solutions in $H^{2m}(\Omega)$. In this section we give necessary and sufficient conditions in order that the group $\{e^{tM^{-1}L}: -\infty < t < +\infty\}$ of bounded linear operators in $H^{2m}(\Omega; \{B_j\})$ satisfy

$$\|e^{tM^{-1}L}\|_{2m} \leqq Ce^{-at}, \qquad t > 0,$$

with positive constants a and C; here $\|e^{tM^{-1}L}\|_{2m}$ denotes the norm of the operator $e^{tM^{-1}L}$. We assume throughout this section that $(\mathscr{M}, \{B_j\}, \Omega)$ is a regular elliptic boundary value problem satisfying the spectrum condition. Since we shall have to deal with the formal adjoints of $\mathscr{L}$ and $\mathscr{M}$ we suppose in addition that the *coefficients* ℓ_α *and* m_α *are of class* $C^{|\alpha|}(\bar{\Omega})$.

The key to our results is the following (more or less) well-known criterion for exponential stability of a group generated by a bounded operator. For completeness we include its simple proof.

LEMMA 3.1. *Let* A *be a bounded operator on a Banach space. In order that there exist positive constants* a *and* C *such that* $\|e^{tA}\| \leqq Ce^{-at}$, $t > 0$, *it is necessary and sufficient that the spectrum of* A *lie in the open left half-plane.*

Proof. The operator e^{tA} may be represented as the Dunford integral

$$e^{tA} = \frac{1}{2\pi i}\int_{\Gamma} e^{t\lambda}R(\lambda, A)\,d\lambda,$$

where $R(\lambda, A)$ is the resolvent of A at λ and Γ is a rectifiable Jordan curve, oriented in the positive sense, surrounding the spectrum of A. Let $\sigma(T)$ denote the spectrum of a linear operator T. If $\sigma(A)$ lies in the open left half-plane, it follows easily from the Dunford integral representation of e^{tA} that $\|e^{tA}\| \leqq Ce^{-at}$, $t > 0$.

Conversely, suppose this inequality holds. We apply the spectral mapping theorem to the effect that

$$\sigma(e^{tA}) = e^{t\sigma(A)}.$$

Since $\|e^{tA}\|$ is $\geqq$ the spectral radius of e^{tA}, we have for $t > 0$,

$$Ce^{-at} \geqq \|e^{tA}\| \geqq \sup_{\lambda \in \sigma(e^{tA})} |\lambda| = \sup_{\mu \in \sigma(A)} [e^{t\,\mathrm{Re}\,(\mu)}]$$

so that the spectrum of A lies in the open left half-plane.

Denote by $\mathscr{L}^*$ and $\mathscr{M}^*$ the formal adjoints of $\mathscr{L}$ and $\mathscr{M}$, respectively, and by L^* and M^* the $L^2(\Omega)$ adjoints of L and M. Since $(\mathscr{M}, \{B_j\}, \Omega)$ is a regular elliptic boundary value problem, it is well known that M^* is the $L^2(\Omega)$ realization of the regular elliptic boundary value problem $(\mathscr{M}^*, \{B_j^*\}, \Omega)$, where $\{B_j^*\}_1^m$ is the system of boundary operators adjoint to $\{B_j\}_1^m$ relative to $\mathscr{M}$ with respect to Green's formula. (See, e.g., [3], [8].) Thus $D(M^*) = H^{2m}(\Omega; \{B_j^*\})$ and $M^*u = \mathscr{M}^*(\cdot, D)u(\cdot)$ for $u \in D(M^*)$.

THEOREM 3.1. *In order that the spectrum of $M^{-1}L$ lie in the open left half-plane the following are necessary and sufficient: For each λ with* $\mathrm{Re}\,\lambda \geqq 0$,

$$(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega) \tag{3.1}$$

is a regular elliptic boundary value problem.

$$\mathrm{Re}\,(\mathscr{L}u, \mathscr{M}u) < [\|\mathscr{L}u\|^2\|\mathscr{M}u\|^2 - |\mathrm{Im}\,(\mathscr{L}u, \mathscr{M}u)|^2]^{1/2} \tag{3.2}$$

for all $u \in H^{2m}(\Omega; \{B_j\}), \quad u \not\equiv 0.$

$$\mathrm{Re}\,(\mathscr{L}^*u, \mathscr{M}^*u) < [\|\mathscr{L}^*u\|^2\|\mathscr{M}^*u\|^2 - |\mathrm{Im}\,(\mathscr{L}^*u, \mathscr{M}^*u)|^2]^{1/2} \tag{3.3}$$

for each λ with $\mathrm{Re}\,\lambda \geqq 0$ *and all* $u \in H^{2m}(\Omega; \{C_j^\lambda\})$, $u \not\equiv 0$, *where* $\{C_j^\lambda\}_1^m$ *is any system of boundary operators adjoint to* $\{B_j\}_1^m$ *relative to* $\mathscr{L} - \lambda\mathscr{M}$.

It is easy to check that the boundary system $\{C_j\}_1^m = \{B_j' - \bar{\lambda}B_j''\}_1^m$ is adjoint to $\{B_j\}_1^m$ relative to $\mathscr{L} - \lambda\mathscr{M}$, where $\{B_j'\}_1^m$ and $\{B_j''\}_1^m$ are adjoint to $\{B_j\}_1^m$ relative to $\mathscr{L}$ and $\mathscr{M}$, respectively, with respect to Green's formula. As is known, any other such system $\{C_j^\lambda\}_1^m$ is equivalent to $\{C_j\}_1^m$, that is, $H^{2m}(\Omega; \{C_j\}) = H^{2m}(\Omega; \{C_j^\lambda\})$.

COROLLARY 3.1. *Suppose* $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ *is a regular elliptic boundary value problem whenever* $\mathrm{Re}\,\lambda \geqq 0$. *Then* $\|e^{tM^{-1}L}\|_{2m} \leqq Ce^{-at}$, $t > 0$, *if and only if the eigenvalues of each of the eigenvalue problems*

$$\mathscr{L}u - \lambda\mathscr{M} = 0, \qquad u \in H^{2m}(\Omega; \{B_j\}), \tag{3.4}$$

$$\mathscr{L}^*u - \lambda\mathscr{M}^*u = 0, \qquad u \in H^{2m}(\Omega, \{C_j^\lambda\}), \tag{3.5}$$

lie in the open left half-plane.

Corollary 3.1 is an immediate consequence of the following lemma and Theorem 3.1.

LEMMA 3.2. *Condition* (3.2) *is equivalent to the condition that the eigenvalues of* (3.4) *lie in the open left half-plane. Condition* (3.3) *is equivalent to the condition that the eigenvalues of* (3.5) *lie in the open left half-plane.*

Proof. We prove only the first statement of the lemma. The second is proved in a similar way. Suppose (3.2) holds and λ is an eigenvalue of (3.4.). Then (3.2) implies

$$\operatorname{Re} \lambda < (|\lambda|^2 - |\operatorname{Im} \lambda|^2)^{1/2} = |\operatorname{Re} \lambda|$$

so that $\operatorname{Re} \lambda < 0$.

Conversely, suppose the eigenvalues of (3.4) all lie in the open left half-plane and suppose (3.2) is violated, i.e., equality holds in (3.2) for some $u \in H^{2m}(\Omega; \{B_j\})$, $u \not\equiv 0$. Then for such a u,

$$\begin{aligned} \|Lu\| \, \|Mu\| \geqq |(Lu, Mu)| &= [(\operatorname{Re}(Lu, Mu))^2 + (\operatorname{Im}(Lu, Mu))^2]^{1/2} \\ &= \|Lu\| \, \|Mu\|, \end{aligned}$$

that is, $|(Lu, Mu)| = \|Lu\| \, \|Mu\|$. This implies $Lu = \lambda Mu$ for some complex λ which satisfies, by assumption, $\operatorname{Re} \lambda < 0$. But since equality holds for u in (3.2) we obtain

$$\operatorname{Re} \lambda = (|\lambda|^2 - |\operatorname{Im} \lambda|^2)^{1/2} = |\operatorname{Re} \lambda|,$$

a contradiction.

Theorem 3.1 and Corollary 3.1 can be greatly simplified if we restrict ourselves to absolutely elliptic problems. A regular elliptic boundary value problem $(A, \{B_j\}, \Omega)$ is called an *absolutely elliptic* boundary value problem if the boundary system $\{B_j\}_1^m$ has the property that the complementing condition is always satisfied no matter what the elliptic operator A of order $2m$ may be (subject to the roots condition if $n = 2$). The Dirichlet problem is an example of an absolutely elliptic problem. Several other examples may be found in [1]. The algebraic structure of all absolutely elliptic boundary value problems was determined by Hörmander [4] who was the first to introduce this class of operators.

THEOREM 3.2. *Suppose for each* λ *with* $\operatorname{Re} \lambda \geqq 0$, $(\mathscr{L} - \lambda \mathscr{M}, \{B_j\}, \Omega)$ *is an absolutely elliptic boundary value problem and that there is a real number* $\theta = \theta(\lambda)$ *such that*

$$(3.6) \qquad (-1)^m \frac{\mathscr{L}'(x, \xi) - \lambda \mathscr{M}'(x, \xi)}{|\mathscr{L}'(x, \xi) - \lambda \mathscr{M}'(x, \xi)|} \neq e^{i\theta}$$

for all real n-vectors ξ *and* $x \in \bar{\Omega}$. *Then* $\|e^{tM^{-1}L}\|_{2m} \leqq Ce^{-at}$ *if and only if* (3.2) *is satisfied or, equivalently, the eigenvalues of* (3.4) *lie in the open left half-plane.*

COROLLARY 3.2. *Suppose for each* λ *with* $\operatorname{Re} \lambda \geqq 0$, $\mathscr{L} - \lambda \mathscr{M}$ *is strongly elliptic in* $\bar{\Omega}$ *and that* $(\mathscr{L} - \lambda \mathscr{M}, \{B_j\}, \Omega)$ *is an absolutely elliptic boundary value problem. Then the conclusion of Theorem* 3.2 *is valid.*

Proof of Theorem 3.1. *Necessity*. If the spectrum of $M^{-1}L$ lies in the left half-plane, a fortiori the eigenvalues lie there. As M^{-1} is a bounded operator from $L^2(\Omega)$ onto $H^{2m}(\Omega; \{B_j\})$, for $f \in L^2(\Omega)$ the equations $(M^{-1}L - \lambda)u = M^{-1}f$ and $Lu - \lambda Mu = f$ are equivalent. Consequently the eigenvalues of (3.4) also lie in the left half-plane. Inequality (3.2) therefore follows from Lemma 3.2. Suppose

$\operatorname{Re}\lambda \geqq 0$. Then $M^{-1}L - \lambda$ maps $H^{2m}(\Omega; \{B_j\})$ onto itself and for all u in this space,

$$\|(M^{-1}L - \lambda)u\|_{2m} \geqq C_\lambda \|u\|_{2m}.$$

It follows that the range of $L - \lambda M$ is all of $L^2(\Omega)$ and

$$\|(\mathscr{L} - \lambda\mathscr{M})u\| \geqq C_\lambda \|u\|_{2m}, \qquad u \in H^{2m}(\Omega; \{B_j\}). \tag{3.7}$$

Since the ellipticity, roots and complementing conditions are necessary for the validity of (3.7) we conclude that $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ is a regular elliptic boundary value problem. Therefore $(L - \lambda M)^*$ is the $L^2(\Omega)$ realization of the regular elliptic boundary value problem $(\mathscr{L}^* - \bar{\lambda}\mathscr{M}^*, \{C_j^\lambda\}, \Omega)$, where $\{C_j^\lambda\}_1^m$ is adjoint to $\{B_j\}_1^m$ relative to $\mathscr{L} - \lambda\mathscr{M}$. If λ is an eigenvalue of (3.5) having nonnegative real part, then $(L - \bar{\lambda}M)^*v = 0$ for some $v \in H^{2m}(\Omega, \{C_j^\lambda\})$, and so for all $u \in H^{2m}(\Omega; \{B_j\})$,

$$(Lu - \bar{\lambda}Mu, v) = (u, (L - \bar{\lambda}M)^*v) = 0.$$

If $\operatorname{Re}\lambda \geqq 0$, the range of $L - \bar{\lambda}M$ is all of $L^2(\Omega)$, and so $v \equiv 0$. Condition (3.3) therefore follows from Lemma 3.2.

Sufficiency. Condition (3.1) implies that for $\operatorname{Re}\lambda \geqq 0$, $L - \lambda M$ is a closed, densely defined operator having closed range in $L^2(\Omega)$. Thus to complete the proof we must show that for such λ the null space of both $L - \lambda M$ and $(L - \lambda M)^*$ is $\{0\}$. However this follows from (3.2), (3.3), Lemma 3.2 and the fact that $(L - \lambda M)^*$ is the $L^2(\Omega)$ realization of the problem $(\mathscr{L}^* - \bar{\lambda}\mathscr{M}^*, \{C_j^\lambda\}, \Omega)$.

Proof of Theorem 3.2. We have only to prove the sufficiency part. If (3.2) holds, then for $\operatorname{Re}\lambda \geqq 0$ the null space of $L - \lambda M$ is $\{0\}$. Since $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ is an absolutely elliptic problem, condition (3.6) implies that the spectrum of $L - \lambda M$ is discrete and consists solely of eigenvalues of finite multiplicity. This has been proved by Agmon [1]. Since zero is not an eigenvalue of $L - \lambda M$ it must therefore be a point of the resolvent set. Thus $L - \lambda M$ maps $H^{2m}(\Omega; \{B_j\})$ onto $L^2(\Omega)$ in a one-to-one way for each λ with $\operatorname{Re}\lambda \geqq 0$, as was to be shown.

Proof of Corollary 3.2. We may suppose $\mathscr{L} - \lambda\mathscr{M}$ is normalized by

$$(-1)^m \operatorname{Re}[\mathscr{L}'(x, \xi) - \lambda\mathscr{M}'(x, \xi)] \geqq \delta|\xi|^{2m} \tag{3.8}$$

for all real n-vectors ξ and $x \in \bar{\Omega}$, where $\delta = \delta(\lambda) > 0$. But (3.8) implies

$$\cos\{\arg(-1)^m[\mathscr{L}'(x, \xi) - \lambda\mathscr{M}'(x, \xi)]\} \geqq \varepsilon > 0$$

for all such ξ and x, that is, (3.6) is satisfied for each $\theta \notin (-\pi/2, \pi/2)$.

Remark. If $\mathscr{M}$ and $\mathscr{L}$ are given in divergence form

$$\mathscr{M}(x, D) = \sum_{\substack{|\alpha| \leqq m \\ |\beta| \leqq m}} D^\alpha(m_{\alpha\beta}(x)D^\beta), \tag{3.9}$$

$$\mathscr{L}(x, D) = \sum_{\substack{|\alpha| \leqq m \\ |\beta| \leqq m}} D^\alpha(\ell_{\alpha\beta}(x)D^\beta), \tag{3.10}$$

the smoothness assumptions on the coefficients stated at the beginning of this section may be replaced by : *$m_{\alpha\beta}$ and $\ell_{\alpha\beta}$ are of class $C^{|\alpha|}(\bar{\Omega}) \cap C^{|\beta|}(\bar{\Omega})$*. All the results of this section remain true with no change in their proofs.

4. Exponential stability of solutions in $C^{2m+\rho}(\bar{\Omega})$. In this section we assume $(\mathscr{M}, \{B_j\}, \Omega)$ is a regular elliptic boundary value problem satisfying the spectrum condition and conditions (S_ρ) of § 2. In addition we require *the coefficients* ℓ_α *and* m_α *to be of class* $C^{|\alpha|+\rho}(\bar{\Omega})$. Under these conditions we study exponential stability of the group $\{e^{t\tilde{A}}: -\infty < t < +\infty\}$ of bounded linear operators in $C^{2m+\rho}(\bar{\Omega};\{B_j\})$. We recall that $\tilde{A}$ is a bounded linear operator on $C^{2m+\rho}(\bar{\Omega};\{B_j\})$ defined by $\tilde{A} = \tilde{M}^{-1}\tilde{L}$, where $\tilde{M}$ and $\tilde{L}$ are the restrictions of M and L, respectively, to $C^{2m+\rho}(\bar{\Omega};\{B_j\})$.

In what follows, a, $\tilde{a}$, C and $\tilde{C}$ will denote positive constants and $|e^{t\tilde{A}}|_{2m+\rho}$ the norm of the operator $e^{t\tilde{A}}$.

THEOREM 4.1. $|e^{t\tilde{A}}|_{2m+\rho} \leqq \tilde{C}e^{-\tilde{a}t}$, $t > 0$, *if and only if* $\|e^{tA}\|_{2m} \leqq Ce^{-at}$, $t > 0$.

THEOREM 4.2. *In order that* $|e^{t\tilde{A}}|_{2m+\rho} \leqq \tilde{C}e^{-\tilde{a}t}$, *the following are necessary and sufficient*:

$$\text{(4.1)} \qquad \textit{For } \operatorname{Re}\lambda \geqq 0, \qquad (\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$$

is a regular elliptic boundary problem.

$$\text{(4.2)} \qquad \operatorname{Re}(\mathscr{L}u, \mathscr{M}u) < [\|\mathscr{L}u\|^2\|\mathscr{M}u\|^2 - |\operatorname{Im}(\mathscr{L}u, \mathscr{M}u)|^2]^{1/2}$$

for all $u \in C^{2m+\rho}(\bar{\Omega};\{B_j\})$, $u \not\equiv 0$.

$$\text{(4.3)} \qquad \operatorname{Re}(\mathscr{L}^*u, \mathscr{M}^*u) < [\|\mathscr{L}^*u\|^2\|\mathscr{M}^*u\|^2 - |\operatorname{Im}(\mathscr{L}^*u, \mathscr{M}^*u)|^2]^{1/2}$$

for each λ *with* $\operatorname{Re}\lambda \geqq 0$ *and all* $u \in C^{2m+\rho}(\bar{\Omega};\{C_j^\lambda\})$, $u \not\equiv 0$.

The system $\{C_j^\lambda\}_1^m$ in (4.3) is the same as in § 3.

COROLLARY 4.1. *Suppose* $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ *is a regular elliptic boundary value problem whenever* $\operatorname{Re}\lambda \geqq 0$. *Then* $|e^{t\tilde{A}}|_{2m+\rho} \leqq \tilde{C}e^{-\tilde{a}t}$ *if and only if the eigenvalues of each of the problems*

$$\text{(4.4)} \qquad \mathscr{L}u - \lambda\mathscr{M}u = 0, \qquad u \in C^{2m+\rho}(\bar{\Omega};\{B_j\}),$$

$$\text{(4.5)} \qquad \mathscr{L}^*u - \lambda\mathscr{M}^*u = 0, \qquad u \in C^{2m+\rho}(\bar{\Omega}, \{C_j^\lambda\}),$$

lie in the open left half-plane.

Proof. Just as in Lemma 3.2, we see that conditions (4.2) and (4.3) are equivalent to the condition that the eigenvalues of (4.4) and (4.5) lie in the left half-plane.

For absolutely elliptic problems we have the following.

THEOREM 4.3. *Suppose for each* λ *with* $\operatorname{Re}\lambda \geqq 0$, $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ *is an absolutely elliptic boundary value problem satisfying* (3.6). *Then* $|e^{t\tilde{A}}|_{2m+\rho} \leqq \tilde{C}e^{-\tilde{a}t}$ *if and only if* (4.2) *is satisfied or, equivalently, the eigenvalues of* (4.4) *lie in the open left half-plane.*

COROLLARY 4.2. *Suppose for each* λ *with* $\operatorname{Re}\lambda \geqq 0$, $\mathscr{L} - \lambda\mathscr{M}$ *is strongly elliptic in* $\bar{\Omega}$ *and that* $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ *is an absolutely elliptic boundary value problem. Then the conclusion of Theorem* 4.3 *is valid.*

Proof of Theorem 4.1. Suppose $\|e^{tA}\|_{2m} \leqq Ce^{-at}$, $t > 0$, so that the spectrum of A lies in the left half-plane. If $\operatorname{Re}\lambda \geqq 0$ and $f \in C^\rho(\bar{\Omega})$, there is a unique $u \in H^{2m}(\Omega;\{B_j\})$ satisfying $(\mathscr{L} - \lambda\mathscr{M})u = f$. But $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ is a regular elliptic boundary value problem (by Theorem 3.1) and satisfies the conditions of Theorem 2.1. Thus $u \in C^{2m+\rho}(\bar{\Omega};\{B_j\})$. Since $\tilde{M}$ maps $C^{2m+\rho}(\bar{\Omega};\{B_j\})$ boundedly onto $C^\rho(\bar{\Omega})$ it follows that λ is in the resolvent set of $\tilde{A}$. Thus the spectrum of $\tilde{A}$ lies in the left half-plane.

Conversely, suppose this condition is satisfied. As in the proof of Theorem 3.1, one finds that for $\operatorname{Re}\lambda \geqq 0$, the range of $\tilde{L} - \lambda\tilde{M}$ is all of $C^\rho(\bar{\Omega})$ and

$$|(\mathscr{L} - \lambda\mathscr{M})u|_\rho \geqq C_\lambda |u|_{2m+\rho}, \qquad u \in C^{2m+\rho}(\bar{\Omega}; \{B_j\}).$$

These inequalities imply that $(\mathscr{L} - \lambda\mathscr{M}, \{B_j\}, \Omega)$ is a regular elliptic boundary value problem. We now apply Corollary 3.1 to obtain the desired result. If $\operatorname{Re}\lambda \geqq 0$ and $\mathscr{L}u - \lambda\mathscr{M}u = 0$ for some $u \in H^{2m}(\Omega; \{B_j\})$, another application of Theorem 2.1 gives $u \in C^{2m+\rho}(\bar{\Omega}; \{B_j\})$, i.e., $(\tilde{A} - \lambda)u = 0$. Thus $u \equiv 0$. If $(\mathscr{L}^* - \bar{\lambda}\mathscr{M}^*)v = (L - \bar{\lambda}M)^*v = 0$ for some $v \in H^{2m}(\Omega; \{C_j^\lambda\})$, then for all $u \in C^{2m+\rho}(\bar{\Omega}; \{B_j\})$,

$$0 = (u, (L - \bar{\lambda}M)^*v) = (\tilde{L}u - \bar{\lambda}\tilde{M}u, v).$$

Since the range of $\tilde{L} - \bar{\lambda}\tilde{M}$ is dense in $L^2(\Omega)$, we conclude $v \equiv 0$.

Proof of Theorem 4.2. If $|e^{t\tilde{A}}|_{2m+\rho} \leqq \tilde{C}e^{-\tilde{a}t}$, then $\|e^{tA}\|_{2m} \leqq Ce^{-at}$ so that (3.1)–(3.3) hold, and, a fortiori, (4.1)–(4.3).

Conversely suppose (4.1)–(4.3) are true. We show that (3.1)–(3.3) also must hold. The desired conclusion then will follow from Theorems 3.1 and 4.1.

If $\operatorname{Re}\lambda \geqq 0$, condition (4.1) and Theorem 2.1 show that the eigenvalue problems (3.4) and (4.4) are equivalent. Consequently, (4.2) implies (3.2). Moreover $(L - \lambda M)^*$ is the $L^2(\Omega)$ realization of the problem $(\mathscr{L}^* - \bar{\lambda}\mathscr{M}^*, \{B_j' - \bar{\lambda}B_j''\}, \Omega)$, where $\{B_j'\}_1^m$ and $\{B_j''\}_1^m$ are adjoint to $\{B_j\}_1^m$ relative to $\mathscr{L}$ and $\mathscr{M}$, respectively, with respect to Green's formula (see the paragraph following Theorem 3.1). The coefficients in this problem also satisfy the conditions of Theorem 2.1 by virtue of the smoothness assumption (S_ρ) and the conditions imposed on ℓ_α and m_α at the beginning of this section. Consequently, for $\operatorname{Re}\lambda \geqq 0$ solutions of the eigenvalue problem (3.5) belong to $C^{2m+\rho}(\bar{\Omega}; \{C_j^\lambda\})$ so that the problems (3.5) and (4.5) are equivalent for such λ.

Proof of Theorem 4.3. This result, and its corollary, follow easily from Theorem 4.1 and the results of § 3.

Remark. If $\mathscr{M}$ and $\mathscr{L}$ are given in the divergence forms (3.9) and (3.10), the conditions on the coefficients stated at the beginning of this section may be replaced by: *$m_{\alpha\beta}$ and $\ell_{\alpha\beta}$ are of class* $C^{|\alpha|+\rho}(\bar{\Omega}) \cap C^{|\beta|+\rho}(\bar{\Omega})$.

5. Example. Suppose $\mathscr{M}$ and $\mathscr{L}$ are given in the divergence forms (3.9) and (3.10) with coefficients $m_{\alpha\beta}$ and $\ell_{\alpha\beta}$ of class $C^{|\alpha|+\rho}(\bar{\Omega}) \cap C^{|\beta|+\rho}(\bar{\Omega})$, where Ω is assumed to be of class $C^{2m+\rho}$. We assume also that $\mathscr{M}$ is formally self-adjoint and elliptic in $\bar{\Omega}$, that $\mathscr{L}$ is strongly elliptic there and that these operators are normalized by

$$(-1)^{m+1}\mathscr{M}'(x, \xi) \geqq C_0|\xi|^{2m}, \qquad (-1)^m \operatorname{Re}\mathscr{L}'(x, \xi) \geqq C_1|\xi|^{2m}$$

for $x \in \bar{\Omega}$, $\xi \neq 0$, where C_0 and C_1 are positive. Then according to Garding's inequality there are positive constants k_L, K_L, k_M, and K_M such that

$$\operatorname{Re}(\mathscr{L}u, u) \geqq K_L\|u\|_m^2 - k_L\|u\|^2, \tag{5.1}$$

$$(\mathscr{M}u, u) \leqq -K_M\|u\|_m^2 + k_M\|u\|^2 \tag{5.2}$$

for all $u \in C^{2m}(\bar{\Omega})$ satisfying on $\partial\Omega$ the Dirichlet conditions

$$B_ju = \left(\frac{\partial}{\partial n}\right)^{j-1} u = 0, \qquad j = 1, 2, \cdots, m. \tag{5.3}$$

Set $k = \max(k_L, k_M)$ and consider the equation

$$(\mathscr{M} - k)\frac{\partial u}{\partial t} - (\mathscr{L} + k)u = 0 \tag{5.4}$$

in the cylinder $\Omega \times [0, \infty)$, together with the boundary conditions (5.3) on the wall of the cylinder. We show that the conditions of Corollary 4.2 are satisfied for the operators $\mathscr{L} + k, \mathscr{M} - k$ and the boundary operators $B_j = (\partial/\partial n)^{j-1}, j = 1, \cdots, m$.

THEOREM 5.1. *With the assumptions of this section, let $u(t) = e^{tA}u(0)$ be a solution of* (5.3), (5.4) *in $C'(R^+, C^{2m+\rho}(\bar{\Omega}))$. Then*

$$\|u(t)\|_{2m} \leqq Ce^{-at}\|u(0)\|_{2m}, \qquad t > 0,$$

and

$$|u(t)|_{2m+\rho} \leqq \tilde{C}e^{-\tilde{a}t}|u(0)|_{2m+\rho}, \qquad t > 0,$$

with positive constants a, $\tilde{a}$, C and $\tilde{C}$.

Proof. (i) $(\mathscr{M} - k, \{(\partial/\partial n)^{j-1}\}, \Omega)$ is a regular elliptic boundary value problem satisfying the spectrum condition. In fact, the roots condition holds since the coefficients of $\mathscr{M}$ are real. Also, the Dirichlet boundary conditions satisfy the complementing condition no matter what the elliptic operator of order $2m$ may be. In addition, the Dirichlet boundary operators are self-adjoint, that is, the system of boundary operators adjoint to $\{(\partial/\partial n)^{j-1}\}_i^m$ relative to any elliptic operator of order $2m$ is again the Dirichlet system. Since $\mathscr{M} - k$ is formally self-adjoint, it follows that the $L^2(\Omega)$-realization M_k of the boundary value problem $(\mathscr{M} - k, \{(\partial/\partial n)^{j-1}\}_i^m, \Omega)$ is a self-adjoint operator, $M_k = M_k^*$. From (5.2) it follows that the null space of M_k is $\{0\}$ and, therefore, zero is in the resolvent set of M_k.

(ii) For $\operatorname{Re} \lambda \geqq 0$, $((\mathscr{L} + k) - \lambda(\mathscr{M} - k), \{(\partial/\partial n)^{j-1}\}, \Omega)$ is an absolutely elliptic boundary value problem. To prove this we have only to check the ellipticity and roots condition. But for $\operatorname{Re} \lambda \geqq 0$ we have

$$\begin{aligned}(-1)^m \operatorname{Re}(\mathscr{L}'(x, \xi) - \lambda\mathscr{M}'(x, \xi)) &\geqq (C_1 + (\operatorname{Re}\lambda)C_0)|\xi|^{2m} \\ &\geqq C_1|\xi|^{2m},\end{aligned}$$

that is, $(\mathscr{L} + k) - \lambda(\mathscr{M} - k)$ is a strongly elliptic operator in $\bar{\Omega}$. As is well known, strong ellipticity implies the roots condition.

Let L_k be the realization in $L^2(\Omega)$ of the regular elliptic boundary value problem $(\mathscr{L} + k, \{(\partial/\partial n)^{j-1}\}, \Omega)$.

(iii) The eigenvalues of the problem

$$L_k u - \lambda M_k u = 0$$

lie in the open left half-plane.

In fact, suppose λ is an eigenvalue of $(L_k - \lambda M_K)u = 0$. From (5.1) we have

$$0 < \operatorname{Re}(L_k u, u) = (\operatorname{Re}\lambda)(M_k u, u).$$

Thus $\operatorname{Re}\lambda < 0$ on account of (5.2). Theorem 5.1 therefore follows from Corollary 4.2.

Remark. In the special case where

$$\mathscr{M}(x, D) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(m_{ij}(x)\frac{\partial}{\partial x_j}\right) - m(x), \qquad m(x) > 0,$$

$$\mathscr{L}(x, D) = \ell(x) - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\ell_{ij}(x)\frac{\partial}{\partial x_j}\right), \qquad \ell(x) \geqq 0,$$

and (m_{ij}) and (ℓ_{ij}) are real, symmetric, positive definite matrices, Theorem 5.1 applies with the constant $k = 0$ in (5.4), and we obtain the main result of [12].

REFERENCES

[1] S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math., 15 (1962), pp. 119–147.

[2] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Ibid., 12 (1959), pp. 623–727.

[3] F. Browder, *Estimates and existence theorems for elliptic boundary value problems*, Proc. Nat. Acad. Sci. U.S.A., 45 (1959), pp. 365–372.

[4] L. Hörmander, *On the regularity of the solutions of boundary problems*, Acta Math., 99 (1958), pp. 225–264.

[5] N. Ikebe, *Smoothness of the generalized solutions of linear elliptic partial differential equations*, Math. Rep. General Ed. Dept. Kyushu Univ., 3 (1965), pp. 19–26; 4(1966/67), pp. 11–17.

[6] J. Lagnese, *General boundary value problems for differential equations of Sobolev type*, this Journal, 3 (1972), pp. 105–119.

[7] V. Maslennikova, *L_p-estimates and the asymptotic behavior as $t \to \infty$ of a solution of the Cauchy problem for a Sobolev system*, Proc. Steklov Inst. Math., 103 (1968), pp. 117–141.

[8] M. Schechter, *General boundary value problems for elliptic partial differential equations*, Comm. Pure Appl. Math., 12 (1959), pp. 457–486.

[9] R. E. Showalter, *Well-posed problems for a partial differential equation of order $2m + 1$*, this Journal, 1 (1970), pp. 214–231.

[10] ———, *The Sobolev equation. II*, Applicable Anal., to appear.

[11] R. E. Showalter and T. W. Ting, *Pseudoparabolic partial differential equations*, this Journal, 1 (1970), pp. 1–26.

[12] ———, *Asymptotic behavior of solutions of pseudoparabolic partial differential equations*, Annali. Math., to appear.

[13] T. Zelenjak, *The behavior for $t \to \infty$ of a problem of S. L. Sobolev*, Soviet Math. Dokl., 2 (1961), pp. 956–958.

[14] ———, *On the asymptotic behavior of the solutions of a mixed problem*, Differential Equations, 2 (1966), pp. 47–64.

[15] ———, *The behavior at infinity of solutions of a certain mixed problem*, Ibid., 5 (1969), pp. 1676–1689.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

CHARACTERIZATIONS OF σ-TYPE ZERO POLYNOMIAL SETS*

ARUN VERMA†

Abstract. In this note two characterizations of σ-type zero polynomial sets are obtained. These characterizations are generalizations of two known results for the Appell polynomial sets.

1. Rainville [1], generalizing the well-known classification of simple polynomial sets like Appell, Sheffer A-type m (for details see Sheffer [2]), introduced σ-type m classification of simple polynomial sets as follows:

Let $\{p_n(x)\}$ be a simple set of polynomials that belongs to the operator

$$J(x, \sigma) = \sum_{k=0}^{\infty} T_k(x)\sigma^{k+1}, \quad \sigma = D \prod_{i=1}^{q} [xD + \beta_i - 1], \quad D \equiv \frac{d}{dx}$$

(i.e., $J(x, \sigma)p_n(x) = p_{n-1}(x)$), where β_i are constants not equal to zero or a negative integer, and $T_k(x)$ are polynomials of degree $\leqq k$. We say that this set is of σ-type m if the maximum degree of $T_k(x)$ is $m, m = 0, 1, 2, \cdots$.

Rainville [1] has shown that the simple set of polynomials $p_n(x)$ associated with the operator J is of the σ-type zero if and only if $p_n(x)$ has a generating function of the form

$$\sum_{n=0}^{\infty} p_n(x)t^n = A(t)\, {}_0F_q[\,\cdot\,; (\beta_q); xH(t)],$$

where

$$J(H(t)) = H(J(t)) = t,$$

$$A(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad a_0 \neq 0,$$

$$H(t) = \sum_{n=0}^{\infty} h_n t^{n+1}, \qquad h_0 \neq 0.$$

($A(t)$ is called the determining function of $\{p_n(x)\}$.)

It is clear from the definition that if $q = 0$, then σ-type zero polynomial sets are of the Sheffer A-type zero, which in turn incorporate the Appell set of polynomials.

Furthermore, we know that the necessary and sufficient condition for a given simple polynomial set $\{p_n(x)\}$ to be of Appell type is that there exist a sequence of constants a_n such that

$$p_n(x) = \sum_{k=0}^{n} a_{n-k}x^k.$$

In this note it is intended to prove a similar characterization for σ-type zero polynomials. The note is concluded by mentioning a characterization of σ-type

* Received by the editors March 23, 1971, and in final revised form January 31, 1972.

† College of Arts and Sciences, Pahlavi University, Shirsz, Iran. Now at Department of Mathematics, Lucknow University, Lucknow, India.

zero polynomials in terms of Stieltjes integrals analogous to the one proved by Sheffer [3] for Sheffer A-type zero polynomials.

2. We use the following notation: $[a]_0 = 1, [a]_n = a(a+1)\cdots(a+n-1)$; $[a]^0 = 1, [a]^n = a(a-1)\cdots(a-n+1); n = 1, 2, \cdots$. (a_r) denotes a sequence of r numbers $a_1, a_2, \cdots, a_r$ and $[(b_r)+a]_n = \prod_{s=1}^{r}[b_s + a]_n$ and

$$[b + (a_r)]^n = \prod_{s=1}^{r} [b + a_s]^n.$$

3. Let $\sigma = D\prod_{i=1}^{q}[xD + \beta_i - 1]$. Then

$$\sigma^{k+1}x^n = [n]^{k+1}[n + (\beta_q) - 1]^{k+1}x^{n-k-1}.$$

Let $\{p_n(x)\}$ be a simple polynomial set of σ-type zero and let

$$J(x, \sigma) = \sum_{k=0}^{\infty} a_k \sigma^{k+1},$$

$a_0 = 1$, be the operator associated with this set. If we put $p_n(x) = \sum_{k=0}^{n} C(n, k)x^k$, $C(n, n) = 1/[1]_n[(\beta_q)]^n$, then we have

$$\begin{aligned} J(x, \sigma)p_n(x) &= \sum_{l=0}^{n-1} x^l \sum_{k=0}^{n-l-1} [k + l + 1]^{k+1}[k + (\beta_q) + l]^{k+1} a_k C(n, k + 1 + l) \\ &= p_{n-1}(x) = \sum_{l=0}^{n-1} C(n-1, l)x^l. \end{aligned} \tag{3.1}$$

Now if we set

$$D(n, k) = [1]_k[(\beta_q)]_k C(n, k), \qquad k = 0, 1, \cdots, n-1,$$
$$D(n, n) = [1]_n[(\beta_q)]_n C(n, n),$$

and use the fact that

$$[(\beta_q)]_l[k + (\beta_q) + l]^{k+1}/[(\beta_q)]_{k+l+1} = 1,$$

and equate the coefficients of x^l on both sides of (3.1), then we obtain

$$\sum_{k=0}^{n-l-1} a_k D(n, k + 1 + l) = D(n-1, l), \qquad l = 0, 1, 2, \cdots, n-1, \quad n = 1, 2, \cdots, \tag{3.2}$$

where $a_0 = 1$, $D(n, j) = 0, j > n$ and $D(n, n) = 1$.

Let $\{k_l\}$ be a sequence of arbitrary numbers. Then the general solution of (3.2) is given by

$$D(n, n-j) = \sum_{s=0}^{j} \frac{[n]_{j-s}(-)^{j-s}}{[1]_{j-s}} \sum_{r+1}^{s+1} \binom{j-s}{r-1} a_1^{j-s-r+1} A_{r,s} \tag{3.3}$$

with

$$A_{1,s} = k_s,$$
$$A_{r,s} = \sum{}' a_{i_1}a_{i_2}\cdots a_{i_{r-1}}k_{i_r}, \qquad r = 2, 3, \cdots, s+1, \tag{3.4}$$

where $\sum'$ means the sum over those $r-1$ numbers $a_i \in \{a_2, a_3, \cdots, a_{s-r+3}\}$ and one number $k_i \in \{k_0, k_1, \cdots, k_{s-r+1}\}$ for which the sum of the suffixes equals $s+r-1$.

In order to prove this let $A_n = \sum_{h=0}^{n-1} a_h N^h$, where $a_0 = 1$ and N is the $n \times n$ matrix with entries 1 on the first superdiagonal and all other entries zero. Clearly A_n is a nonsingular matrix because it is an upper triangle matrix with entries 1 on the principal diagonal. Further, let

$$\hat{D}_{n-h} = \{D(n-h, 1-h), D(n-h, 2-h), \cdots, D(n-h, n-h)\}^T, \qquad h = 0, 1, \cdots, n,$$

where $D(n-h, j)$, $j = -1, -2, \cdots, 1-h$, are new quantities which together with $D(n-h, j)$, $j \geqq 0$, satisfy

$$\hat{D}_{n-h} = A_n \hat{D}_{n-h+1}. \tag{3.5}$$

This is (3.2) together with a recurrence formula for the new quantities $D(n-h, j)$, $j \leqq 0$.

It follows from (3.5) that

$$\hat{D}_n = (A_n)^{-n} \hat{D}_0. \tag{3.6}$$

Now choosing $D(0, 1-h) = k_{h-1}$ arbitrary for $h = 1, 2, \cdots, n$, we have

$$\hat{D}_0 = \{k_{n-1}, k_{n-2}, \cdots, k_0\}^T. \tag{3.7}$$

Further, since

$$A_n = I + \sum_{h=1}^{n-1} a_h N^h,$$

where I is the $n \times n$ unit matrix, we have

$$(A_n)^{-n} = \sum_{j=0}^{n} \frac{[n]_j}{j!} (-)^j \left(\sum_{h=1}^{n-1} a_h N^h \right)^j. \tag{3.8}$$

Now we expand the last power by means of

$$\left(\sum_{h=1}^{n-1} a_h N^h \right)^j = \sum_{l=0}^{n-1} \binom{j}{l} a_1^{j-l} \sum{}' a_{h_1} a_{h_2} \cdots a_{h_l} N^{h_1 + h_2 + \cdots + h_l + j - l}, \tag{3.9}$$

where $\sum'$ is extended over all combinations of integers $h_1, h_2, \cdots, h_l$ with $2 \leqq h_m \leqq n-1$, taking the order into account and not excluding the cases with equal entries.

Then substituting the value of $(\sum_{h=1}^{n-1} a_h N^h)^j$ from (3.9) in (3.8) and in turn using this value of $(A_n)^{-n}$ and (3.7) in (3.6) we have the desired solution for $D(n, n-j)$.

It may be remarked that in (3.3) all the terms have nonnegative powers of a_1 because those with negative powers are multiplied with binomial coefficients equal to zero.

Thus we have shown that if the set $\{p_n(x)\}$ is of the σ-type zero, then $p_n(x)$ must have the form:

$$(3.10)\quad p_n(x) = \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)![(\beta_q)]_{n-k}} \sum_{s=0}^{k} \frac{(-)^{k-s}[n]_{k-s}}{[1]_{k-s}} \sum_{r=1}^{s+1} \binom{k-s}{r-1} a_1^{k-s-r+1} A_{r,s},$$

where the $A_{r,s}$ are given by (3.4).

Conversely, if $p_n(x)$ is given by (3.10), then working backwards it is easy to see that $\{p_n(x)\}$ is of the σ-type zero. Hence we obtain the following theorem.

THEOREM 1A. *The necessary and sufficient condition that the simple polynomial set $\{p_n(x)\}$ be of σ-type zero is that there exist sequences of constants $\{k_l\}$ and $\{a_j\}$ such that $p_n(x)$ is expressible as* (3.10).

It might be of interest to note that if instead of (3.9) we use the expansion

$$\left(\sum_{h=1}^{n-1} a_h N^h\right)^j = \sum_{s=j}^{n-1} \left(\sum\nolimits_1 a_{i_1} a_{i_2} \cdots a_{i_j}\right) N^s,$$

where $\sum_1$ is extended over all combinations of integers $i_1, i_2, \cdots, i_j$ with $1 \leqq i_m \leqq s - j + 1$ such that the sum of the suffixes equals s and not excluding the cases with equal entries, and the same procedure as that for Theorem 1A is adopted, then the following characterization for σ-type zero polynomials is obtained.

THEOREM 1B. *A necessary and sufficient condition that the simple polynomial set $\{p_n(x)\}$ be of σ-type zero is that there exist constants $\{k_l\}$ and $\{a_j\}$ such that $p_n(x)$ is expressible as*

$$p_n(x) = \sum_{l=0}^{n} \frac{x^{n-l}}{[1]_{n-l}[(\beta_q)]_{n-l}} \sum_{j=0}^{l} \frac{(-)^j[n]_j}{[1]_j} \sum_{s=j}^{n-1} k_{l-s}\left(\sum\nolimits_1 a_{i_1} a_{i_2} \cdots a_{i_j}\right).$$

Setting $q = 0$ in the above theorems we obtain an explicit representation for Sheffer A-type zero polynomials.

4. We conclude the note by mentioning a characterization of σ-type zero polynomial sets analogous to the one proved by Sheffer [3] for Sheffer A-type zero polynomials.

THEOREM 2. *A simple polynomial set $\{p_n(x)\}$ is of σ-type zero if and only if there exists a function $\beta(t)$ of bounded variation on $(0, \infty)$ such that*

(i) $b_n = \int_0^\infty t^n \, d\beta(t)$ *exists for* $n = 0, 1, 2, \cdots$,
(ii) $b_0 \neq 0$,
(iii) $p_n(x) = \int_0^\infty B_n(x + t)\, d\beta(t)$,

where ${}_0F_q[\cdot\,; (\beta_q); xH(t)] = \sum_{n=0}^\infty B_n(x)t^n$ *and* $J(H(t)) = H(J(t)) = t$, *J being the operator associated with the polynomial set.*

The determining function is then

$$A(t) = \int_0^\infty {}_0F_q[\cdot\,; (\beta_q); xH(t)]\, d\beta(t).$$

The proof runs on exactly the same lines as that of Sheffer [3] for A-type zero polynomials and is omitted.

Acknowledgment. The author wishes to thank the referee for many constructive suggestions which led to the present form of this note.

REFERENCES

[1] E. D. RAINVILLE, *Special Functions*, Macmillan, New York, 1965.
[2] I. M. SHEFFER, *Some properties of polynomial sets of type zero*, Duke Math. J., 5 (1939), pp. 590–622.
[3] ———, *Note on Appell polynomials*, Bull. Amer. Math. Soc., 51 (1945), pp. 739–744.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

A CONCAVE PROPERTY OF THE HYPERGEOMETRIC FUNCTION WITH RESPECT TO A PARAMETER*

JOSEPH B. KADANE†

Abstract. The hypergeometric function is shown to be logarithmically concave in integer values of one of its parameters. The methods used are probabilistic.

THEOREM. *Let m, i and g be positive integers satisfying $3 \leqq i + 1 \leqq g$, and let z be a negative real number. Then*

$$\{{}_2F_1[-m, i : g : z]\}^2 > {}_2F_1[-m, i + 1 ; g ; z]\, {}_2F_1[-m, i - 1 ; g ; z].$$

We first establish the following lemma concerning the evaluation of the generating function of the negative hypergeometric distribution.

LEMMA.

$$\sum_{j=0}^{k} \binom{b + j - 1}{j} \binom{k + a - j - 1}{k - j} s^j \Big/ \binom{a + b + k - 1}{k} = {}_2F_1[-k, b ; a + b ; 1 - s] \tag{1}$$

for all positive integers k, and all real s, and positive real values of a and b.

Proof of Lemma. Skellam [2] has shown that if X follows a binomial distribution with parameters p and k, and if p is integrated with respect to the normalized beta function

$$\frac{p^{b-1}(1 - p)^{a-1}\, dp}{B(a, b)},$$

then the unconditional distribution of X is negative hypergeometric, that is,

$$\Pr\{X = j\} = \binom{b + j - 1}{j} \binom{k + a - j - 1}{k - j} \Big/ \binom{a + b + k - 1}{k}.$$

The left-hand side of (1), denoted below by I, is then the probability generating function of the negative hypergeometric distribution. Thus

$$\begin{aligned} I = \mathscr{E}(s^X) &= \mathscr{E}_p\{(s^X|p)\} = \mathscr{E}_p(1 - p + ps)^k \\ &= \frac{1}{B(a, b)} \int_0^1 [1 - p(1 - s)]^k p^{b-1}(1 - p)^{a-1}\, dp \\ &= {}_2F_1[-k, b ; a + b ; 1 - s]. \end{aligned}$$

See [3, p. 20]. This proves the lemma.

Proof of Theorem. The essence of the proof is to use two theorems proved elsewhere [1], one on the existence of a probability distribution with a certain property, the other giving an inequality relating to such a distribution.

* Received by the editors October 19, 1971, and in revised form February 22, 1972.

† Department of Statistics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213.

Let $s = 1 + z$, $h = g - 1$ and $n = m + g - 1$. Theorem 3 of [1] states that there is a probability distribution F such that $a_{i,n}$, the expected value of the ith largest of a sample of size n drawn independently from F, satisfies

$$a_{i,n} = s^{i-1} \quad \text{for all } i, \quad 1 \leqq i \leqq n.$$

By use of a standard recurrence relation, quoted in [1, (4)], the expected value of the ith largest of some smaller sample of size h can be deduced as follows: For $1 \leqq i \leqq g \leqq n$,

$$\begin{aligned} a_{i,h} &= \sum_{j=0}^{n-h} \binom{i+j-1}{j}\binom{n-j-i}{n-h-j} a_{i+j,n} \Big/ \binom{n}{h} \\ &= \sum_{j=0}^{n-h} \binom{n-j-i}{n-h-j}\binom{i+j-1}{j} s^{i+j-1} \Big/ \binom{n}{h} \qquad (2) \\ &= s^{i-1}\, {}_2F_1[h-n, i; h+1; 1-s] \end{aligned}$$

on using the lemma with $k = n - h$, $b = i$ and $a = h - i + 1$.

Theorem 4 of [1] states that if $a_{i,n} = s^{i-1}$ for all $i = 1, \cdots, n$, then

$$a_{i,h}^2 > a_{i-1,h}a_{i+1,h} \quad \text{for } i = 2, \cdots, h-1 \quad \text{and } h \leqq n-1.$$

Applying (2), we obtain

$$\begin{aligned} &s^{2i-2}\{{}_2F_1[h-n, i; h+1; 1-s]\}^2 \\ &> s^{i-2}\{{}_2F_1[h-n, i-1; h+1; 1-s]\}\, s^{i}\{{}_2F_1[h-n, i+1; h+1; 1-s]\}. \end{aligned}$$

The theorem now follows by substituting for h, n and s.

Remark. An analytic proof of the theorem has been shown to the author by Dr. Tyson of the Center for Naval Analyses.

REFERENCES

[1] J. B. KADANE, *A moment problem for order statistics*, Ann. Math. Statist., 42 (1971), pp. 745–751.

[2] J. G. SKELLAM, *A probability distribution derived from the binomial distribution by regarding the probability of success as variable between sets of trials*, J. Roy. Statist. Soc. Ser. B, 10 (1948), pp. 257–261.

[3] L. J. SLATER, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

A NEGATIVE DEFINITE EQUILIBRIUM AND ITS INDUCED CONE OF GLOBAL EXISTENCE FOR THE RICCATI EQUATION*

R. S. BUCY AND J. RODRIGUEZ-CANABAL†

Abstract. The domain of global existence for the Riccati equation is shown to include a cone with vertex at the negative definite equilibria and containing the positive definite equilibria, under regularity conditions.

We consider the autonomous matrix Riccati equation

$$dP/dt = FP + PF' - PH'HP + GG', \quad P_0 = \Gamma, \tag{1}$$

where [1] $F \in M_{n,n}(R)$, $H \in M_{s,n}(R)$, $G \in M_{n,r}(R)$, $\Gamma = \Gamma' \in M_{n,n}(R)$ ($M_{l,m}(R)$ is the set of $l \times m$ matrices with real coefficients). Equation (1) arises in the linear filtering problem (see [1]) with the physical constraint Γ positive semidefinite (i.e., $\Gamma \geq 0$). Our purpose in this note is to show that solutions of (1) globally exist in a certain region of Γ's larger than the cone $\tilde{C}$ of nonnegative definite matrices, where we established global existence previously in [1]. We shall find it necessary to assume that the triple (H, F, G) is completely controllable and completely observable. The applications which motivated consideration of the global existence problem for $\Gamma \notin \tilde{C}$ was a study of the conjugate point structure of a Jacobi accessory problem related to min-max control (see [3]).

Our method consists of showing that (1) possesses a *negative* definite equilibria P_- and employing a rather simple comparison theorem of Reid to show solutions of (1) exist for $\Gamma - P_- \geq 0$, a cone $\tilde{C}_-$ containing C as $-P_-$ is positive definite. It is shown by more intricate arguments in [3], that for $\Gamma \notin C_-$ solutions of (1) have finite escape time. Further, $P_- = -S^{-1}$ with S, the positive definitive matrix used in [2], to show that the infinite lag filtering error is $(P_+^{-1} + S)^{-1}$ with P_+ the positive definite equilibrium of (1), the Wiener filtering error covariance matrix. Also, as is noted in [2], S^{-1} has physical interpretation as the steady state error variance of the estimate of the present state given future observations.

Results. We will assume the (H, F, G) completely controllable and completely observable and denote this assumption by A_1. The following theorem then holds.

THEOREM 1. *Suppose* A_1 *holds. Then* (1) *has two equilibria* P_+ *and* P_- *with* P_+ *and* $-P_-$ *strictly positive definite. Further, with*

$$\mathbf{H} = \begin{bmatrix} -F' & H'H \\ GG' & F \end{bmatrix},$$

* Received by the editors November 2, 1971.

† Department of Aerospace Engineering, University of Southern California, University Park, Los Angeles, California 90007. This work was supported by the U.S. Air Force, Office of Aerospace Research, under Grant AF-AFOSR-71-214 and the U.S. Army Research Office under Grant DA-ARO-D31-124-71-637.

[1] Prime denotes transpose.

(2) $$\Delta(\mathbf{H})\begin{bmatrix} I \\ P_- \end{bmatrix} = 0,$$

(3) $$(-P_+, I)\Delta(\mathbf{H}) = 0,$$

where $\det(\lambda I - \mathbf{H}) = (-1)^n\Delta(\lambda)\Delta(-\lambda)$ *with* $\Delta(\lambda)$ *a Hurwitz polynomial.*

Proof. The existence of P_+ in the interior of $\tilde{C}$ is implied by the results in [1, Chap. 5], and the Bass–Roth theorem [1, p. 105] implies (3). Consider the system (G', F', H'); it satisfies A_1 and hence, as before, the equation

(4) $$F'S + SF - SGG'S + H'H = 0$$

possesses a unique positive definite solution S. This solution S determines P_- as $-S^{-1}$, and from (4), P_- is an equilibrium solution of (1), which is negative definite by construction. Now the Hamiltonian associated with (G', F', H') is $\mathbf{H}^* = \begin{bmatrix} -F & GG' \\ H'H & F' \end{bmatrix} = \mathbf{H}'$ so that

$$\det(\lambda I - \mathbf{H}^*) = \det(\lambda I - \mathbf{H})$$

and Δ is the same for $\mathbf{H}$ as for $\mathbf{H}^*$. Now the Bass–Roth result applied to $\mathbf{H}^*$ gives

$$(-S, I)\Delta(\mathbf{H}^*) = 0$$

or

$$(I, -S^{-})\Delta(\mathbf{H}^*) = 0$$

as S is positive definite or

$$\Delta(\mathbf{H}^*)'\begin{bmatrix} I \\ P_- \end{bmatrix} = 0,$$

but $\Delta(\mathbf{H}^*)' = [\Delta(\mathbf{H}')]' = \Delta(\mathbf{H})$ so that (2) holds.

Remarks. The Bass–Roth type results (2) and (3) allow explicit determination by *linear* equations of P_+ and P_- or equivalently S. As mentioned in the introduction, S has important filtering applications and interpretations, see [2]. Note that P_- is the unique negative definite equilibrium of (1) as S is the unique positive definite solution of (4). Now our main results have the following form.

THEOREM 2. *Suppose* A_1 *holds. Then the solution of* (1), $\pi(t, \Gamma, t_0)$, *exists for all* $t > t_0$ *when* $\Gamma \in \tilde{C}_- = \{\Gamma \in M_{n,n}(R) | \Gamma = \Gamma', \Gamma + S^{-1} \geq 0\}$. *Further for* $\Gamma \in \tilde{C}_-$,

$$\pi(t, \Gamma, t_0) \geq P_- = -S^{-1}.$$

Proof. We note that P_- is a solution of (1) and that $\Gamma \geq P_-$. Now, since A_1 holds the comparison theorem of Reid implies our result (see Reid [4, Corollary p. 198 or Lemma 2.3]).

Remarks. Notice that Theorem 2 is a remarkable generalization of the one-dimensional situation, where (1) has the form

(5) $$\frac{dP}{dt} = 2fp - \frac{p^2}{r} + q,$$

with r and q positive, and C_- is the half-line $(r(f - \sqrt{f^2 + q/r}), \infty)$. Further solutions of (5) for $\gamma \notin (r(f - \sqrt{f^2 + q/r}), \infty)$ escape to infinity in finite time, and this also generalizes (see [3]). Theorem 2 can be interpreted to imply that a certain variational problem is free of focal points for certain end conditions.

Conclusions. We have shown that in a completely controllable and completely observable system the associated Riccati equation possesses exactly one positive definite equilibrium P and exactly one negative definite equilibrium P_-. Further, the cone C_- induced by P_- has the property that the set $\pi(t, \tilde{C}_-, t_0)$ is bounded below by P_- in the ordering of C for all $t > t_0$. In a future paper, we shall discuss the equilibria of (1) which are neither positive nor negative definite.

Added in revision. After this paper was submitted the authors noted that Theorem 1 was independently discovered by Willems in [5]. One of the referees also pointed this out.

REFERENCES

[1] R. S. Bucy and P. D. Joseph, *Filtering for Stochastic Processes with Applications to Guidance*, Interscience, New York, 1968.

[2] R. S. Bucy, *The Riccati equation and its bounds*, J. Comput. Systems Sci., to appear.

[3] J. Rodriguez-Canabal, Thesis, Department of Aerospace Engineering, Univ. of Southern California, Los Angeles, 1972.

[4] W. T. Reid, *Monotoneity properties of solutions of Hermitian Riccati matrix differential equations*, this Journal, 1 (1970), pp. 195–213.

[5] J. C. Willems, *Least squares stationary optimal control and the algebraic Riccati equation*, IEEE Trans. Automatic Control, AC-6 (1971), pp. 621–634.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

PERTURBATIONS IN NONLINEAR SYSTEMS*

NOAL C. HARBERTSON†

Abstract. If a certain system of nonlinear differential equations has a bounded solution $x(t)$, then for the same system subject to a small perturbation the existence of a solution $y(t)$ which lies "close" to $x(t)$ is established.

1. Introduction. In [7] May has shown that if a certain nonlinear system of differential equations possesses a solution $x(t)$ bounded on the entire real line, then the same system subject to a small perturbation has a solution $y(t)$ which is "close" to $x(t)$ on the entire real line. The method used to prove this involves first finding (using hypotheses called (H_1)) a solution $y(t)$ of an improper integral equation and then showing (using rather restrictive additional hypotheses called (H_2)) that $y(t)$ was a solution of the perturbed differential equation. The objective of this paper is to prove the same result using only hypotheses (called (H)) which are weaker than (H_1) and an alternative method of constructing $y(t)$. In our construction of $y(t)$ we do not employ an improper integral equation. We do show, however, that if (H_1) holds ((H_1) implies our (H)), the $y(t)$ obtained by our procedure satisfies the improper integral equation used by May. We also discuss the relation between our results and earlier work by Marlin and Struble [6] and Fennell and Proctor [3] on perturbed systems similar to ours.

Our proofs utilize the Schauder–Tikhonov theorem, a generalization of the variation of parameters formula, and techniques similar to those introduced for linear systems by Brauer and Wong [1] and Hallam and Heidel [4]. Fennell and Proctor [3] have also used similar methods in treating nonlinear systems.

2. Preliminary considerations. Consider the differential equations

$$x' = f(t, x), \tag{1}$$

$$y' = f(t, y) + \varepsilon g(t, y). \tag{2}$$

We assume f and g are continuous n-functions defined on $R \times \Omega$, where Ω is an open connected subset of R^n. In (2), ε denotes a nonnegative scalar parameter. We also assume that $f_x(t, x) = (\partial f/\partial x)(t, x)$ exists and is continuous on $R \times \Omega$. For $t_0 \in R$, $c \in \Omega$, the solution $x(t)$ of (1) that satisfies $x(t_0) = c$ is denoted by $x(t, t_0, c)$. The principal matrix solution $\Phi(t, t_0, c) = (\partial x/\partial c)(t, t_0, c)$ is the solution of the linear variational equation

$$Z' = f_x(t, x(t, t_0, c))Z,$$

which satisfies $\Phi(t_0, t_0, c) = I$, where I is the $n \times n$ identity matrix.

Let D be a bounded subregion of Ω such that the closure of D is contained in Ω. If $x(t)$ is a fixed bounded solution of (1) defined on R that lies in D and has no limit

* Received by the editors August 31, 1971, and in revised form January 5, 1972.

† Department of Mathematics, Fresno State College, Fresno, California 93710. This work was supported by the U.S. Army Research Office, Durham.

points on the boundary of D, then there exists a $d > 0$ such that

$$\{x_0 : |x_0 - x(t)| \leqq d \text{ for some } t \in R\} \subseteq D.$$

If m is a natural number, we denote by $\mathscr{C}_m$ the set of all continuous n-functions defined on $[-m, m]$, and define the space $\mathscr{S}_m$ by

$$\mathscr{S}_m = \{y \in \mathscr{C}_m : |y(t) - x(t)| \leqq d, \ -m \leqq t \leqq m\},$$

with the sup norm. (We use $|\cdot|$ to denote any appropriate vector or matrix norms.)

We suppose that (1) may be written as a pair of uncoupled equations

$$x_1' = f_1(t, x_1), \tag{3}$$

$$x_2' = f_2(t, x_2), \tag{4}$$

where f_1 is a k-function, f_2 is an $(n - k)$-function, x_1 a k-vector and x_2 is an $(n - k)$-vector. With similar notation we can write (2) as

$$y_1' = f_1(t, y_1) + \varepsilon g_1(t, y), \tag{5}$$

$$y_2' = f_2(t, y_2) + \varepsilon g_2(t, y). \tag{6}$$

We assume that, for arbitrary $t_0 \in R$ and $z = (z_1, z_2) \in \bar{D}$ (again z_1 is a k-vector and z_2 an $(n - k)$-vector), the solutions $x_1(t, t_0, z_1)$ and $x_2(t, t_0, z_2)$ of (3) and (4) exist and are defined on the half lines $[t_0, \infty)$ and $(-\infty, t_0]$ respectively. We also assume that Ω can be factored in the form $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 is a subregion of real k-space and Ω_2 is a subregion of real $(n - k)$-space. Furthermore we suppose that $x_1(t, t_0, z_1) \in \Omega_1$ for $t \geqq t_0$ and $x_2(t, t_0, z_2) \in \Omega_2$ for $t \leqq t_0$. It follows that the solutions $\Phi_1(t, t_0, z_1)$ and $\Phi_2(t, t_0, z_2)$ of the linear variational equations corresponding to (3) and (4) exist and are defined for $t \geqq t_0$ and $t \leqq t_0$ respectively.

Finally, we make use of the following hypotheses.

(H) There exists a positive constant J such that for every natural number m we have

$$\left| \int_t^m \Phi_1(t, s, y_1(s)) g_1(s, y(s))\, ds \right| \leqq J \quad \text{for } m \geqq t \geqq -m$$

and

$$\left| \int_{-m}^t \Phi_2(t, s, y_2(s)) g_2(s, y(s))\, ds \right| \leqq J \quad \text{for } -m \leqq t \leqq m$$

and for all $y(t) = (y_1(t), y_2(t)) \in \mathscr{S}_m$.

3. Existence of solutions of (2) "close" to $x(t)$.

THEOREM 1. *Let $x(t)$ be a fixed bounded solution of* (1) *defined on R that lies in D without limit points on the boundary of D and let* (H) *hold. Then for $\varepsilon \leqq d/J$ there exists a solution $y(t)$ of* (2) *defined on R such that*

$$\sup_{t \in R} |y(t) - x(t)| \leqq d.$$

Proof. For simplicity we take the case $f(t, x) = f_1(t, x_1)$ and omit subscripts. Using the Schauder–Tikhonov theorem (see [2]) we first show that there is a function $y_m \in \mathscr{S}_m$ which is a solution of (2) on $[-m, m]$. Note that $\mathscr{S}_m$ with the sup norm is a convex space and consider the transformation $\mathscr{T}$ defined on $\mathscr{S}_m$ by

$$\mathscr{T}y(t) = x(t) - \varepsilon \int_t^m \Phi(t, s, y(s))g(s, y(s))\, ds. \tag{7}$$

Since $\varepsilon \leqq d/J$ we have from (H) and (7) that

$$|\mathscr{T}y(t) - x(t)| \leqq \varepsilon \left| \int_t^m \Phi(t, s, y(s))g(s, y(s))\, ds \right| \leqq \varepsilon J \leqq d \tag{8}$$

for $-m \leqq t \leqq m$. If $y(t), z(t) \in \mathscr{S}_m$, then

$$\begin{aligned} |\mathscr{T}y(t) - \mathscr{T}z(t)| &= \varepsilon \left| \int_t^m [\Phi(t, s, y(s))g(s, y(s)) - \Phi(t, s, z(s))g(s, z(s))]\, ds \right| \\ &\leqq \varepsilon \left| \int_t^m [\Phi(t, s, y(s)) - \Phi(t, s, z(s))]g(s, y(s))\, ds \right| \\ &\quad + \varepsilon \left| \int_t^m \Phi(t, s, z(s))[g(s, y(s)) - g(s, z(s))]\, ds \right|. \end{aligned} \tag{9}$$

From the uniform continuity of $\Phi(t, s, w)$ on $[-m, m] \times [-m, m] \times \bar{D}$ and the uniform continuity of $g(s, w)$ on $[-m, m] \times \bar{D}$ it follows from (9) that $\mathscr{T}$ is a continuous mapping on $\mathscr{S}_m$. Thus (8) and (9) together imply that $\mathscr{T}$ is a continuous mapping from $\mathscr{S}_m$ into $\mathscr{S}_m$. We also have that the functions in the image set $\mathscr{T}\mathscr{S}_m$ are an equicontinuous family. To prove this let $t_1, t_2 \in [-m, m]$ and (without loss of generality) suppose $t_2 \geqq t_1$. Using (7) we may write

$$\begin{aligned} |\mathscr{T}y(t_1) - \mathscr{T}y(t_2)| &\leqq |x(t_1) - x(t_2)| + \varepsilon \left| \int_{t_2}^m \Phi(t_2, s, y(s))g(s, y(s))\, ds \right. \\ &\qquad \left. - \int_{t_1}^m \Phi(t_1, s, y(s))g(s, y(s))\, ds \right| \\ &\leqq |x(t_1) - x(t_2)| + \varepsilon \int_{t_1}^{t_2} |\Phi(t_1, s, y(s))g(s, y(s))|\, ds \\ &\quad + \varepsilon \int_{t_2}^m |[\Phi(t_2, s, y(s)) - \Phi(t_1, s, y(s))]g(s, y(s))|\, ds. \end{aligned}$$

Again using the uniform continuity of Φ and g and the uniform continuity of $x(t)$ on $[-m, m]$ it follows easily from this last inequality that the image set $\mathscr{T}\mathscr{S}_m$ is an equicontinuous family. From the Schauder–Tikhonov theorem we conclude that there exists at least one fixed point $y_m(t)$ of $\mathscr{T}$ in $\mathscr{S}_m$. Accordingly, there is a $y_m(t) \in \mathscr{S}_m$ that satisfies

$$y_m(t) = x(t) - \varepsilon \int_t^m \Phi(t, s, y_m(s))g(s, y_m(s))\, ds \tag{10}$$

for $-m \leqq t \leqq m$. Equation (10) is a generalization of the well-known variation of parameters formula, and it has been shown (see [6] for example) that since $y_m(t)$ satisfies (10) it is a solution of the differential equation (2) on $[-m, m]$.

We now construct a sequence of solutions of (2) that converges uniformly on compact subsets of R. For any natural number M consider for $t \in [-M, M]$

the sequence $\{y_m(t)\}_{m=M}^{\infty}$ of fixed points (and therefore solutions of (2)) given by (10). Since $x(t)$ is bounded on R it follows from (H) and (10) that $\{y_m(t)\}_{m=M}^{\infty}$ is uniformly bounded on $[-M, M]$. Since each $y_m(t)$ is also a solution of the differential equation (2) it follows that $\{y_m(t)\}_{m=M}^{\infty}$ is an equicontinuous sequence for $t \in [-M, M]$. From Ascoli's theorem there exists a subsequence $\{y_{m_1}(t)\}_{m_1=1}^{\infty}$ of $\{y_m(t)\}_{m=M}^{\infty}$ that converges uniformly on $[-M, M]$. By considering the sequence $\{y_{m_1}(t)\}_{m_1=M+1}^{\infty}$ we can construct by the same procedure a subsequence $\{y_{m_2}(t)\}_{m_2=1}^{\infty}$ of $\{y_{m_1}(t)\}_{m_1=M+1}^{\infty}$ that converges uniformly on the interval $[-M-1, M+1]$. Furthermore, it is obvious that $\{y_{m_1}(t)\}_{m_1=1}^{\infty}$ and $\{y_{m_2}(t)\}_{m_2=1}^{\infty}$ both converge to the same limit on the interval $[-M, M]$. In fact, by induction we can construct for each natural number l a sequence $\{y_{m_l}(t)\}_{m_l=1}^{\infty}$ of solutions of (2) which converges uniformly on $[-M-l, M+l]$ and with the property that $\{y_{m_{l-1}}(t)\}_{m_{l-1}=1}^{\infty}$ and $\{y_{m_l}(t)\}_{m_l=1}^{\infty}$ converge to the same limit on $[-M-l+1, M+l-1]$.

Now define $y(t)$ on R as the limit as m tends to infinity of the diagonal sequence $\{y_{m_m}(t)\}_{m=1}^{\infty}$. It is clear that for t in any compact subset of R the diagonal sequence converges uniformly to $y(t)$. It follows that $y(t)$ itself is a solution of the differential equation (2) on R. We also have

$$y(t) - x(t) = \lim_{m\to\infty} [y_{m_m}(t) - x(t)],$$

and for $t \in [-m, m]$,

$$|y_{m_m}(t) - x(t)| \leqq d,$$

so

$$\sup_{t\in R} |y(t) - x(t)| \leqq d.$$

Thus we have proved Theorem 1.

4. A comparison of (H) and (H_1) and a theorem concerning an improper integral equation. Let $\mathscr{C}$ denote the continuous n-functions defined on R and define

$$\mathscr{S} = \{y \in \mathscr{C} : |y(t) - x(t)| \leqq d, t \in R\}.$$

The hypotheses (H_1) used by May in [7] are as follows.

(H_1) There exist functions $J_1(T, t)$ and $J_2(T, t)$ defined for $T \geqq t$ and $T \leqq t$, respectively, with the following properties:

(i) $$\int_T^{\infty} |\Phi_1(t, s, y_1(s))g_1(s, y(s))|\, ds \leqq J_1(T, t) \quad \text{for } T \geqq t,$$

and

$$\int_{-\infty}^{T} |\Phi_2(t, s, y_2(s))g_2(s, y(s))|\, ds \leqq J_2(T, t) \quad \text{for } T \leqq t$$

and for all $y \in \mathscr{S}$.

(ii) $$\lim_{T\to\infty} J_1(T, t) = 0 \quad \text{and} \quad \lim_{T\to-\infty} J_2(T, t) = 0,$$

uniformly for t in compact subsets of R.

(iii) There exists a constant J such that

$$J_1(t, t) \leqq J \quad \text{and} \quad J_2(t, t) \leqq J$$

for all $t \in R$.

It is obvious that (H_1) implies our (H) so we have an immediate corollary to Theorem 1.

COROLLARY 1. *Let $x(t)$ be a fixed bounded solution of* (1) *defined on R that lies in D without limit points on the boundary of D, and let* (H_1) *hold. Then for $\varepsilon \leqq d/J$ there exists a solution $y(t)$ of* (2) *defined on R such that*

$$\sup_{t \in R} |y(t) - x(t)| \leqq d.$$

Proof. Since (H_1) implies (H) the proof in Theorem 1 suffices.

We can also prove a theorem which relates our $y(t)$ to the techniques used in [3], [5], [7].

THEOREM 2. *Let the hypotheses in Corollary* 1 *hold. Then the solution $y(t)$ of* (2) *in Corollary* 1 *satisfies the improper integral equation*

$$y(t) = x(t) - \int_t^\infty \Phi(t, s, y(s))g(s, y(s))\, ds.$$

(In stating and proving this theorem we are still assuming that $f(t, x) = f_1(t, x_1)$ and omitting subscripts.) We know that the solution $y(t)$ of (2) in Corollary 1 is the limit of the diagonal sequence $\{y_{m_m}(t)\}_{m=1}^\infty$, where each $y_{m_m}(t)$ satisfies (10) and the convergence is uniform for t in compact subsets of R. Since $y(t)$ obviously lies in $\mathscr{S}$ we know from (H_1) that the improper integral

$$\int_t^\infty \Phi(t, s, y(s))g(s, y(s))\, ds$$

exists. We now show that

$$\lim_{m \to \infty} \int_t^m \Phi(t, s, y_{m_m}(s))g(s, y_{m_m}(s))\, ds = \int_t^\infty \Phi(t, s, y(s))g(s, y(s))\, ds. \tag{11}$$

Given $\eta > 0$ and for $t \in [-k, k]$, k an arbitrary natural number, choose $T > k$ such that $J(T, t) < \eta/6$. (This is possible since (ii) in (H_1) holds.) For any $m > T$ we can write

$$\begin{aligned}
&\left| \int_t^\infty \Phi(t, s, y(s))g(s, y(s))\, ds - \int_t^m \Phi(t, s, y_{m_m}(s))g(s, y_{m_m}(s)) ds \right| \\
&\quad = \left| \int_m^\infty \Phi(t, s, y(s))g(s, y(s))\, ds + \int_t^m \Phi(t, s, y(s))g(s, y(s))\, ds \right. \\
&\qquad \left. - \int_t^m \Phi(t, s, y_{m_m}(s))g(s, y_{m_m}(s))\, ds \right| \\
&\quad \leqq J(m, t) + \int_T^m |\Phi(t, s, y(s))g(s, y(s))|\, ds \\
&\qquad + \int_T^m |\Phi(t, s, y_{m_m}(s))g(s, y_{m_m}(s))|\, ds \qquad \text{(cont.)}
\end{aligned}$$

$$+ \left| \int_t^T [\Phi(t, s, y(s))g(s, y(s)) - \Phi(t, s, y_{m_m}(s))g(s, y_{m_m}(s))]\, ds \right|$$

$$\leqq J(m, t) + J(T, t) + J(T, t)$$

$$+ \int_t^T |\Phi(t, s, y(s))g(s, y(s)) - \Phi(t, s, y_{m_m}(s))g(s, y_{m_m}(s))|\, ds.$$

It follows from this last inequality, the uniform continuity of $\Phi(t, s, w)$ on $[-T, T] \times [-T, T] \times \bar{D}$, and the uniform convergence of $y_{m_m}(t)$ on $[-T, T]$ that there exists a natural number M such that for $m > M$,

$$\left| \int_t^\infty \Phi(t, s, y(s))g(s, y(s))\, ds - \int_t^m \Phi(t, s, y_{m_m}(s))g(s, y_{m_m}(s))\, ds \right| < \eta.$$

Thus (11) holds. From (10) and (11) we have for every $t \in R$ (recall that k was arbitrary in the proof of (11)) that

$$y(t) = x(t) - \int_t^\infty \Phi(t, s, y(s))g(s, y(s))\, ds, \tag{12}$$

as was required. Equation (12) is of interest since it was used directly in the proofs in [3], [6], [7]. In fact with his stronger hypotheses May in [7] was able to show that every function $y \in \mathscr{S}$ that satisfies (12) is a solution of (2). (We would not expect this result to hold with our weaker hypotheses.) A detailed discussion of (H_1) together with some more "natural" conditions related to (H_1) can be found in [6], [7] and especially [5].

4. Theorem 1, Corollary 1 and the theorems in [3], [6], [7]. Theorem 1 and Corollary 1 are very similar to results in [3], [6], [7]. A comparison of the two assumptions (H_1) and (H_2) used in [7] with our (H) shows that we have substantially improved on the results in [7]. Comparison with the results in [3] and [6] is somewhat more involved. In [3], [6] assumptions are made which imply that the improper integral on the right-hand side of (12) exists and tends to zero for large t. This would most commonly hold only if the perturbation term tended to zero for large t. The assumption we have made in (H) clearly allows for truly persistent perturbations which do not necessarily tend to zero. Of course the stronger assumptions in [3], [6] also allow for truly asymptotic results while we can only show that $y(t)$ and $x(t)$ are "close" in the sense defined in Theorem 1. Also, in [3], [6] the differential equations considered do not explicitly contain the parameter ε in their perturbation terms. We include the parameter ε in our equation (2) since to assume that solutions of (5) and (6) exist (see § 2) on arbitrary half-lines would not be reasonable without some compensating (generally small) parameter so that the perturbations are not "too" persistent. Furthermore, it is not uncommon for systems with a parameter to arise naturally or to have a parameter introduced by some procedure (such as the method of averaging). Finally, we should comment that the techniques used in obtaining our theorems are very similar to those in [1], [3], [4]. The basic difference between all of these papers lies in the assumptions about the differential equations on which these techniques are used.

REFERENCES

[1] FRED BRAUER AND J. S. W. WONG, *On asymptotic behavior of perturbed linear systems*, J. Differential Equations, 6 (1969), pp. 142–153.

[2] W. A. COPPEL, *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.

[3] R. E. FENNELL AND T. G. PROCTOR, *On asymptotic behavior of perturbed nonlinear systems*, Proc. Amer. Math. Soc., to appear.

[4] T. G. HALLAM AND J. W. HEIDEL, *The asymptotic manifolds of a perturbed linear system of differential equations*, Trans. Amer. Math. Soc., 149 (1970), pp. 233–241.

[5] N. HARBERTSON AND R. A. STRUBLE, *Integral manifolds for perturbed nonlinear differential equations*, Applicable Anal., 1 (1971), pp. 241–278.

[6] J. A. MARLIN AND R. A. STRUBLE, *Asymptotic equivalence of nonlinear systems*, J. Differential Equations, 6 (1969), pp. 578–596.

[7] L. E. MAY, *Perturbations in fully nonlinear systems*, this Journal, 1 (1970), pp. 376–391.

SIAM J. MATH. ANAL.
Vol. 3, No. 4, November 1972

THE BEHAVIOR OF OSCILLATORY SOLUTIONS OF $x''(t) + p(t)g(x(t)) = 0$*

STEPHEN R. BERNFELD† AND JAMES A. YORKE‡

Abstract. Various quantitative properties of oscillatory solutions of the scalar second order nonlinear differential equation $x'' + p(t)g(x) = 0$ are obtained under appropriate hypotheses on p and g. In particular, letting $\{t_i\}_{i=1}^{\infty}$, $0 < t_i < t_{i+1}$, $t_i \to \infty$ as $i \to \infty$, be the zeros of any solution $x(t)$, we obtain inequalities on $J_i \stackrel{\text{def}}{\equiv} \int_{t_i}^{t_{i+1}} g(x(t))\,dt$ which yield asymptotic behavior on $x(t)$. For example, it is shown that $\lim_{t\to\infty} \int_0^t g(x(s))\,ds$ exists and is finite; moreover, assuming an added growth condition on $g(x)/x$, we have then that $\lim_{t\to\infty}\int_0^t x(s)\,ds$ exists and is finite.

1. Introduction. In this paper we investigate the behavior of oscillatory solutions of the scalar second order nonlinear ordinary differential equation

$$x''(t) + p(t)g(x(t)) = 0. \tag{E}$$

We always assume that solutions of (E) are unique (given $x(t_0)$, $x'(t_0)$). Assume

$$g: R \to R \quad \text{is continuous} \quad \text{and} \quad yg(y) > 0 \quad \text{for } y \neq 0. \tag{1.1}$$

Moreover, we assume

$$\begin{aligned} &p:[0, \infty) \to (0, \infty) \text{ is piecewise continuous nondecreasing,} \\ &p(t) \to \infty \quad \text{as } t \to \infty. \end{aligned} \tag{1.2}$$

A solution $x(t)$ of (E) is said to be *oscillatory* if the set of zeros of x is a countable sequence of points $\{t_i\}_{i=1}^{\infty}$ such that

$$t_i < t_{i+1} \quad \text{and} \quad t_i \to \infty \quad \text{as } i \to \infty.$$

If in addition to (1.1) and (1.2) we assume

$$G(r) \to \infty \quad \text{as } r \to \infty \tag{1.3}$$

where $G(r) = \int_0^r g(x)\,dx$, then it is well known [2] that all solutions of (E) are bounded and oscillatory.

For the second order linear equation

$$x'' + p(t)x = 0, \tag{L}$$

where $p(t)$ satisfies (1.2), then it is well known that the amplitudes of any oscillatory solution $x(t)$ form a nonincreasing sequence and that $t_{i+1} - t_i \to 0$ as $i \to \infty$, where once again $x(t_i) = 0$ (see [4, p. 226] and [7], for example). Using this, Hartman [3, Cor. 3.1, p. 513] has proved that $\lim_{t\to\infty} \int_0^t x(s)\,ds$ exists and is finite with the assumption $\lim_{t\to\infty} x(t) = 0$.

* Received by the editors July 30, 1971, and in revised form February 15, 1972.

† University of Missouri, Columbia, Missouri 65201. The work of this author was supported in part by a University of Missouri Summer Research Grant.

‡ Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland 20742. The work of this author was supported in part by the National Science Foundation under Grant NSF-GP-27284.

In Theorem 1 we obtain a useful description of the behavior of oscillatory solutions of (E) on small intervals by using comparison-type techniques. These methods are quite different from those used in analyzing (L) since we cannot use the Sturm theorems. We show, in fact,

$$\frac{|2x'(t_i)|}{p(t_i)} \geqq \left| \int_{t_i}^{t_{i+1}} g(x(s))\, ds \right| \geqq \frac{|2x'(t_{i+1})|}{p(t_{i+1})}$$

(thus implying $|\int_{t_{i+1}}^{t_{i+2}} g(x(s))\, ds| \leqq |\int_{t_i}^{t_{i+1}} g(x(s))\, ds|$)

and

$$\lim_{i \to \infty} \left| \int_{t_i}^{t_{i+1}} g(x(s))\, ds \right| = 0,$$

where $x(s)$ is any oscillatory solution of (E) whose zeros are $\{t_i\}, t_i < t_{i+1}$. An immediate consequence is that $\lim_{t\to\infty} \int_0^t g(x(s))\, ds$ exists and is finite. For $g(x) = x$ we extend Hartman's result; the assumption "$x(t) \to 0$ as $t \to \infty$" is not needed. By imposing additional restrictions on g we show in Theorem 3 that $\lim_{t\to\infty} \int_0^t x(s)\, ds$ exists and is finite.

As mentioned before, for (L) it has been proved that if $p \in C'$ and satisfies (1.2), then the values of $|x(t)|$ at which $x'(t) = 0$ of any oscillatory solution form a nonincreasing sequence. Others later conjectured that if p is continuous, then all solutions $x(t)$ of (L) satisfy

$$x(t) \to 0 \quad \text{as } t \to \infty. \tag{1.4}$$

Examples have since been given in [5] showing that (1.4) need not occur. It is in fact quite easy to construct a piecewise constant p (dropping the continuity conditions on p) in which $p(t) \to \infty$ as $t \to \infty$ and such that there is a solution which does not satisfy (1.4). Theorems have been proved for (L) (see [1, p. 88]) which guarantee (1.4) by imposing various regularity conditions on p, which essentially insure that p does not behave like a step function. Rather than imposing more and more elaborate conditions on p, we shall change the manner in which all solutions must approach zero. Namely for any L^1-function $\psi : [0, \infty) \to R$ and any continuous function $y : [0, \infty) \to R$ we define a type of convolution operator for $t \geqq 0$,

$$(y * \psi)(t) = \int_0^\infty y(t + s)\psi(s)\, ds. \tag{1.5}$$

If y is a bounded rapidly oscillating physical phenomenon, then one might not be able to measure $y(t)$ but one may be able to measure $(y * \psi)(t)$ since we are essentially smoothing y. (Of particular interest are those ψ which are zero except in a small neighborhood of zero and $\int_0^\infty \psi = 1$.) We are then led to the following definition of the asymptotic behavior of any function.

DEFINITION. A function $y : [0, \infty) \to R$ is said to be *weakly asymptotic to zero* if for every $\psi \in L^1([0, \infty))$,

$$y * \psi(t) \to 0 \quad \text{as } t \to \infty,$$

where $y * \psi(t)$ satisfies (1.5).

In Theorem 2 we show that under conditions (1.1), (1.2) and (1.3), $g(x(t))$ is weakly asymptotic to zero and that under more restrictive conditions on g each solution $x(t)$ is weakly asymptotic to zero.

For a survey of results see [8].

2. Results. We shall present our main results.

THEOREM 1. *Assume conditions* (1.1) *and* (1.3) *and that* $p:[0, \infty) \to (0, \infty)$ *is piecewise continuous and nondecreasing. Let* $\{t_i\}_{i=1}^{\infty}$, $t_i < t_{i+1}$, *be the set of zeros of any solution* x. *Define*

$$J_i = \int_{t_i}^{t_{i+1}} g(x(t))\,dt.$$

Then the following inequalities are satisfied:

$$\frac{2|x'(t_i)|}{p(t_i)} \geqq |J_i| \geqq \frac{2|x'(t_{i+1})|}{p(t_{i+1})}, \qquad |J_{i+1}| \leqq |J_i|. \tag{2.1}$$

If in addition, (1.2) *holds, then*

$$|J_i| \to 0 \quad \textit{as } i \to \infty.$$

COROLLARY 1. *Assume conditions* (1.1), (1.2), *and* (1.3). *Then for every solution* $x(t)$ *of* (E), $\lim_{t\to\infty} \int_0^t g(x(s))\,ds$ *exists and is finite.*

THEOREM 2. *Assume conditions* (1.1), (1.2) *and* (1.3). *Then for every solution* $x(t)$ *of* (E), $g(x(t))$ *is weakly asymptotic to zero.*

THEOREM 3. *Assume conditions* (1.1) *and* (1.2) *are satisfied. Assume* g *is an odd function and* $g(x)/x$ *is nonincreasing for* $x > 0$. *Then for every oscillatory solution* $x(t)$ *of* (E),

$$\lim_{T\to\infty} \int_0^T x(t)\,dt \quad \textit{exists and is finite.} \tag{2.2}$$

Remark. Every solution of

$$x'' + p(t)x^{1/(2n+1)} = 0, \qquad n = 0, 1, 2, \cdots,$$

where (1.2) holds, satisfies (2.2) since $g(x) = x^{1/(2n+1)}$ satisfies the hypotheses of Theorem 3. For $n = 0$ we obtain (L) which as we have mentioned has been analyzed by Hartman.

Example 1. We now show that (2.2) need not hold without the hypothesis used in Theorem 3 that g is odd. Define

$$g(x) = \begin{cases} x, & x > 0, \\ 2x, & x \leqq 0. \end{cases} \tag{2.3}$$

Consider the equation

$$x'' + kg(x) = 0, \qquad x(0) = 0, \qquad x'(0) = 1. \tag{2.4}$$

Let $\{t_i\}_{i=0}^{\infty}$ be the zeros of the solution $x(t)$ of (2.4), where $t_0 = 0$. Then $x(t) > 0$ for $t \in (t_{2i}, t_{2i+1})$ and $x(t) < 0$ for $t \in (t_{2i+1}, t_{2i+2})$. Moreover, an easy calculation yields the results

(2.5) $$\int_{t_{2i}}^{t_{2i+1}} x(t)\,dt = 2/k, \qquad \int_{t_{2i+1}}^{t_{2i+2}} x(t)\,dt = -1/k.$$

We assume g satisfies (2.3) and construct a piecewise continuous function p as follows. Let $p(t) = 1$ for $t \in [s_0, s_2)$, where $s_0 = 0$ and s_1 and s_2 are the first three zeros of the solution $y(t)$ satisfying

$$y'' + g(y) = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

Define $p(t) = i + 1$ for $t \in [s_{2i}, s_{2i+2})$, where s_{2i}, s_{2i+1} and s_{2i+2} are consecutive zeros of $y(t)$ satisfying

$$y'' + (i + 1)g(y) = 0, \qquad y(s_{2i}) = 0, \quad y'(s_{2i}) = 1.$$

Since $s_{2i+2} - s_{2i} = \pi(i + 1)^{-1/2} + \pi(2i + 2)^{-1/2}$, we have $\sum_{i=0}^{\infty} (s_{2i+2} - s_{2i}) = \infty$, and hence $p(t)$ is defined for $t \in [0, \infty)$. Hence it follows that since $y(s_i) = 0$ and $y'(s_i) = 1$ for all i, $y(t)$ satisfies

$$y'' + p(t)g(y) = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

Using (2.5) we have

$$\int_{s_{2i}}^{s_{2i+1}} y(t)\,dt = \frac{2}{i+1}$$

and

$$\int_{s_{2i+1}}^{s_{2i+2}} y(t)\,dt = \frac{-1}{i+1}.$$

Then for any n,

$$\int_{0}^{s_{2n+2}} x(t)\,dt = \sum_{i=0}^{n} \frac{1}{i+1},$$

so $\int_0^\infty x(t)\,dt$ is not finite. Of course, from Corollary 1 we know $\int_0^\infty g(x(t))\,dt$ exists. In fact, a calculation yields $\int_0^\infty g(x(t))\,dt = 0$.

Remark. It is curious that the proof of Theorem 1 (and all the results that follow from it) use the assumption that solutions of (E) are unique and we do not know if the results remain true without this assumption. We also do not know any examples of p and g satisfying (1.2) and (1.1) respectively for which solutions of (E) fail to be unique!

3. Proof of Theorem 1.

LEMMA 1. *Assume conditions* (1.1) *and* (1.2) *are satisfied. Let* $x(t)$ *be an oscillatory solution of* (E) *and let* $\{t_i\}_{i=1}^{\infty}$ *be the consecutive zeros of* $x(t)$. *Then*

(3.1) $$\frac{|x'(t_i)|}{p(t_i)} \text{ tends monotonically to } 0 \quad \text{as } i \to \infty.$$

Proof. Letting $r(t)$ be any function we shall use the following Dini derivatives:

$$D^+r(t) = \limsup_{h\to 0^+} \frac{r(t+h)-r(t)}{h},$$

$$D^-r(t) = \liminf_{h\to 0^+} \frac{r(t+h)-r(t)}{h}.$$

Since $p(t)$ is a nondecreasing function, $D^-p(t) \geqq 0$. Define

$$V(t) = \frac{(x'(t))^2}{2p(t)} + G(x(t)), \tag{3.2}$$

where $G(r) = \int_0^r g(u)\,du$. Then

$$\begin{aligned} D^+V(t) &= \frac{x'(t)x''(t)}{p(t)} - \frac{(x'(t))^2 D^-p(t)}{2p^2(t)} + g(x(t))x'(t) \\ &= \frac{-(x'(t))^2 D^-p(t)}{2p^2(t)} \leqq 0. \end{aligned}$$

Since p is piecewise continuous and V decreases at the discontinuities of p it follows that V is nonincreasing. Hence $V(t_i)$ is nonincreasing. From (3.2) and (1.2) we have

$$\left(\frac{x'(t_i)}{p(t_i)}\right)^2 = 2\frac{V(t_i)}{p(t_i)} \to 0 \quad \text{as } i \to \infty.$$

We now prove two comparison results.

LEMMA 2. *Let $x(t)$ and $y(t)$ be two C^2-functions with*

$$\begin{aligned} &x(s_1) = y(s_1), \\ &x'(s_1) = 0, \qquad y'(s_1) > 0. \end{aligned} \tag{3.3}$$

Assume x and y are strictly increasing on $[t_1, s_1]$ and $[u_1, s_1]$ respectively with

$$x(t_1) = y(u_1) = 0. \tag{3.4}$$

Assume x'' and y'' are negative on $(t_1, s_1]$ and $(u_1, s_1]$, respectively. Moreover, assume

$$y''(u) < x''(t), \tag{3.5}$$

when $t \in (t_1, s_1)$ and $u \in (u_1, s_1)$ satisfy

$$x(t) = y(u). \tag{3.6}$$

Then for such t and u,

$$x'(t) < y'(u). \tag{3.7}$$

Furthermore,

$$u_1 > t_1 \quad \text{and} \quad x(u) > y(u) \quad \text{for all } u \in (u_1, s_1). \tag{3.8}$$

Moreover, if g is any function satisfying (1.1), *then*

$$\int_{t_1}^{s_1} g(x(t))\,dt > \int_{u_1}^{s_1} g(y(s))\,ds. \tag{3.9}$$

Proof. Since x and y are strictly increasing with $x(t_1) = y(u_1) = 0$ and $x(s_1) = y(s_1)$, there is a unique continuous differentiable one-to-one, onto mapping $T:[u_1, s_1] \to [t_1, s_1]$ such that $x(T(u)) = y(u)$ for $u \in [u_1, s_1]$. We can in fact define

$$T(u) = x^{-1}(y(u)). \tag{3.10}$$

From (3.5) we have for $u \in (u_1, s_1]$,

$$y''(u) < x''(T(u)),$$

and we first must show (3.7) holds; that is, for $u \in (u_1, s_1]$,

$$x'(T(u)) < y'(u). \tag{3.11}$$

For $u = s_1$, (3.11) holds. Let u_2 be the smallest number in $[u_1, s_1]$ such that (3.11) holds for $u \in (u_2, s_1]$. If $u_2 = u_1$, then (3.11) is proved. If $u_2 \neq u_1$, then

$$x'(T(u_2)) = y'(u_2). \tag{3.12}$$

From (3.10),

$$\frac{d}{du}y(u) = \frac{d}{du}x(T(u)) = x'(T(u))T'(u),$$

so

$$T'(u) = \frac{y'(u)}{x'(T(u))}. \tag{3.13}$$

Using (3.12) we have $T'(u_2) = 1$, and defining

$$\Delta(u) = x'(T(u)) - y'(u)$$

we obtain

$$\begin{aligned}\Delta'(u_2) &= x''(T(u_2))T'(u_2) - y''(u_2)\\ &= x''(T(u_2)) - y''(u_2) > 0.\end{aligned}$$

Since $\Delta(u_2) = 0$ and $\Delta'(u_2) > 0$, we have that $\Delta(u) > 0$ for $u \in (u_2, u_2 + \delta]$ for some $\delta > 0$; that is, $x'(T(u)) > y'(u)$ for $u \in (u_2, u_2 + \delta]$. But this is a contradiction since (3.11) holds for $u \in (u_2, s_1]$. Thus (3.11) holds for all $u \in (u_1, s_1]$.

We now show (3.8) holds. Since $x' > 0$ on $[t_1, s_1)$ and $y' > 0$ on $[u_1, s_1]$ we have from (3.11) and (3.13) that

$$T'(u) > 1. \tag{3.14}$$

Then for $u \in [u_1, s_1)$,

$$s_1 - T(u) = \int_u^{s_1} T'(r)\,dr > \int_u^{s_1} dr = s_1 - u,$$

thus implying $u > T(u)$. In particular, $u_1 > t_1$ and since x and y are strictly increasing, we have for $u \in [u_1, s_1)$,

$$x(u) > x(T(u)) = y(u),$$

thus proving (3.8).

We now show (3.9) holds. Observe

$$g(y(u))T'(u)\,du = g(x(T(u))\,d(T(u)). \tag{3.15}$$

Using (3.14) and (3.15), we have

$$\begin{aligned}\int_{u_1}^{s_1} g(y(u))\,du &< \int_{u_1}^{s_1} g(y(u))T'(u)\,du \\ &= \int_{u_1}^{s_1} g(x(T(u))\,d(T(u)) \\ &= \int_{t_1}^{s_1} g(x(t))\,dt,\end{aligned}$$

thus concluding the proof of Lemma 2.

LEMMA 3. *Assume* (1.1) *holds and* $p:[0,\infty)\to(0,\infty)$ *is piecewise continuous. Let* $x(t)$ *be an oscillatory solution of* (E) *in which* t_1, t_2 *are two consecutive zeros of* $x(t)$ *with* $t_1 < t_2$ *and* $x(t) > 0$ *for* $t \in (t_1, t_2)$. *Let* $s_1 \in (t_1, t_2)$ *be the point where the maximum of* $|x(t)|$ *occurs. Let* $r:[t_1, s_1] \to R$ *be continuous. Assume there is a unique solution* $y_0(t)$ *for*

$$y'' + r(t)g(y) = 0, \qquad y(s_1) = x(s_1), \quad y'(s_1) = 0, \tag{R}$$

where $r(t) \geqq p(s_1) \geqq p(t)$. *Let* u_1 *be the first zero of* y_0 *less than* s_1 *and* $y_0 > 0$ *on* $(u_1, s_1]$. *Then* $t_1 \leqq s_1$ *and*

$$\begin{aligned}&x(t) \geqq y_0(t) \quad \text{for } t \in [u_1, s_1], \\ &\int_{t_1}^{s_1} g(x(t))\,dt \geqq \int_{u_1}^{s_1} g(y_0(t))\,dt.\end{aligned} \tag{3.16}$$

Proof. For $\varepsilon > 0$ let $r_\varepsilon(t) = r(t) + \varepsilon$ and let y_ε be a solution of

$$y'' + r_\varepsilon(t)g(y) = 0, \quad y(s_1) = x(s_1), \quad y'(s_1) = \varepsilon.$$

On $[t_1, s_1]$ and $[u_\varepsilon, s_1]$ where u_ε is the first zero of y_ε less than s_1, we thus have the hypotheses of Lemma 2 holding. Hence,

$$t_1 < u_\varepsilon < s_1, \quad x(t) > y_\varepsilon(t) \quad \text{on } [u_\varepsilon, s_1], \tag{3.17}$$

and

$$\int_{t_1}^{s_1} g(x(t))\,dt > \int_{u_\varepsilon}^{s_1} g(y_\varepsilon(t))\,dt. \tag{3.18}$$

Since $y_0(t)$ is the unique solution of (R) on $[u_1, s_1]$ we have by continuous dependence,

$$\begin{aligned}&y_\varepsilon(t) \to y_0(t) \quad \text{as } \varepsilon \to 0 \text{ uniformly for } t \in [u_1, s_1], \\ &u_\varepsilon \to u_1 \text{ as } \varepsilon \to 0.\end{aligned} \tag{3.19}$$

By use of (3.17), (3.18), (3.19) it follows that (3.16) holds, thus concluding the proof of Lemma 3.

We are now able to prove Theorem 1.

Proof. Assume conditions (1.1) and (1.3) hold and that $p:(0, \infty) \to (0, \infty)$ is piecewise continuous and nondecreasing. Let $\{t_i\}_{i=1}^{\infty}$, $t_i < t_{i+1}$ be the set of zeros of any solution x. Define

$$J_i = \int_{t_i}^{t_{i+1}} g(x(t))\,dt.$$

We prove

$$\frac{2|x'(t_i)|}{p(t_i)} \geqq |J_i| \geqq \frac{2|x'(t_{i+1})|}{p(t_{i+1})} \tag{3.20}$$

by first showing

$$\left|\int_{t_i}^{s_i} g(x(t))\,dt\right| \geqq \left|\int_{s_i}^{t_{i+1}} g(x(t))\,dt\right|, \tag{3.21}$$

where $s_i \in (t_i, t_{i+1})$ and $x'(s_i) = 0$. Define

$$r(t) = p(2s_i - t) \quad \text{for } t \in [t_i, s_i).$$

From the monotonicity of $p(t)$ we have $r(t) \geqq p(t)$. Let $y_0(t)$ be a solution of

$$y'' + r(t)g(y), \qquad y(s_i) = x(s_i), \quad y'(s_i) = 0. \tag{R}$$

Note that if $y_1(t)$ is another solution on $[t_i, s_i]$, then $x_0(t) = y_0(2s_i - t)$ and $x_1(t) = y_1(2s_i - t)$ are solutions of (E) with $x_0(s_i) = x_1(s_i) = x(s_i)$ and $x_0'(s_i) = x_1'(s_i) = 0$. Since solutions of (E) are unique, $x_0 \equiv x_1$ and $y_0 \equiv y_1$, so (R) has a unique solution. Hence $y_0(t)$ satisfies the hypotheses of Lemma 3. Letting u_i denote the first zero of $y_0(t)$ less than s_i we have $u_i \geqq t_i$ and from (3.16) conclude

$$\int_{t_i}^{s_i} g(x(t))\,dt \geqq \int_{u_i}^{s_i} g(y_0(t))\,dt. \tag{3.22}$$

Moreover from the definition of $r(t)$ and the uniqueness of solutions of (E) we have $y_0(t) = x(2s_i - t)$ for $t \in [2s_i - t_{i+1}, s_i]$; thus $u_i = 2s_i - t_{i+1}$ which implies $s_i - u_i = t_{i+1} - s_i$. Therefore,

$$\int_{u_i}^{s_i} g(y_0(t))\,dt = \int_{s_i}^{t_{i+1}} g(x(t))\,dt,$$

and with the use of (3.22) we have proved (3.21).

We now prove (3.20). Observing that

$$|x'(t_i)| = |x'(s_i) - x'(t_i)| = \left|\int_{t_i}^{s_i} x''(t)\,dt\right|$$

$$= \left|\int_{t_i}^{s_i} p(t)g(x(t))\,dt\right| \geqq p(t_i)\left|\int_{t_i}^{s_i} g(x(t))\,dt\right|;$$

and using (3.21) we have

$$2|x'(t_i)| \geqq p(t_i)\left[\int_{t_i}^{s_i} g(x(t))\,dt + \int_{s_i}^{t_{i+1}} g(x(t))\,dt\right]$$

$$= p(t_i)\left|\int_{t_i}^{t_{i+1}} g(x(t))\,dt\right|,$$

thus proving the first inequality in (3.20). The inequality is reversed for $x'(t_{i+1})$; that is,

$$2|x'(t_{i+1})| = 2|x'(t_{i+1}) - x'(s_i)| = 2\left|\int_{s_i}^{t_{i+1}} p(t)g(x(t))\,dt\right|$$

$$\leqq 2p(t_{i+1})\left|\int_{s_i}^{t_{i+1}} g(x(t))\,dt\right| \leqq p(t_{i+1})\left|\int_{t_i}^{t_{i+1}} g(x(t))\,dt\right|,$$

proving the second inequality in (3.20).

We now claim

$$\operatorname{sgn} J_{i+1} = -\operatorname{sgn} J_i, \tag{3.23}$$

$$|J_{i+1}| \leqq |J_i|, \tag{3.24}$$

$$J_i \to 0 \quad \text{as } i \to \infty \tag{3.25}$$

(using the additional hypothesis (1.2)), and hence, from the alternating series test, $\sum_{i=1}^{\infty} J_i$ exists and is finite, thus implying $\int_0^\infty g(x(t))\,dt$ exists and is finite.

Since $x(t)$ is oscillatory, $g(x(t))$ is oscillatory, and hence (3.23) holds.

From (3.20) we immediately obtain (3.24).

We need now only to prove (3.25). With the use of (3.20) and Lemma 1 we obtain

$$|J_i| \leqq \frac{2|x'(t_i)|}{p(t_i)} \to 0 \quad \text{as } i \to \infty,$$

thus proving (3.25) which concludes the proof of Theorem 1 and Corollary 1.

Remark. J. S. W. Wong has suggested an alternative proof of Corollary 1 which does not make use of Theorem 1. This proof uses an additional (stronger) hypothesis that p is absolutely continuous. We now present his proof since the techniques are of some interest.

Alternative proof. Divide the terms of (E) by $p(t)$ and integrate the first term by parts. This gives

$$\left.\frac{x'}{p}\right|_0^t + \int_0^t \frac{x'(s)p'(s)}{p^2(s)}\,ds + \int_0^t g(x(s))\,ds = 0. \tag{3.26}$$

From Lemma 1, $|x'(t)^2/p(t)|$ is bounded by some number B, so $x'(t)^2/p^2$ tends to 0 as $t \to \infty$. Therefore the first term of (3.26) tends to a constant as $t \to \infty$. To prove the corollary, that $\lim_{t\to\infty} \int_0^t g(x(s))\,ds$ exists and is finite, it is sufficient to prove that the integrand, $x'(s)p'(s)/p(s)^2$ in equation (3.26) is absolutely integrable on $[T, \infty)$ for some T. To see this, observe that

$$\int_T^\infty \left|\frac{x'(s)p'(s)}{p^2(s)}\right| ds \leqq B^{1/2}\int_T^\infty \frac{p'(s)}{p^{3/2}(s)}\,ds = B^{1/2}\int_T^\infty p^{-3/2}\,dp < \infty.$$

Therefore the result is proved.

4. Proof of Theorem 2.

LEMMA 4. *Assume conditions* (1.1) *and* (1.2) *hold and that g is odd. Assume* $x(t)$ *is an oscillatory solution of* (E) *and let* $\{t_i\}_{i=1}^{\infty}$ *be the successive zeros of* $x(t)$. *Let* s_i

be the point where the maximum of $|x(t)|$ *occurs on the interval* (t_i, t_{i+1}). *Then* $|x(s_{i+1})| \leqq |x(s_i)|$ *for all i.*

Proof. Define

$$V(t) = (x'(t))^2/2p(t) + G(x(t)).$$

It was shown in Lemma 1 that V is a nonincreasing function. Hence $V(s_{i+1}) \leqq V(s_i)$. Since $x'(s_i) = 0$,

$$G(x(s_{i+1})) \leqq G(x(s_i)).$$

Using (1.1), and the fact that g is odd we have $|x(s_{i+1})| \leqq |x(s_i)|$.

Remark. Using the proof of Lemma 4 observe that if (1.1) and (1.2) are satisfied (without the assumption g is odd), then $G(x(s_{i+1})) \leqq G(x(s_i))$ implies $|x(s_{i+2})| \leqq |x(s_i)|$, so $|x(t)|$ is bounded.

Proof of Theorem 2. Let $x(t)$ be any solution of (E). From Theorem 1, $\int_0^\infty g(x(t))\,dt < \infty$; hence

$$\lim_{t\to\infty} \int_a^b g(x(t+s))\,ds = 0 \quad \text{for any } b > a \geqq 0. \tag{4.1}$$

Moreover, since $x(t)$ is bounded (as a consequence of the remark following Lemma 4) and g is continuous, there exists an $M > 0$ such that $|g(x(t))| \leqq M$ for $t \in [0, \infty)$. Let ψ be any function in $L^1[0, \infty)$. For any $\varepsilon > 0$ there exists a $\tau > 0$ such that

$$\int_\tau^\infty |\psi(s)|\,ds < \frac{\varepsilon}{4M}.$$

On the interval $[0, \tau]$ there exists a piecewise constant function $\mu(t)$ such that

$$\int_0^\tau |\psi(s) - \mu(s)|\,ds < \frac{\varepsilon}{4M}$$

(see [6, p. 77]). Define $\mu(t) = 0$ for $t > \tau$. Hence,

$$\begin{aligned}\int_0^\infty |\psi(s) - \mu(s)|\,ds &= \int_0^\tau |\psi(s) - \mu(s)|\,ds + \int_\tau^\infty |\psi(s)|\,ds \\ &\leqq \frac{\varepsilon}{4M} + \frac{\varepsilon}{4M} = \frac{\varepsilon}{2M}.\end{aligned}$$

Thus,

$$\begin{aligned}\left|\int_0^\infty g(x(t+s))\psi(s)\,ds\right| &= \left|\int_0^\infty g(x(t+s))(\psi(s) - \mu(s))\,ds \right.\\ &\qquad \left. + \int_0^\infty g(x(t+s))\mu(s)\,ds\right| \\ &\leqq \frac{\varepsilon}{2} + \left|\int_0^\tau g(x(t+s))\mu(s)\,ds\right|.\end{aligned} \tag{4.2}$$

Since $\mu(s)$ is a piecewise constant function there exists a finite number N of values $\{C_i\}_{i=1}^N$ of μ on $[0, \tau]$, and a set of $\{a_i\}_1^{N+1}$ where $0 = a_1 < \cdots < a_i < \cdots < a_{N+1} = \tau$ such that

$$\mu(t) = C_i \quad \text{for } t \in [a_i, a_{i+1}).$$

Using (4.1), there exists $\{T_i\}_{i=1}^N$ such that

$$\left|C_i \int_{a_i}^{a_{i+1}} g(x(t+s))\,ds\right| < \frac{\varepsilon}{2N} \quad \text{for } t > T_i. \tag{4.3}$$

Let $T = \max(T_1, \cdots, T_N)$. For $t > T$, we have, using (4.3),

$$\left|\int_0^\tau g(x(t+s))\mu(s)\,ds\right| = \left|\sum_{i=1}^N C_i \int_{a_i}^{a_{i+1}} g(x(t+s))\,ds\right|$$
$$\leqq \sum_{i=1}^N \left|C_i \int_{a_i}^{a_{i+1}} g(x(t+s)\,ds\right| < \frac{\varepsilon}{2}.$$

From (4.2),

$$|(g(x) * \psi)(t)| = \left|\int_0^\infty g(x(t+s))\psi(s)\,ds\right| < \varepsilon \quad \text{for } t > T.$$

Hence, since ε is arbitrary, $g(x)$ is weakly asymptotic to zero.

5. Proof of Theorem 3.

Proof of Theorem 3. Let $x(t)$ be an oscillatory solution; let $\{t_i\}_{i=1}^\infty$ be the successive zeros of x; and let $s_i \in (t_i, t_{i+1})$ be those points such that $|x(s_i)|$ is the maximum of $|x(t)|$ for $t \in (t_i, t_{i+1})$. Define

$$U_i = \int_{t_i}^{t_{i+1}} x(t)\,dt; \tag{5.1}$$

we shall show

$$\operatorname{sgn} U_{i+1} = -\operatorname{sgn} U_i, \tag{5.2}$$

$$|U_{i+1}| \leqq |U_i|, \tag{5.3}$$

$$U_i \to 0 \quad \text{as } i \to \infty, \tag{5.4}$$

and thus conclude that $\int_0^\infty x(t)\,dt$ exists and is finite.

Since $x(t)$ is oscillatory (5.2) follows immediately. Assume i is such that $x(t) > 0$ for $t \in (t_i, t_{i+1})$.

We now prove (5.3). Define

$$r(t) = p(t + (s_{i+1} - s_i)) \quad \text{for } t \in [t_i, t_{i+1}].$$

For $t \in [t_i, t_{i+1}]$, let $z(t)$ be a solution of

$$z'' + r(t)g(z) = 0, \qquad z'(s_i) = 0, \quad z(s_i) = -x(s_{i+1}). \tag{Z}$$

Since g is an odd function, and from the uniqueness of the solutions of (E),

$$z(t) = -x(t + (s_{i+1} - s_i)). \tag{5.5}$$

We now show that for as long as $z(t) > 0$ we have $z(t) \leqq x(t)$ for $t \in [t_i, t_{i+1}]$. From Lemma 4,

$$z(s_i) = |x(s_{i+1})| \leqq x(s_i).$$

Consider the two cases $z(s_i) < x(s_i)$ and $z(s_i) = x(s_i)$.

Case 1. Assume $z(s_i) < x(s_i)$. Define

$$\eta = (x/z)';$$

hence,

$$\begin{aligned}(z^2\eta)' &= (zx' - xz')' \\ &= zx'' - xz'' = -zpg(x) + xrg(z) \\ &= xz\left(-p\frac{g(x)}{x} + r\frac{g(z)}{z}\right).\end{aligned} \tag{5.6}$$

There exists a $\delta > 0$ such that $0 < z(t) < x(t)$ for $t \in J \overset{\text{def}}{\equiv} (s_i - \delta, s_i + \delta)$. Since $g(x)/x$ is nonincreasing,

$$\frac{g(z(t))}{z(t)} \geqq \frac{g(x(t))}{x(t)} \quad \text{for } t \in J,$$

and with $r(t) \geqq p(t)$ we have from (5.6),

$$(z^2\eta)' \geqq 0. \tag{5.7}$$

Since $\eta(s_i) = 0$ and from (5.7), we conclude

$$z^2(t)\eta(t) \geqq 0 \quad \text{for } t \in (s_i, s_i + \delta),$$
$$z^2(t)\eta(t) \leqq 0 \quad \text{for } t \in (s_i - \delta, s_i),$$

and thus

$$\eta(t) \geqq 0 \quad \text{for } t \in [s_i, s_i + \delta),$$
$$\eta(t) \leqq 0 \quad \text{for } t \in (s_i - \delta, s_i).$$

Therefore $x(t)/z(t)$ has its minimum value at $t = s_i$ at which $x(s_i)/z(s_i) > 1$. Thus, as long as $z \neq 0$, we have that $x(t) > z(t)$.

Case 2. Assume $z(s_i) = x(s_i)$. Let $z_n(t)$ be a solution of the system

$$z'' + r(t)g(z) = 0, \qquad z'(s_i) = 0,$$
$$z(s_i) = x(s_i) - 1/n, \qquad n = 1, 2, \cdots.$$

By our previous arguments we have proved that as long as $z_n(t) > 0$, then $z_n(t) < x(t)$. By continuous dependence $z_n(t) \to z(t)$ uniformly for $t \in [t_i, t_{i+1}]$. Hence, for as long as $z(t) > 0$, then $z(t) \leqq x(t)$.

Hence, if a and b, $a < s_i < b$, are the first zeros of z less than s_i and greater than s_i respectively, then $t_i \leqq a < b \leqq t_{i+1}$. From (5.5), $z(t) = -x(t + (s_{i+1} - s_i))$ for all $t \in [a, b]$; hence, $b - a = t_{i+2} - t_{i+1}$, thus yielding

$$t_{i+2} - t_{i+1} \leqq t_{i+1} - t_i. \tag{5.8}$$

Since $x(t) \geqq z(t)$, we also have

$$\int_{t_i}^{t_{i+1}} x(t)\,dt \geqq \int_a^b z(t)\,dt = \left|\int_{t_{i+1}}^{t_{i+2}} x(t)\,dt\right|,$$

thus proving (5.3) for the case where $U_i > 0$. Using these same techniques it follows that (5.3) holds for $U_i < 0$.

To prove (5.4) it is sufficient to show

$$t_{i+1} - t_i \to 0 \quad \text{as } i \to \infty,$$

since $|x(t)|$ is bounded. Define the extended real-valued function

$$h(t) = \frac{g(x(t))}{x(t)} p(t), \qquad t > 0, \quad t \neq t_i,$$

and $h(t_i) = \infty$. From the hypotheses on g there exists a $B > 0$ such that $g(x(t))/x(t) \geqq B$. Then from (1.2), $h(t) \to \infty$ as $t \to \infty$. For any integer k pick an integer $I = I(k)$ so large that for $t \geqq t_I$, $h(t) \geqq k$. Observe that $x(t)$ satisfies

$$x'' + h(t)x = 0,$$

$$x'(s_I) = 0.$$

Let $z(t)$ be a solution of

$$z'' + kz = 0, \quad z(s_I) = x(s_I), \quad z'(s_I) = 0.$$

Since $g(x) = x$ satisfies the hypotheses of the theorem and since $h(t) \geqq k$ for $t \in (t_I, t_{I+1})$, then from our previous analysis we may conclude that $z(t) \geqq x(t)$ for $t \in (t_I, t_{I+1})$. If a and b are any two consecutive zeros of z, then from (5.8),

$$k^{-1/2}\pi = b - a \geqq t_{I+1} - t_I \geqq t_{i+1} - t_i \quad \text{for } i \geqq I.$$

Since k is an arbitrary positive integer, $t_{i+1} - t_i \to 0$ as $i \to \infty$, thus proving (5.4), and hence Theorem 3 is proved.

6. Concluding remarks. If in Theorem 3 the assumption "$g(x)/x$ is non-increasing" is omitted, then it may not be necessarily true that $\int_0^\infty x(t)\,dt$ exists and is finite. One does not expect the same "nice" behavior governing the zeros of oscillatory solutions. However, on finite intervals it may be possible that $\int x(t)\,dt$ behaves well. In particular, one interesting open problem is whether under the weaker hypotheses we have $\int_0^a x(t + s)\,ds \to 0$ as $t \to \infty$ for any $a > 0$ (still assuming g is an odd function).

It seems that our results can be extended to include the equation

$$\text{(Y)} \qquad (r(t)x')' + p(t)g(x) = 0$$

under appropriate general hypotheses on $r(t)$ by transforming (Y) back to (E). Finally, it would be interesting to investigate the nonlinear equation

$$x'' + f(t, x) = 0,$$

where $f:[0, \infty) \times R \to [0, \infty)$ is continuous such that $f(t, x) \nearrow \infty$ as $t \to \infty$ for each x, $xf(t, x) > 0$ for all t and $x \neq 0$, and $f(t, x)$ is an odd function of x for each t. A natural problem then would be to find additional conditions on $f(t, x)$ so as to obtain results similar to those presented in this paper.

Acknowledgments. We would like to thank A. Lasota for his suggestions concerning the definition of weakly asymptotic to zero, W. Leighton for his suggestions which led to a shorter proof of Case 1 of Theorem 3, and P. Hartman for several comments. We also would like to thank D. Haley for his interesting discussions and J. S. W. Wong for the alternative proof of Corollary 1.

REFERENCES

[1] R. BELLMAN, *Perturbation Techniques in Mathematics, Physics, and Engineering*, Holt, New York, 1964.

[2] N. BHATIA, *Some oscillation theorems for second order differential equations*, J. Math. Anal. Appl., 15 (1966), pp. 442–446.

[3] P. HARTMAN, *Ordinary Differential Equations*, John Wiley, New York, 1964.

[4] W. LEIGHTON, *Ordinary Differential Equations*, Wadsworth, Calif., 1970.

[5] G. PRODI, *Un' osser vazione sugl' integrali dell' equazione* $y'' + A(x)y = 0$ *nel caso* $A(x) \to \infty$ *per* $x \to \infty$, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 8 (1950), pp. 462–464.

[6] H. ROYDEN, *Real Analysis*, Macmillan, New York, 1965.

[7] A. WIMAN, *Über eine stabilitätsfrage in der theorie der linearen differentialgleichungen*, Acta Math., 66 (1936), pp. 121–145.

[8] J. S. W. WONG, *On second order nonlinear oscillation*, Funkcial. Ekvac., 11 (1968), pp. 207–234.